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# ROTATIONAL SURFACES IN TERMS OF COORDINATE FINITE CHEN $II$-TYPE

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ABSTRACT. In this study, we first establish several formulae according to the first and second Beltrami operators. We discuss the class of surfaces of revolution in the 3-dimensional Euclidean space $E^3$ without parabolic points, in which the position vector $\boldsymbol{X}$ satisfies $\Delta^{II}\boldsymbol{X} = D\boldsymbol{X}$, with $\Delta^{II}$ is the Laplace operator of the metric $II$ of the surface and $D$ is a square matrix of order 3. We prove that surfaces satisfying the preceding relation are either part of a sphere or catenoid.

## 1. INTRODUCTION

Surfaces of finite type are one of the main topics that attracted the interest of many differential geometers from the moment that B. Y. Chen defined the notion of surfaces of finite $I$-type regarding the first fundamental form $I$ about four decades ago. Many results concerning this subject have been collected in [15].

Let $\boldsymbol{X} : M^2 \to E^3$ be a parametric representation of a surface in $E^3$. Denote by $\Delta^I$ the Laplace operator and by $\boldsymbol{H}$ the mean curvature field of $M^2$. Then, it is well known that [14]

$$\Delta^I \boldsymbol{X} = -2\boldsymbol{H}.$$

Furthermore, T. Takahashi mentioned in [25] that a surface $M^2$ whose position vector $\boldsymbol{z}$ satisfies $\Delta^I\boldsymbol{X} = \lambda\boldsymbol{X}$ is either a minimal with $\lambda = 0$ or $M^2$ lies in an ordinary sphere $S^2$ with a fixed nonzero eigenvalue.

In [19], O. Garay extended T. Takahashi's condition. Specifically, he studied surfaces in $E^3$ satisfying $\Delta^I\boldsymbol{X}_i = \mu_i r_i, i = 1, 2, 3$, with different eigenvalues $\mu_i$, and $(r_1, r_2, r_3)$ are the coordinate functions of $\boldsymbol{z}$. Another general problem was also presented in [17] for which surfaces in $E^3$ satisfying $\Delta^I\boldsymbol{X} = K\boldsymbol{X} + L(\S)$, where $K \in M(3 \times 3); L \in M(3 \times 1)$. It was proved that minimal surfaces, spheres, and circular cylinders are the only surfaces in $E^3$ satisfying $(\S)$. Surfaces satisfying $(\S)$ are said to be of coordinate finite type.

In the framework of the theory of surfaces of finite $I$-type in $E^3$, a general study of the Gauss map was made within this context in [16]. On the other hand, it is also interesting to study surfaces of finite $I$-type in $E^3$ with the property $\Delta^I\boldsymbol{G} = D\boldsymbol{G}$, where $D \in M(3\times 3)$. Surfaces in $E^3$ whose Gauss map is of a coordinate finite $I$-type was investigated by many researchers as one can see in [4, 6–11, 18].

In 2003 authors in [24] followed the ideas of B. Y. Chen, by defining the concept of surfaces of finite type regarding the second or third fundamental forms, and since then much work has been done in this context.

Similarly, one can further study surfaces in $E^3$ with the position vector $\boldsymbol{z}$ satisfies

$$\Delta^J \boldsymbol{X} = D\boldsymbol{X}, \tag{1.1}$$

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where $J = II, III$, and $D \in M(3 \times 3)$.

Regarding the third fundamental form, it was proved that helicoid is the only ruled surface that satisfies (1.1). Meanwhile, for the quadric surfaces, spheres are the only ones with the given property [1]. In [2] it was shown that Scherk's surface is the only translation surface that satisfies (1.1).

In [23], Bendiheba and Bekkar studied the helicoidal surfaces in $E^3$ satisfying $\Delta^J \boldsymbol{X} = D\boldsymbol{X}, J = II, III$.

