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ON SURPRISING APPROXIMATIONS INVOLVING MULTIPLE MATHEMATICAL CONSTANTS

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ABSTRACT. In this article, we present new numerical approximations involving five numerical constants, namely the Pi constant, the Euler constant, the Euler-Mascheroni constant, the Catalan constant, and the golden ratio constant. Four of these constants can be simply approximated by the others with relative accuracy. These results have potential applications in mathematics education by providing an accessible framework for exploring the connections between fundamental constants. In a sense, they also promote a deeper conceptual understanding of numerical analysis.

1. INTRODUCTION

To understand the purpose of this article, it is necessary to introduce five famous mathematical constants. First, the Euler constant, denoted by e, is a fundamental mathematical constant approximately equal to 2.71828. It arises naturally in the study of growth processes, compound interest and calculus, particularly in the context of limits and exponential functions. It can be defined by the following limit:

$$e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n.$$

It also satisfies the following infinite series expansion:

$$e = \sum_{n=0}^{+\infty} \frac{1}{n!}.$$

In terms of notation, we also encounter $e = e^1 = \exp(1)$. We refer to [1] and [8].

The Catalan constant, denoted by G, is an important constant in series and integral calculus, as well as in number theory. It is defined by the following alternating series:

$$G = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.915965.$$

See [2], [10], [3] and [8].

The Euler-Mascheroni constant, denoted by γ, is defined as the limiting difference between the harmonic series and the natural logarithm, as follows:

$$\gamma = \lim_{n \to +\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n) \right) \approx 0.577216.$$

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It appears in number theory, especially in the study of the distribution of prime numbers, and in various areas of analysis. We refer to [1] and [8].

The Pi constant (or simply Pi), denoted by π, is one of the best known mathematical constants. It is approximately equal to 3.14159. It represents the ratio of the circumference of a circle to its diameter. It is fundamental to geometry, trigonometry, and calculus. It also appears in many integral formulas and series. Some common examples of integrals with integrands of different types are as follows:

$$\pi = \int_0^{+\infty} \frac{\log(1+x^2)}{x^2} dx, \quad \pi = \int_0^{+\infty} \frac{1}{\sqrt{x}(1+x)} dx, \quad \pi = 2\int_0^{+\infty} \frac{\sin(x)}{x} dx,$$

$$\pi = \left[\int_{-\infty}^{+\infty} e^{-x^2} dx\right]^2, \quad \pi = 2\int_{-1}^{1} \sqrt{1-x^2} dx.$$

Some known examples using series are as follows:

$$\pi = \sqrt{6\sum_{n=1}^{+\infty} \frac{1}{n^2}}, \quad \pi = 4\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}, \quad \pi = 3 + 4\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{2n(2n+1)(2n+2)},$$

to which we add Ramanujan's beautiful one:

$$\pi = \left[\frac{2\sqrt{2}}{9801}\sum_{n=0}^{+\infty} \frac{(4n)!\,(1103+26390n)}{(n!\,)^4 396^{4n}}\right]^{-1}.$$

See [1], [4] and [5] for more details.

The golden ratio constant (or simply the golden ratio), denoted by φ, is a special number found in geometry and analysis. It is defined algebraically as the positive solution x to the following equation: $x^2 - x - 1 = 0$. It is expressed with $\sqrt{5} \approx 2.23606$ as follows:

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618033.$$

It appears in many geometrical constructions, including the proportions of the regular pentagon and the Fibonacci sequence. See [11], [1] and [8].

The study of these fundamental constants has long been a central theme in mathematics. Uncovering any hidden connections between them could potentially lead to new theorems or reveal surprising interactions between seemingly unrelated areas of mathematics. It could also improve our understanding of some aspects of unsolved problems. There are also potential applications in mathematics education by stimulating curiosity and critical thinking. In particular, it can motivate students to explore mathematical ideas that connect different areas. Therefore, any contribution in this direction is of interest.

