

## NUMERICAL SOLUTION OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATION: METHOD OF LINES AND CRANK-NICHOLSON METHOD

MD ROKNUJJAMAN<sup>1,\*</sup>, MOHAMMAD ASIF AREFIN<sup>2</sup>, AND ATISH KUMAR JOARDER<sup>3</sup>

**ABSTRACT.** The method of lines (MOL) is a solution procedure for solving partial differential equation (PDE) and the Crank-Nicholson method (CNM) is an implicit finite difference method, used to solve the elliptic equation and similar partial differential equations numerically. In this study, the numerical techniques were introduced to solve elliptic partial differential equation (PDE) together with initial and boundary conditions. The Poisson equation was considered as an elliptic PDE. Particularly, two methods were used to solve a two-dimensional Poisson equation numerically, such as the method of lines (MOL) and Crank-Nicholson method (CNM). The implementation of the solutions was done using Microsoft office spreadsheet and MATLAB programming language. Lastly, the numerical solutions were presented graphically which was obtained with method of lines along with Crank-Nicholson method.

### 1. INTRODUCTION

The Method of Lines is a semi-discrete computational approach that estimates space derivatives by converting the problem into a system of initial value ordinary differential equations (ODEs). This method involves to solve the ODE system, allowing for efficient and accurate solutions [1–6]. To enhance accuracy, highly efficient initial value ODE solvers can be employed, allowing for corresponding orders of accuracy in time integration without the requirement for incredibly short time steps. In existing applications, partial differential equations play a crucial role in various fields, with a focus on the sciences and engineering [6–8]. Many studies in scientific and engineering domains are modeled using partial differential equations, including studies in thermoelasticity [8, 9], fluid flow [10, 11], and so on. There is a growing emphasis on the development, analysis, and implementation of numerical methods to solve these problems.

Partial differential equations usually arise when a dependent variable depends on two or more independent variables. Computing analytic solutions for these equations is difficult, leading to the utilization other approaches are used. Numerical methods are widely employed to address engineering problems, while taking boundary conditions into account. Generally, the domain is discretized into a finite number of components, or discretized into small triangles or rectangles. The values of the dependent variable at the

---

<sup>1</sup>DEPARTMENT OF MECHANICAL ENGINEERING, ROCHESTER INSTITUTE OF TECHNOLOGY, ROCHESTER, NY 14620, UNITED STATE

<sup>2</sup>DEPARTMENT OF MATHEMATICS, JASHORE UNIVERSITY OF SCIENCE AND TECHNOLOGY, JASHORE-7408, BANGLADESH

<sup>3</sup>DEPARTMENT OF MATHEMATICS, ISLAMIC UNIVERSITY, KUSHTIA, BANGLADESH

*E-mail addresses:* mdreme@rit.edu, asif.math@just.edu.bd, atishjoardar@yahoo.com.

Submitted on Oct. 17, 2024.

2020 *Mathematics Subject Classification.* 35A01, 65L10, 65L12, 65L20, 65L70.

*Key words and phrases.* initial and boundary conditions, Poisson equation, MOL, CNM.

\*Corresponding author.

nodes or grid points within these components are then computed. Another important class of ODEs originates from the application of the general approach the Method of Lines (MOL) to time-dependent Partial Differential Equations (PDE). In this method, the PDEs system is first spatially discretized, which results in a coupled system of ODEs in time only. Then, any well-established numerical method is applied to achieve an accurate approximate solution to the problem [12]. Two broadly applicable techniques include Finite Difference, which is local methods and Spectral Methods, which are global methods.

In this study, the Method of lines and Crank-Nicholson method are applied to solve two dimensional Poisson equations with Dirichlet boundary conditions using spreadsheet [13]. The MOL is applied to approximate the PDE problem to reduce it to simpler system of ODEs. This system of ODEs is then represented in matrix form which, eigenvalues will be calculated. Therefore, the objective of this research is to propose an efficient numerical method for the numerical solution of PDE. This paper is organized as follows; In Sec. 2, The Method of lines was introduced and discussed. The mathematical formulation of Poisson equation with initial and Dirichlet boundary conditions was explained and the numerical solution procedure were describe in Sec. 3. The discussion is explained in Sec.4 and Section 5 is devoted to the conclusion of this study.