## 2. Fundamentals

Let $\boldsymbol{X} := \boldsymbol{X}(u^1, u^2)$ be a regular parametric representation of a surface $Q$ in $E^3$, and let

$$I = \mathfrak{g}_{km}\, du^k du^m, \quad II = \mathfrak{b}_{km}\, du^k du^m, \quad III = \mathfrak{e}_{km}\, du^k du^m, \quad k, m = 1, 2.$$

For any two differentiable functions $f$ and $h$ on $Q$, the first and second differential parameters of Beltrami are defined by [20]

$$\nabla^J(f, h) := \mathfrak{a}^{km} f_{/k} h_{/m}, \tag{2.1}$$

$$\triangle^J h := -\mathfrak{a}^{km} \nabla^J_k h_{/m} = -\frac{1}{\sqrt{|\mathfrak{a}|}} \frac{\partial h}{\partial u^k} (\sqrt{|\mathfrak{a}|} \mathfrak{a}^{km} \frac{\partial h}{\partial u^m}), \tag{2.2}$$

where $h_{/k} := \frac{\partial h}{\partial u^k}$, $(a^{km})$ denotes the components of the inverse tensor of $(\mathfrak{g}_{km}), (\mathfrak{b}_{km})$ and $(\mathfrak{e}_{km})$ for $J = I, II$, and $III$ respectively, $\mathfrak{a} = det(\mathfrak{a}_{km})$, and $\nabla^J_k$ is the covariant derivative in the $u^k$ direction.

The Gauss and the mean curvature of $Q$ are respectively

$$K = \frac{1}{R_1 R_2} = \frac{\mathfrak{b}}{\mathfrak{g}} = \frac{\mathfrak{e}}{\mathfrak{b}}, \quad 2H = \frac{1}{R_1} + \frac{1}{R_2} = \mathfrak{g}^{ik}\mathfrak{b}_{ik} = \mathfrak{b}^{ik}\mathfrak{e}_{ik}, \tag{2.3}$$

where $\mathfrak{g} = det(\mathfrak{g}_{ik}), \mathfrak{b} = det(\mathfrak{b}_{ik}), \mathfrak{e} = det(\mathfrak{e}_{ik})$ and $R_1, R_2$ are the principal radii of curvature. For simplicity, we put

$$R = \frac{2H}{K} = \mathfrak{e}^{ik}\mathfrak{b}_{ik} = \mathfrak{b}^{ik}\mathfrak{g}_{ik}, \quad Z = 4H^2 - 2K = \mathfrak{g}^{ik}\mathfrak{e}_{ik}. \tag{2.4}$$

The Weingarten equations are

$$\boldsymbol{X}_k = -\mathfrak{g}_{kj}\mathfrak{b}^{jr}\boldsymbol{G}_{/r} = -\mathfrak{b}_{kj}\mathfrak{e}^{jr}\boldsymbol{G}_{/r},$$

$$\boldsymbol{G}_k = -\mathfrak{e}_{kj}\mathfrak{b}^{jr}\boldsymbol{X}_{/r} = -\mathfrak{b}_{kj}\mathfrak{g}^{jr}\boldsymbol{X}_{/r}. \tag{2.5}$$

Following are some useful relations that will be used later

$$\triangle^I \boldsymbol{X} = -2H\boldsymbol{G}, \tag{2.6}$$

$$\triangle^I \boldsymbol{G} = 2grad^I H + Q\boldsymbol{G}, \tag{2.7}$$

$$\triangle^{II} \boldsymbol{X} = \frac{1}{2} K grad^{III} \frac{1}{K} - 2\boldsymbol{G}, \tag{2.8}$$

$$\triangle^{II} \boldsymbol{G} = \frac{1}{2K} grad^I K + 2H\boldsymbol{G}, \tag{2.9}$$

$$\triangle^{III} \boldsymbol{X} = grad^{III} R - R\boldsymbol{G}, \tag{2.10}$$

(2.11) $$\triangle^{III}\boldsymbol{G} = 2\boldsymbol{G}.$$

Let $T_P(Q)$ denote the tangent plane to $Q$ for any point $P \in Q$. Then the Weingarten map is defined to be the linear transformation $S : T_P(Q) \to T_P(Q)$, where for $\boldsymbol{z} = z^i\boldsymbol{X}_{/i}$, then $S(\boldsymbol{z}) = v^i\boldsymbol{X}_{/i}, v^i := b_{jk}g^{ki}z_j$. When $\mathfrak{b}_{12} = \mathfrak{g}_{12} = 0$, we get

$$S(\boldsymbol{z}) = \frac{1}{R_1}z^1\boldsymbol{X}_{/1} + \frac{1}{R_2}z^2\boldsymbol{X}_{/2}.$$