On this topic, notable and elegant approximations include the one in [12]. It connects π and γ as follows:

$$\gamma \approx \frac{\pi}{2e} - 0.00064.$$

In the same spirit, but with more constants involved, there are the approximations established in [6]. In particular, based on [6, Proposition 2.1], the following approximation linking π, γ and G is found:

$$\frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2+\gamma^2)} \approx 1.0000748. \tag{1.1}$$

It is therefore relatively precise. From this, the following approximation was derived in [6, Proposition 2.2]:

$$\pi \approx \frac{2}{G}\left[G^2 + \gamma^2 - \gamma \log\left(\frac{\gamma}{G}\right)\right] \approx 3.141401.$$

In [6, Last equation], there is also a less precise but potentially interesting approximation. It is given by

$$\frac{\pi\gamma - 2\gamma\log(\gamma/G)}{2(G^2+\gamma^2)} \approx 1.000899. \tag{1.2}$$

It is interesting because it is similar in form to Equation (1.1), although there are notable differences in the changes of a constant and a sign. This similarity, combined with a relatively acceptable accuracy and the absence of any "special extra parameters", can, in a sense, arouse mathematical curiosity.

With this in mind, in this article, we find new similar approximations linking five mathematical constants: e, G, γ, π and φ. Surprisingly, it uses the same functional form as in Equation (1.1). The results are presented in the next section, i.e., Section 2. Section 3 gives a discussion of the results obtained. A final section, Section 4, completes the study with other types of exotic approximations still involving e, G, γ, π and φ.

2. Results

We start with an approximation using simple operations and the mathematical constants mentioned above, without any desire for optimality. Then some more accurate approximations are presented, with more accuracy but less mathematical elegance, in a sense.

2.1. An elegant approximation.

The first new approximation result is presented in the proposition below.

Proposition 2.1. *We have*

$$\frac{\pi\varphi + 2\gamma\log(2e)}{6(G^2+\gamma^2)} \approx 1.00068$$

(*followed by:* 25629270502380804434910917459772153056766295502719157865452...).

Numerical evidence of Proposition 2.1. Let us consider the following approximations, with 8 decimals:

$$\pi \approx 3.14159265, \quad \varphi \approx 1.61803398, \quad e \approx 2.71828182, \quad G \approx 0.91596559$$

and

$$\gamma \approx 0.57721566.$$

So, after a basic calculus, we find that

$$\frac{\pi\varphi + 2\gamma\log(2e)}{6(G^2+\gamma^2)} \approx \frac{3.14159265 \times 1.61803398 + 2 \times 0.57721566 \times \log(2 \times 2.71828182)}{6(0.91596559^2 + 0.57721566^2)}$$
$$\approx \frac{7.0378257900338579}{7.0330252812797022} \approx 1.0006826.$$

The other decimals of the stated result are obtained with more precise approximations. □

Using the approximation in [12], we notice that $2e \approx \pi/\gamma$. Based on Proposition 2.1, this inspires the following result:

$$\frac{\pi\varphi + 2\gamma\log(\pi/\gamma)}{6(G^2+\gamma^2)} \approx 1.00086$$

(followed by: 673587940798343183440343306076089326831486019163665204477724...).

It is a little less sharp. However, the similarity in form with Equations (1.1) and (1.2) is surprising enough to justify this remark.

Note that, in Proposition 2.1, the constant e can disappear with basic logarithmic developments, and the result can be rewritten as follows:

$$\frac{\pi\varphi + 2\gamma[\log(2) + 1]}{6(G^2 + \gamma^2)} \approx 1.00068.$$

The constant $\log(2) \approx 0.69314718$ is also special in mathematics. It is an irrational and transcendental number, and can be expressed in many ways. See [7], and the references therein.

Based on Proposition 2.1, we can derive the following unexpected approximations of some constants as a function of others:

- $\pi\varphi + 2\gamma \log(2e) \approx 6(G^2 + \gamma^2)$.
- For π, we have

  $$\pi \approx \frac{1}{\varphi}\left[6(G^2 + \gamma^2) - 2\gamma \log(2e)\right] \approx 3.1386$$

  (followed by: $2579231870802043612783799934361607356675767823940173204449 69\ldots$).
- For φ, the approximate form is very similar to the result above. We just have to swap the places π and φ, and we get

  $$\varphi \approx \frac{1}{\pi}\left[6(G^2 + \gamma^2) - 2\gamma \log(2e)\right] \approx 1.6165$$

  (followed by: $0594775099696526362705369367484802086113340970953247769044 74\ldots$).
- For G, after some manipulations, we get

  $$G \approx \sqrt{\frac{1}{6}\left[\pi\varphi + 2\gamma \log(2e)\right] - \gamma^2} \approx 0.9164$$

  (followed by: $0223161694276282354122061003317998299272753182713693906954 61\ldots$).
- For γ, after a few manipulations of the root polynomial type, we get

  $$\gamma \approx \frac{1}{6}\left[\sqrt{6\pi\varphi - 36G^2 + [\log(2e)]^2} + \log(2e)\right] \approx 0.5785$$

  (followed by: $6851910143901556037579948230478227732240253054356759748719 07\ldots$).