## 2. THE METHOD OF LINES

The numerical approach known as the method of lines involves solving partial differential equations by discretizing all dimensions except one and then integrating the semi-discrete problem as a system of ordinary differential equations (ODEs) [1–6]. An advantage of this method is it allows the solution to take advantage of the sophisticated general-purpose methods and software that have been developed for numerically integrating ODEs. The Method of lines is applicable for the partial differential equations; the method usually proves to be quite efficient. The method of lines is well-suited for solving partial differential equations. It is important that the PDE problem is well-posed as an initial value (Cauchy) problem in at least one dimension. This requirement is essential because the ODE integrators employed are designed for solving initial value problems.

Additionally, this method applied to the equations involves a time variable ( $t$ ) and one or more space variables ( $x_1, x_2, x_m$ ), transforms them into a system of ordinary differential equations (ODEs) [14]. The part of the equations related to space variables is discretized, resulting in an approximation of the PDE in the form of a system of ODEs. This system of ODEs can be directly integrated using a standard ODE code. The MOL is often chosen for its ability to solve general and complex PDEs with relative ease and acceptable efficiency. Additionally, MOL suggest the advantage of using theoretical knowledge from ODEs to address PDEs, and the convenience of powerful ODE solvers [15–22]. The connection between partial and ordinary differential equations was known by Lagrange in 1759 [3], leading to the derivation of a system of ordinary differential equations.

$$(1) \quad \left. \begin{aligned} y_1'' &= K^2(-2y_1 + y_2) \\ y_2'' &= K^2(y_1 - 2y_2 + y_3) \\ &\vdots \\ y_n'' &= K^2(y_{n-1} - 2y_n). \end{aligned} \right\}$$

The D'Alemberts equation observed by Lagrange is given by

$$(2) \quad \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

With a finite number of mass points attached to a light string, the same equations (1) are found. It is also noted that Fourier In 1807 [3] motivated by the problem of heat conduction and obtained the equation

$$(3) \quad y_i' = K^2 (y_{i-1} - 2y_i + y_{i+1}) \quad i = 1, 2, \dots, N$$

The heat equation was obtained by considering the value of  $N$  to be large

$$(4) \quad \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Nevertheless, the numerical application of the method of lines (MOL) was first applied by Rothe in 1930 [23]. He considered the method to address a parabolic equation.

$$(5) \quad u_{xx} = R(x, t) u_t + S(x, t, u),$$

With  $R > 0, 0 \leq x \leq 1, 0 \leq t \leq T$ . He used the approach to estimate the equation after discretizing the time variables in equation (5).

$$(6) \quad u_{n+1}'' = R(x, t_{n+1}) \frac{u_{n+1} - u_n}{h} + S(x, t_{n+1}, u_n), \quad t_n = nh$$

Equation (6) represents a transversal approach for integrating the ordinary differential equations (ODEs) along lines parallel to the  $x$ -axis. In other words, transversal designs results to boundary value problems, while longitudinal systems give rise to initial value problems. In this paper, we have established a solution methodology for partial differential equations (PDEs) using the Method of Lines (MOL) along with the Crank-Nicolson Method (CNM). Subsequently, we have examined numerical solutions of the Poisson equation through the application of this solution procedure involving MOL and CNM.