Firstly, we recall the following:

**Theorem 2.1.** *For a sufficient differentiable function $A(u^1, u^2)$ on $Q$, the following relations satisfied*

(1) $\nabla^{II}(A, \boldsymbol{G}) + grad^I A = 0,$
(2) $\nabla^{II}(A, \boldsymbol{X}) + grad^{III} A = 0,$
(3) $\nabla^{III}(A, \boldsymbol{X}) + \frac{1}{K}grad^I A = R\nabla^{II}(A, \boldsymbol{X}),$
(4) $S(grad^I A) + \nabla^I(A, \boldsymbol{G}) = 0.$

*Proof.* [3,24]

(1) From (2.1), and using the Weingarten equations (2) we obtain

$$grad^I A := \nabla^I(A, \boldsymbol{X}) = \mathfrak{g}^{ik}A_{/i}\boldsymbol{X}_k = -A_{/i}\mathfrak{g}^{ik}\mathfrak{g}_{kj}\mathfrak{b}^{jr}\boldsymbol{G}_{/r} = -\mathfrak{b}^{ir}A_{/i}\boldsymbol{G}_{/r} = -\nabla^{II}(A, \boldsymbol{G}),$$

which is (1).

(2) Similarly, from (2.1), and (2) we have

$$\nabla^{II}(A, \boldsymbol{X}) = \mathfrak{b}^{ik}A_{/k}\boldsymbol{X}_i = -\mathfrak{b}^{ik}\mathfrak{b}_{ij}\mathfrak{e}^{jr}A_{/k}\boldsymbol{G}_{/r} = -\mathfrak{e}^{kr}A_{/k}\boldsymbol{G}_{/r} = -\nabla^{III}(A, \boldsymbol{G}) = -grad^{III}A.$$

(3) Taking into consideration the well-known relation

$$K\mathfrak{e}^{ik} - 2H\mathfrak{b}^{ik} + \mathfrak{g}^{ik} = 0,$$

then

$$\mathfrak{e}^{ik} = R\mathfrak{b}^{ik} - \frac{1}{K}\mathfrak{g}^{ik}.$$

From (2.1), and taking into account the last equation we obtain

$$\nabla^{III}(A, \boldsymbol{X}) = \mathfrak{e}^{ik}A_{/k}\boldsymbol{X}_i = (R\mathfrak{b}^{ik} - \frac{1}{K}\mathfrak{g}^{ik})A_{/k}\boldsymbol{X}_i = R\nabla^{II}(A, \boldsymbol{X}) - \frac{1}{K}grad^I A.$$

Hence

$$\nabla^{III}(A, \boldsymbol{X}) + \frac{1}{K}grad^I A = R\nabla^{II}(A, \boldsymbol{X}).$$

(4) Without loss of generality, we suppose that $\mathfrak{g}_{12} = \mathfrak{b}_{12} = 0$, then

$$\begin{aligned} S(grad^I A) &= S(\mathfrak{g}^{ik}A_{/k}\boldsymbol{X}_i) = \frac{\mathfrak{b}_{11}}{\mathfrak{g}_{11}}\mathfrak{g}^{11}A_{/1}\boldsymbol{X}_{/1} + \frac{\mathfrak{b}_{22}}{\mathfrak{g}_{22}}\mathfrak{g}^{22}A_{/2}\boldsymbol{X}_{/2} \\ &= \frac{\mathfrak{b}_{11}}{(\mathfrak{g}_{11})^2}A_{/1}\boldsymbol{X}_{/1} + \frac{\mathfrak{b}_{22}}{(\mathfrak{g}_{22})^2}A_{/2}\boldsymbol{X}_{/2}. \end{aligned}$$

On the other hand

$$\nabla^I(A, \boldsymbol{G}) = \mathfrak{g}^{ik}A_{/k}\boldsymbol{G}_i = \mathfrak{g}^{11}A_{/1}\boldsymbol{G}_{/1} + \mathfrak{g}^{22}A_{/2}\boldsymbol{G}_{/2}.$$