We see that the approximations are relatively accurate to the rounded second decimal, and for three decimals for φ.

2.2. **More precise results.** During our numerical investigations, in the same spirit as Proposition 2.1, other more precise but somewhat less elegant approximations were found. These are presented below.

2.2.1. *First improved result.* The result below is the analogue of Proposition 2.1, but with a slight modification in the logarithmic term, which gives a more accurate approximation.

Proposition 2.2. *We have*

$$\frac{\pi\varphi + 2\gamma \log[2\log(15)]}{6(G^2 + \gamma^2)} \approx 1.000063$$

(*followed by:* $5570367825173774006658364556917036679074248983579067851280\ldots$).

The proof is based on a direct calculation of the left term, taking into account the right values of the mathematical constants, as in the proof of Proposition 2.1.

If we compare this result with Proposition 2.1, the constant e in the logarithmic term has been replaced by $\log(15) \approx 2.7080502$. Note that e and $\log(15)$ are relatively close numerically; we have $\log(15) - e \approx$

-0.010231. As an aside, we can also write this rough approximation as the following sum of the logarithms of the first three odd integers:

$$\log(15) = \log(1) + \log(3) + \log(5) \approx e,$$

with the following optimized approximation: $\log(1+\epsilon) + \log(3) + \log(5) - e \approx 0.00000064523$, with $\epsilon = 0.01028480129872$.

However, since the constant $\log(15)$ is not particularly distinguished in the mathematical literature, we lose some understanding.

The following surprising approximations of some constants as a function of other constants are derived from Proposition 2.2:

- $\pi\varphi + 2\gamma\log[2\log(15)] \approx 6(G^2+\gamma^2)$.
- For π, we have
$$\pi \approx \frac{1}{\varphi}\left\{6(G^2+\gamma^2) - 2\gamma\log[2\log(15)]\right\} \approx 3.1413$$
(followed by: $1639347741079334283817152505699931259262224099836525432546 47\ldots$).
- For φ, we get
$$\varphi \approx \frac{1}{\pi}\left\{6(G^2+\gamma^2) - 2\gamma\log[2\log(15)]\right\} \approx 1.6178$$
(followed by: $917047873130373875123406007305295351290509061795965391868362\ldots$).
- For G, after some manipulations, we obtain
$$G \approx \sqrt{\frac{1}{6}\left\{\pi\varphi + 2\gamma\log[2\log(15)]\right\} - \gamma^2} \approx 0.9160$$
(followed by: $062605817828072450490912548804594050007892189611652520191005\ldots$).
- For γ, finding the root polynomial gives
$$\gamma \approx \frac{1}{6}\left[\sqrt{6\pi\varphi - 36G^2 + \{\log[2\log(15)]\}^2} + \log[2\log(15)]\right] \approx 0.5773$$
(followed by: $416298755932090201175267651034861720750488380729037298633026\ldots$).

The approximations obtained are clearly better than those derived from Proposition 2.1.

2.2.2. *Second improved result.* Similarly, the result below is the analogue of Proposition 2.1, but with the use of a numerical value that is independent of the mathematical constant involved. This constant, chosen by an optimal criterion, slightly modifies the logarithmic term to give an excellent approximation.

Proposition 2.3. *Let us consider the following constant, with an exact value:*

$$\rho = 1.9917$$

which we can call "the link constant". Then we have

$$\frac{\pi\varphi + 2\gamma\log(\rho e)]}{6(G^2+\gamma^2)} \approx 0.9999999$$

(*followed by:* $464912292864326965415641952301619002691170015626133608877\ldots$).

Again, the proof is based on a direct calculation of the left term, as in the proof of Proposition 2.1.

The result obtained is of exceptional precision, but the link constant has been artificially determined for this purpose. For this reason, we lose some mathematical understanding and elegance. We can see that $\rho \approx 2$ is consistent with the result of Proposition 2.1.