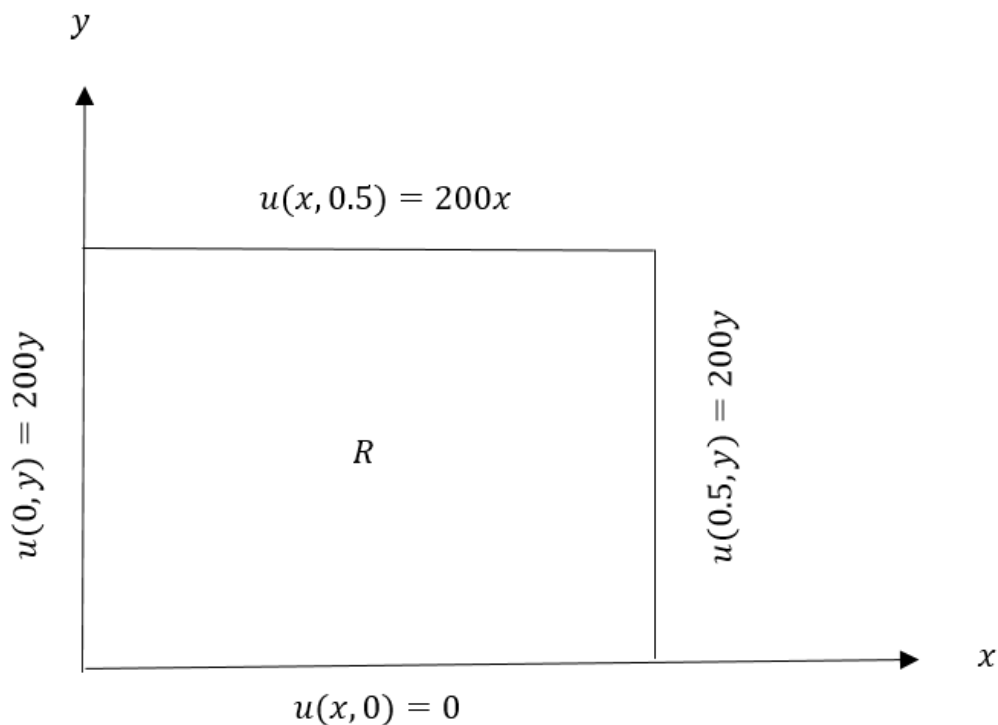


FIGURE 1. Rectangular region  $R$  with boundary conditions.

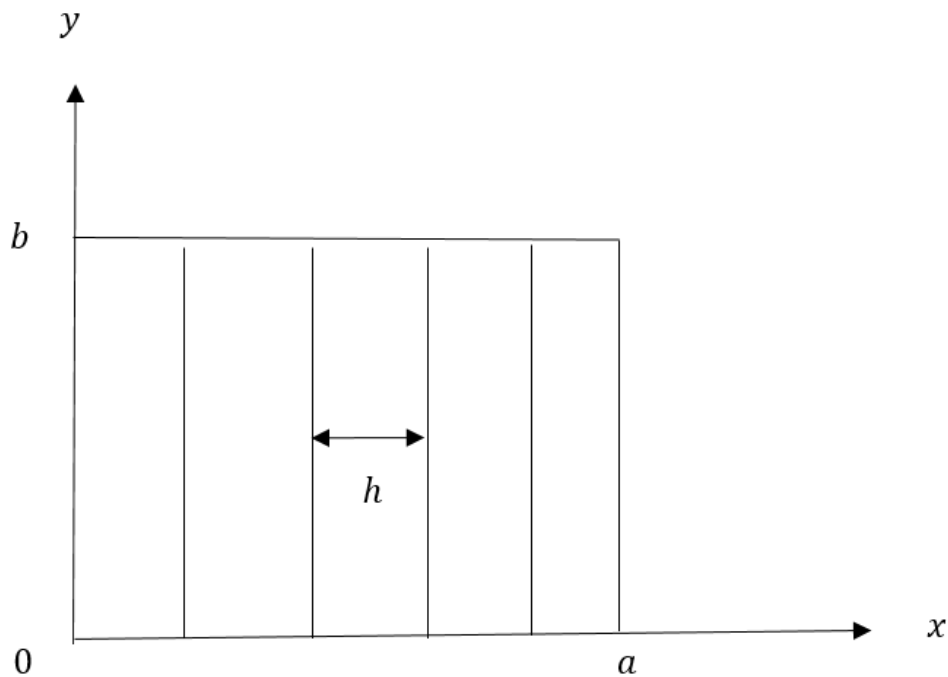


FIGURE 2. Illustration of discretization in the x-directions.