On account of $\mathfrak{g}_{12} = \mathfrak{b}_{12} = 0$, then Weingarten equations (2.5) become $\boldsymbol{G}_{/1} = \frac{\mathfrak{b}_{11}}{\mathfrak{g}_{11}}\boldsymbol{X}_{/1}$ and $\boldsymbol{G}_{/2} = \frac{\mathfrak{b}_{22}}{\mathfrak{g}_{22}}\boldsymbol{X}_{/2}$, so the last equation become

$$\nabla^I(A, \boldsymbol{G}) = -\frac{\mathfrak{b}_{11}}{(\mathfrak{g}_{11})^2}A_{/1}\boldsymbol{X}_{/1} - \frac{\mathfrak{b}_{22}}{(\mathfrak{g}_{22})^2}A_{/2}\boldsymbol{X}_{/2} = -S(grad^I A).$$

□

Denote by $<,>$ the Euclidean inner product and $w = - < \boldsymbol{X}, \boldsymbol{G} >$ the support function of $Q$. We prove the following theorem

**Theorem 2.2.** *The support function $w$ of $Q$, satisfies the following relations*

(1) $\triangle^I w = Zw - 2H - < grad^I H, \boldsymbol{X} >$,
(2) $\triangle^{II} w = 2Hw - \frac{1}{2K} < grad^I K, \boldsymbol{X} > -2$,
(3) $\triangle^{III} w = 2w - R$.

*Proof.* (1) Using (2.2),we get

$$\begin{aligned}
\triangle^I w &= -\mathfrak{g}^{in}\nabla^I_n w_{/i} = \mathfrak{g}^{in}\nabla^I_n < \boldsymbol{X}_{/i}, \boldsymbol{G} > + \mathfrak{g}^{in}\nabla^I_n < \boldsymbol{X}, \boldsymbol{G}_{/i} > \\
&= < \mathfrak{g}^{in}\nabla^I_n \boldsymbol{X}_{/i}, \boldsymbol{G} > + < \boldsymbol{X}, \mathfrak{g}^{in}\nabla^I_n \boldsymbol{G}_{/i} > + 2\mathfrak{g}^{in} < \boldsymbol{X}_{/i}, \boldsymbol{G}_{/n} > \\
&= - < \triangle^I \boldsymbol{X}, \boldsymbol{G} > - < \boldsymbol{X}, \triangle^I \boldsymbol{G} > + 2\mathfrak{g}^{in} < \boldsymbol{X}_{/i}, \boldsymbol{G}_{/n} > .
\end{aligned}$$

From (2.3), (2.5), (2.6) and (2.7), the last equation becomes

$$\begin{aligned}
\triangle^I w &= < 2H\boldsymbol{G}, \boldsymbol{G} > - < \boldsymbol{X}, 2grad^I H + Q\boldsymbol{G} > + 2\mathfrak{g}^{in} < \boldsymbol{X}_{/i}, -\mathfrak{e}_{nj}\mathfrak{b}^{jr}\boldsymbol{X}_{/r} > \\
&= Zw - 2H - < grad^I H, \boldsymbol{X} > .
\end{aligned}$$

(2) From (2.2),we have

$$\begin{aligned}
\triangle^{II} w &= -b^{ik}\nabla^{II}_k w_{/i} = \mathfrak{b}^{ik}\nabla^{II}_k < \boldsymbol{X}_{/i}, \boldsymbol{G} > + \mathfrak{b}^{ik}\nabla^{II}_k < \boldsymbol{X}, \boldsymbol{G}_{/i} > \\
&= < \mathfrak{b}^{ik}\nabla^{II}_k \boldsymbol{X}_{/i}, \boldsymbol{G} > + < \boldsymbol{X}, \mathfrak{b}^{ik}\nabla^{II}_k \boldsymbol{G}_{/i} > + 2\mathfrak{b}^{ik} < \boldsymbol{X}_{/i}, \boldsymbol{G}_{/k} > \\
&= - < \triangle^{II} \boldsymbol{X}, \boldsymbol{G} > - < \boldsymbol{X}, \triangle^{II} \boldsymbol{G} > + 2\mathfrak{b}^{ik} < \boldsymbol{X}_{/i}, \boldsymbol{G}_{/k} > .
\end{aligned}$$