Again, we derive the following approximations of some constants as a function of others:

- $\pi\varphi + 2\gamma \log(\rho e) \approx 6(G^2 + \gamma^2)$.
- For π, we have

$$\pi \approx \frac{1}{\varphi}\left[6(G^2+\gamma^2) - 2\gamma\log(\rho e)\right] \approx 3.14159$$

(followed by: $28861736228049594677626728321953579739233497843805488665171\ldots$).

- For φ, we find that

$$\varphi \approx \frac{1}{\pi}\left[6(G^2+\gamma^2) - 2\gamma\log(\rho e)\right] \approx 1.61803$$

(followed by: $41085389900519339753353316756988060306533634356201476042499\ldots$).

- For G, after some manipulations, we get

$$G \approx \sqrt{\frac{1}{6}\left[\pi\varphi + 2\gamma\log(\rho e)\right] - \gamma^2} \approx 0.91596$$

(followed by: $55599393474456136394642391816654736440297309397255959437399\ldots$).

- For γ, we obtain

$$\gamma \approx \frac{1}{6}\left[\sqrt{6\pi\varphi - 36G^2 + [\log(\rho e)]^2} + \log(\rho e)\right] \approx 0.57721$$

(followed by: $55588519768993570390289840444739128490675783995171011744913\ldots$).

The approximations obtained are much better than those given in Propositions 2.1 and 2.2.

3. Discussion

In this article, several numerical approximations have been established linking five mathematical constants, namely e, G, γ, π and φ. Their interest remains mainly mathematical, showing how a formula seems to numerically link constants with an "a priori" unexpected connection between them. This may lead to some open questions and discussions about these aspects. The similarity in form of the main approximation term, i.e., a ratio involving the logarithmic function and the square of the constants, also remains a kind of interrogation. These results could also improve mathematics education by encouraging students to explore the surprising connections between fundamental mathematical concepts and their potential implications.

4. Complements

During this research, we found several curious approximations that go beyond the scope of the previous study and do not seem to have been mentioned in the literature. A selection of the most original ones are listed below.

We have

$$\pi - \sqrt{\pi + \sqrt{\pi}} - G \approx 0.0088$$

(followed by: $62185887348665017267546813280240430151653916752319697628377 6\ldots$).

This implies the following new reasonable approximation of G in function of π:

$$G \approx \pi - \sqrt{\pi + \sqrt{\pi}}.$$

The following exotic result involving π, γ, G, φ and e also holds:

$$\log(\pi\gamma G\varphi e) - 2 \approx -0.011$$

(followed by: $37407802797846675451746224604221201235993182403900350972418 8\ldots$), or, equivalently,

$$\log(\pi\gamma G\varphi) - 1 \approx -0.011$$
$$(\text{followed by: } 374078027978466754517462246042212012359931824039003509724188\ldots).$$

We also find that

$$\pi^{\gamma/\varphi} - \frac{3}{2} \approx 0.00436$$
$$(\text{followed by: } 319135156699606603884063488316993521623910516163073252794 13\ldots),$$

giving the following reasonable approximation:

$$\pi^{\gamma/\varphi} \approx \frac{3}{2}.$$

In the same spirit, we have

$$\gamma^{\varphi/\pi} - \frac{3}{4} \approx 0.00349$$
$$(\text{followed by: } 518761692143229226381846731635206667800853272171843031160 97\ldots),$$

giving the following reasonable approximation:

$$\gamma^{\varphi/\pi} \approx \frac{3}{4}.$$

We also get

$$\varphi^{\gamma/\pi} - \frac{11}{10} \approx -0.0075$$
$$(\text{followed by: } 589256083874087329866103777239956956174480518704350589156 2\ldots),$$

giving the following reasonable approximation:

$$\varphi^{\gamma/\pi} \approx \frac{11}{10}.$$

Some simple approximations are now presented. We have

$$\pi - (\pi - 2)[1 + \sqrt{\pi}] + \frac{\gamma}{25} \approx -0.00033$$
$$(\text{followed by: } 166842461447626422253183324923917683910460087271037382954 9\ldots).$$

Without the last term, we get

$$\pi - (\pi - 2)[1 + \sqrt{\pi}] \approx -0.023420$$
$$(\text{followed by: } 295020675790688483015436545336418525478038469654326060239\ldots).$$

We hope that some of these results will inspire further research and new questions to be asked about the numerical aspects of the mathematical constants.

Competing interests. The authors declare no competing interests.

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