### 3. MATHEMATICAL FORMULATION

The governing equation is given by [15, 16, 20, 22, 24, 25]

$$(7) \quad \Delta u = f(x, y)$$

where the Laplace operator is defined by  $\Delta = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ , Equation (7) is an elliptic second order linear partial differential equation. In equation (7), the dependent variable  $u(x, y)$  is a function of its arguments and depends on the independent variables  $x$  and  $y$ . The function  $f(x, y)$  is known as a source function. To compute the numerical solution of equation (7), we divide the interval into an equal parts with step length  $h = \Delta x = \frac{a}{N+1}$  and considered a plane region defined,  $R = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$

The initial condition;  $u(x, 0) = f(x)$  ;  $a < x < b$ ,

We also impose the Dirichlet boundary condition

$$u(0, y) = 0, \quad u(x, 0) = 0 ; \quad u(x, 0.5) = 200x, \quad u(0.5, y) = 200y.$$

In the region denoted as  $R$ , focus to Dirichlet boundary conditions as illustrated in Figure 1, the dimensions of the variables are associated with the Cartesian coordinate system. Precisely, the length along with the variable  $X$  extends along the positive horizontal  $X$ -axis, while the length along the variable  $Y$  extends along the positive vertical  $Y$ -axis. This geometric alignment provides a clear framework for understanding the spatial distribution and boundary conditions within the designated region  $R$ .

The first step is discretization of the  $x$ -variable. The region is divided into strips by  $N$  dividing straight lines (hence the name method of lines) parallel to the  $y$ -axis. Since we are discretizing along  $x$ , we now split the region  $R$  in Figure 1 into a finite number of rectangular elements. We divide the interval into  $N$  equal parts with step length  $h = \Delta x = \frac{a}{N+1}$  will split the plane region  $R$  into 16 rectangular elements. The nodes and the sides in and on the region,  $R$  can be classified into two groups, viz. interior and exterior.

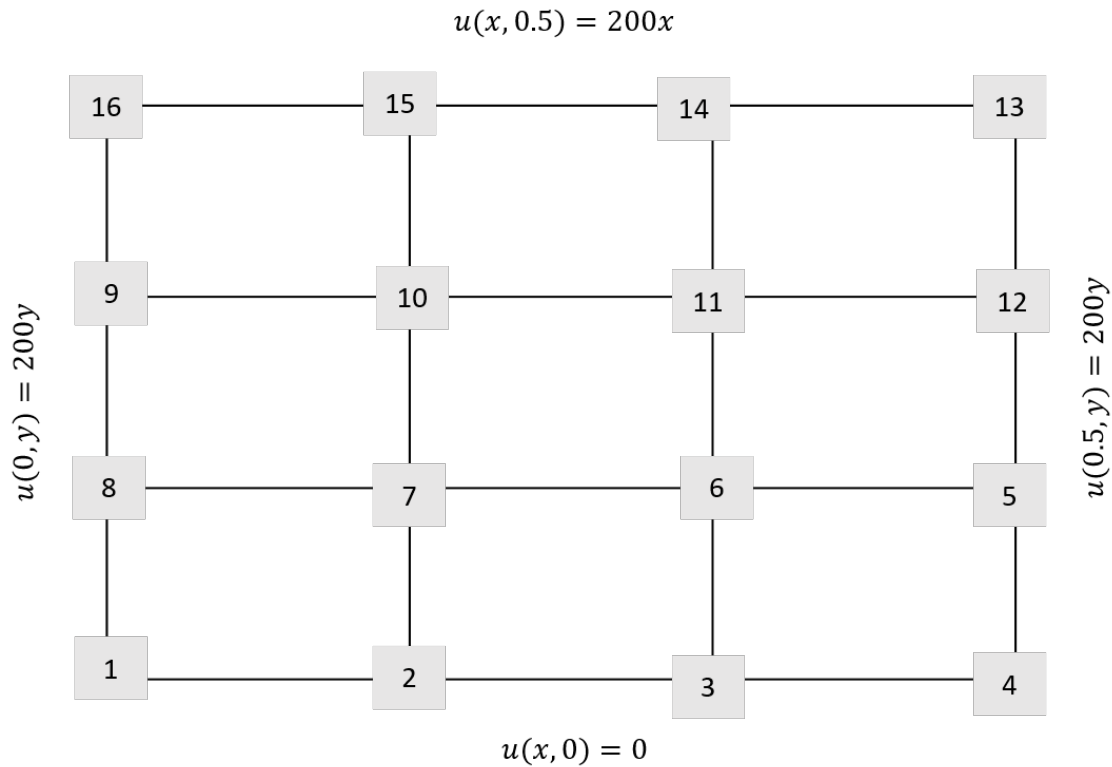


FIGURE 3. Geometric shape with specified Boundary condition.

The interior nodes and sides lie inside the region  $R$  while the exterior nodes and sides lie on the boundary of the region  $R$ . Further, all the interior nodes and sides are common to adjacent rectangular elements. Of course, this fact is exempted for the exterior nodes and sides. The nodes of the region  $R$  are numbered and shown in Figure 3.

**3.1. Numerical Solutions.** Let us illustrate this by an example of poisson equation [24, 25].

$$(8) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

Applying the Crank-Nicholson scheme when  $x$  is independent variable [14]

$$(9) \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{2h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}]$$

Where  $h$  is the spacing of discretized lines [16]

$$h = \Delta x = \frac{a}{N+1}$$

Apply the MOL approximation in Equation (8) becomes,

$$(10) \quad \frac{\partial^2 u_{(i,j)}}{\partial y^2} = \frac{1}{2h^2} [-u_{i+1,j} + 2u_{i,j} - u_{i-1,j} - u_{i+1,j+1} + 2u_{i,j+1} - u_{i-1,j+1}] + f(x, y)$$

Now using the figure which domain is divided into squares of 0.125-unit size, the boundary condition implies that,

$$u_{1,0} = u_{2,0} = u_{3,0} = u_{0,1} = u_{0,2} = u_{0,3} = 0$$

$$u_{1,4} = u_{4,1} = 25, u_{2,4} = u_{4,2} = 50$$

$$\text{and } u_{3,4} = u_{4,3} = 75$$

Let,  $f(x, y) = 1$

Now for each  $i, j=1,2,3, \dots, N$  and applying the boundary conditions;

Considering the node (1,1), then the equation (10) becomes,

$$\begin{aligned} \frac{\partial^2 u_{(1,1)}}{\partial y^2} &= \frac{1}{2h^2} [-u_{2,1} + 2u_{1,1} - u_{0,1} - u_{2,2} + 2u_{1,2} - u_{0,2}] + 1, \\ \frac{\partial^2 u_1}{\partial y^2} &= \frac{1}{2h^2} [-u_2 + 2u_1 - 0 - u_5 + 2u_4 - 0] + 1, \\ (11) \quad \frac{\partial^2 u_1}{\partial y^2} &= \frac{1}{2h^2} [-u_2 + 2u_1 - u_5 + 2u_4] + 1 \end{aligned}$$

Similarly, considering node (1,1), (1,2), (1,3), (2,1), (2,2), (2,3) and (3,1), (3,2), (3,3), then the equation (10) becomes

$$(12) \quad \frac{\partial^2 u_4}{\partial y^2} = \frac{1}{2h^2} [-u_5 + 2u_4 + u_7 - u_8] + 1$$

$$(13) \quad \frac{\partial^2 u_7}{\partial y^2} = \frac{1}{2h^2} [-u_8 + 2u_7] + 1$$

$$(14) \quad \frac{\partial^2 u_2}{\partial y^2} = \frac{1}{2h^2} [-u_3 + 2u_2 - u_1 - u_6 + 2u_5 - u_4] + 1$$

$$(15) \quad \frac{\partial^2 u_5}{\partial y^2} = \frac{1}{2h^2} [-u_6 + 2u_5 - u_4 - u_9 + 2u_8 - u_7] + 1$$

$$(16) \quad \frac{\partial^2 u_8}{\partial y^2} = \frac{1}{2h^2} [-u_9 + 2u_8 - u_7] + 1$$

$$(17) \quad \frac{\partial^2 u_3}{\partial y^2} = \frac{1}{2h^2} [-25 + 2u_3 - u_2 - 50 + 2u_6 - u_5] + 1$$

$$(18) \quad \frac{\partial^2 u_6}{\partial y^2} = \frac{1}{2h^2} [-50 + 2u_6 - u_5 - 75 + 2u_9 - u_8] + 1$$

$$(19) \quad \frac{\partial^2 u_9}{\partial y^2} = \frac{1}{2h^2} [-75 + 2u_9 - u_8 + 2 \times 75 - 50] + 1$$

Again, we consider the Crank-Nicholson Scheme when  $y$  is independent variable

$$(20) \quad \frac{\partial^2 u}{\partial y^2} = \frac{1}{2h^2} [u_{i,j+1} - 2u_{i,j} + u_{i,j-1} + u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1}]$$

Substitute equations (9) and (10) into (8) is given by

$$(21) \quad -u_{i+1,j} - 4u_{i,j} + u_{i-1,j} + 2u_{i+1,j+1} - u_{i,j+1} + u_{i-1,j+1} + u_{i,j-1} + u_{i+1,j-1} = 2h^2 f(x, y)$$

Now, apply the boundary condition for  $i, j = 1, 2, 3, \dots, N$  and obtain the system of ODEs. Considering nodes (1,2), (1,3), (2,1), (2,2), (2,3) and (3,1), (3,2), (3,3), then the equation (21) becomes

$$-u_{2,1} - 4u_{1,1} + u_{0,1} + 2u_{2,2} - u_{1,2} + u_{0,1} + u_{1,0} + u_{2,0} = 2(.125)^2$$

$$(22) \quad 4u_1 + u_2 + u_4 - 2u_5 = 2(.125)^2$$

$$(23) \quad u_1 + u_2 - 4u_4 - u_5 - u_7 + 2u_8 = 2(.125)^2$$

$$(24) \quad u_4 + u_5 - 4u_7 - u_8 + 100 - 25 = 2(.125)^2$$

$$(25) \quad u_1 - 4u_2 - u_3 + u_4 - u_5 + 2u_6 = 2(.125)^2$$

$$(26) \quad u_2 + u_3 + u_4 - 4u_5 - u_6 + u_7 - u_8 + 2u_9 = 2(.125)^2$$

$$(27) \quad u_5 + u_6 + u_7 - 4u_8 - u_9 + 2 \times 75 - 50 + 25 = 2(.125)^2$$

$$(28) \quad u_2 - 4u_3 + u_5 - u_6 - 25 + 2 \times 50 = 2(.125)^2$$

$$(29) \quad u_3 + u_5 - 4u_6 + u_8 - u_9 - 50 + 2 \times 75 + 25 = 2(.125)^2$$

$$(30) \quad u_6 + u_8 - 4u_9 - 75 + 0 - 75 + 50 + 50 = 2(.125)^2$$

The above equations represent a system of ODEs. By considering Matlab code the required solution for the system of ODEs is given by

$U = [-10.4132, 20.8238, 13.1834, 20.8238, -0.0182, 43.0405, 13.1834, 43.0405, 9.0124]$  Substitute the above solution in equations (11) given by

$$\frac{\partial^2 u_1}{\partial y^2} = \frac{1}{2(.125)^2} [0.0041 - .0012 - .0082] + 1$$

By integrating on both sides,

$$u_1 = -0.4152y^2 + c_1y + c_2$$

Applying the boundary condition and using the solution of the system of equations, the required solution is given by

$$(31) \quad u_1 = -0.4152y^2 + 0.1038y - 10.4651$$

Similarly, substitute the above solution and apply the boundary conditions in equations (12)-(19), the final solutions are given by

$$(32) \quad u_2 = 0.1944y^2 - 0.072896y + 20.829874$$

$$(33) \quad u_3 = -1198.895y^2 + 449.6347y - 24.294338$$

$$(34) \quad u_4 = 0.1731y^2 - 0.108186y + 20.785935$$

$$(35) \quad u_5 = 0.6267y^2 + 0.391685y - 0.15529$$

$$(36) \quad u_6 = -2002.7163y^2 + 1251.69y - 144.7122$$

$$(37) \quad u_7 = 0.6971y^2 - 0.2614y + 13.1834$$

$$(38) \quad u_8 = 0.63782y^2 - 0.239182y + 43.0405$$

$$(39) \quad u_9 = 400.905y^2 - 150.339375y + 9.0124$$

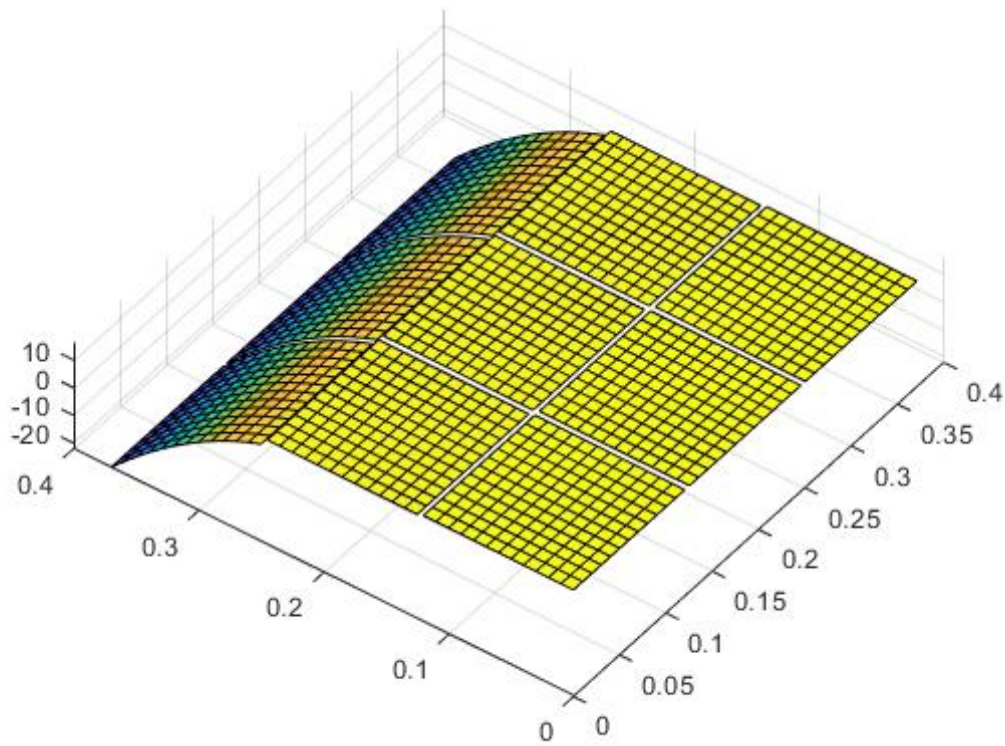


FIGURE 4. Illustration of numerical solutions to the 2D Poisson Equation using the Method of Lines (MOL) and Crank-Nicolson Method (CNM)

#### 4. DISCUSSIONS

In this study, we provide a basic introduction of the Method of Lines (MOL) and the Crank-Nicholson method (CNM) for solving elliptic partial differential equations (PDEs). The graph in Figure 4 represents the numerical solution of the Poisson equation using the Method of Lines (MOL) and Crank-Nicolson Method (CNM).

## 5. CONCLUSION

In this paper we have introduced method of lines and CNM to implementations of numerical methods for solving 2D Poisson equation with initial and Dirichlet boundary conditions. Other types of boundary conditions and right-hand side function of the Poisson equation could be considered to solve the same problem using method of lines. Number of physical problems have irregularly shaped boundaries and boundary conditions are expressed using derivatives. Boundary conditions which are expressed using differential equations are difficult to handle using implicit finite difference techniques because each boundary condition involving a derivative must be approximated by a difference quotient at the grid points. The result of this method is illustrated by the availability of high-quality numerical algorithms for the solution of systems of ODEs. This study, illustrate the MOL approach for solving the two-dimensional Poisson equation with boundary condition. The numerical results confirmed the efficiency, reliability and accuracy of this procedure and this auspicious performance is achieved with very little increased computational effort. Therefore, it was concluded that the use of MOL along with CNM in the numerical solution for the Poisson equation and other partial differential equations is a very efficacious and adequate procedure.

**Competing interests.** The author declares no competing interests.

## REFERENCES

- [1] G.G. Hall, J.M. Watt, Modern numerical methods for ordinary differential equations, Clarendon Press, 1976.
- [2] A.M. Loeb, W.E. Schiesser, Stiffness and accuracy in the method of lines integration of partial differential equations, in: R.A. Willoughby (Ed.), *Stiff Differential Systems*, Springer US, Boston, MA, 1974: pp. 229–243.
- [3] W.E. Schiesser, G.W. Griffiths, *A compendium of partial differential equation models: method of lines analysis with Matlab*, Cambridge University Press, 2009.
- [4] Md. Roknujjaman, G. Sekine, H. Kyotoh, Analysis of wave propagation and pulsation in an elastic tube using a diaphragm pump, *Phys. Fluids* 34 (2022), 123612.
- [5] W.E. Schiesser, Method of lines solution of the Korteweg-de vries equation, *Comput. Math. Appl.* 28 (1994), 147–154.
- [6] S. Hamdi, W.H. Enright, W. E Schiesser, J.J. Gottlieb, Exact solutions and conservation laws for coupled generalized Korteweg–de vries and quintic regularized long wave equations, *Nonlinear Anal.: Theory Methods Appl.* 63 (2005), e1425–e1434.
- [7] H. Kyotoh, G. Sekine, M. Roknujjaman, Propagation of the rim under a liquid-curtain breakup, *J. Fluid Mech.* 945 (2022), A12.
- [8] I.G. Rosen, C.H.F. Su, An approximation theory for the identification of linear thermoelastic systems, Report No. 90-24, ICASE, 1990.
- [9] W. Day, A decreasing property of solutions of parabolic equations with applications to thermoelasticity, *Q. Appl. Math.* 40 (1983), 468–475.
- [10] Md. Roknujjaman, H. Kyotoh, A. Yohei, A. Yasuhisa, Experimental and theoretical analysis of pulsatile flow in elastic tubes: wave propagation and attenuation, *Phys. Fluids* 35 (2023), 123608.
- [11] R. Khatun, M. Roknujjaman, M.A. Al Mohit, Numerical investigation on magnetohydrodynamics (MHD) free convection fluid flow over a vertical porous plate with induced magnetic field, *Theor. Phys.* 4 (2018), 15–26.
- [12] N. Nasir, Z.B. Ibrahim, K.I. Othman, M. Suleiman, Numerical solution of first order stiff ordinary differential equations using fifth order block backward differentiation formulas, *Sains Malays.* 41 (2012), 489–492.
- [13] M.A. Lau, S.P. Kuruganty, Spreadsheet implementations for solving boundary-value problems in electromagnetics, *Spreadsheets Educ.* 4 (2010), 1.
- [14] M. Roknujjaman, M. Asaduzzaman, On the solution procedure of partial differential equation (PDE) with the methods of lines (MOL) using Crank-Nicholson method (CNM), *Amer. J. Appl. Math.* 6 (2018), 1–7.
- [15] I. Ayraanci, N. Selçuk, MOL solution of DOM for transient radiative transfer in 3-D scattering media, *J. Quant. Spectrosc. Radiat. Transf.* 84 (2004), 409–422.
- [16] A. Bratsos, The solution of the two-dimensional Sine-Gordon equation using the method of lines, *J. Comput. Appl. Math.* 206 (2007), 251–277.
- [17] A.G. Bratsos, The solution of the boussinesq equation using the method of lines, *Comput. Methods Appl. Mech. Eng.* 157 (1998), 33–44.
- [18] S. Hamdi, W. Enright, Y. Ouellet, W. Schiesser, Method of lines solutions of the extended boussinesq equations, *J. Comput. Appl. Math.* 183 (2005), 327–342.

- [19] H. Han, Z. Huang, The direct method of lines for the numerical solutions of interface problem, *Comput. Methods Appl. Mech. Eng.* 171 (1999), 61–75.
- [20] T. Koto, Method of lines approximations of delay differential equations, *Comput. Math. Appl.* 48 (2004), 45–59.
- [21] S. Pamuk, A. Erdem, The method of lines for the numerical solution of a mathematical model for capillary formation: The role of endothelial cells in the capillary, *Appl. Math. Comput.* 186 (2007), 831–835.
- [22] S. Mirhosseini, H. Rahami, A. Kaveh, Analytical solution of Laplace and Poisson equations using conformal mapping and Kronecker products, *Int. J. Civil Eng.* 14 (2016), 369–377.
- [23] E. Rothe, Zweidimensionale parabolische randwertaufgaben als grenzfall eindimensionaler randwertaufgaben, *Math. Ann.* 102 (1930), 650–670.
- [24] D. Jones, J. South Jr, E. Klunker, On the numerical solution of elliptic partial differential equations by the method of lines, *J. Comput. Phys.* 9 (1972), 496–527.
- [25] G.D. Smith, *Numerical solution of partial differential equations: finite difference methods*, Clarendon Pres, 1985.