From (2.5), (2.8) and (2.9), the last equation becomes

$$\begin{aligned}
\triangle^{II} w &= - < \boldsymbol{X}, \frac{1}{2K} grad^I K + 2H\boldsymbol{G} > - < \frac{1}{2} K grad^{III} \frac{1}{K} - 2\boldsymbol{G}, \boldsymbol{G} > \\
&\quad + 2\mathfrak{b}^{ik} < \boldsymbol{X}_{/i}, \boldsymbol{G}_{/k} > \\
&= 2Hw - < \boldsymbol{X}, \frac{1}{2K} grad^I K > + 2 + 2\mathfrak{b}^{ik} < \boldsymbol{X}_{/i}, -\mathfrak{b}_{kj}\mathfrak{g}^{jr}\boldsymbol{X}_{/r} > \\
&= 2Hw - < \boldsymbol{X}, \frac{1}{2K} grad^I K > -2.
\end{aligned}$$

(3) From (2.2),we have

$$\begin{aligned}
\triangle^{III} w &= -\mathfrak{e}^{ik}\nabla^{III}_k w_{/i} = \mathfrak{e}^{ik}\nabla^{III}_k < \boldsymbol{X}_{/i}, \boldsymbol{G} > + \mathfrak{e}^{ik}\nabla^{III}_k < \boldsymbol{X}, \boldsymbol{G}_{/i} > \\
&= < \mathfrak{e}^{ik}\nabla^{III}_k \boldsymbol{X}_{/i}, \boldsymbol{G} > + < \boldsymbol{X}, \mathfrak{e}^{ik}\nabla^{III}_k \boldsymbol{G}_{/i} > + 2\mathfrak{e}^{ik} < \boldsymbol{X}_{/i}, \boldsymbol{G}_{/k} > \\
&= - < \triangle^{III} \boldsymbol{X}, \boldsymbol{G} > - < \boldsymbol{X}, \triangle^{III} \boldsymbol{G} > + 2\mathfrak{e}^{ik} < \boldsymbol{X}_{/i}, \boldsymbol{G}_{/k} > .
\end{aligned}$$

From (2.4), (2.5), (2.10) and (2.11), the last equation becomes

$$\begin{aligned}
\triangle^{III} w &= - < grad^{III} R - R\boldsymbol{G}, \boldsymbol{G} > - < \boldsymbol{X}, 2\boldsymbol{G} > + 2\mathfrak{e}^{ik} < \boldsymbol{X}_{/i}, \boldsymbol{G}_{/k} > \\
&= R + 2w + 2\mathfrak{e}^{ik} < \boldsymbol{X}_{/i}, -\mathfrak{b}_{kj}\mathfrak{g}^{jr}\boldsymbol{X}_{/r} > = 2w - R.
\end{aligned}$$

□

*Remark* 2.3. New results can be drawn by defining the first and second Beltrami operators using the definition of the fractional vector operators [22]. Some applications can be found in [12, 13].

In the next paragraph, we mainly focus on surfaces of finite $II$-type by studying surfaces of revolution in $E^3$ of which their position vector $\boldsymbol{z}$ satisfies the relation

$$\Delta^{II}\boldsymbol{X} = D\boldsymbol{X}. \tag{2.12}$$

Our main result is the following:

**Theorem 2.4.** *The only surfaces of revolution in $E^3$ whose position vector $\boldsymbol{z}$ satisfies relation (2.12) are the spheres and the catenoids.*

Firstly, we show that the mentioned surfaces in the abstract indeed satisfy (2.12). Consider a sphere $Q$ of radius $c$ centered at the origin. Then

$$K = \frac{1}{c^2}, \quad \boldsymbol{G} = -\frac{1}{c}\boldsymbol{X}.$$

Hence, by (2.8) it is $\Delta^{II}\boldsymbol{X} = \frac{2}{c}\boldsymbol{X}$. Therefore, $Q$ satisfies (2.12) with $D = \frac{2}{c}I_3$, where $I_3$ is the identity matrix. On a catenoid, a parametric representation is the following

$$S : \boldsymbol{X}(u,v) = (c\cosh\frac{u}{c}\cos v, c\cosh\frac{u}{c}\sin v, u),$$

where $u \in [-\pi,\pi), v \in \mathbb{R}$ and $c$ is a non-zero real constant. Let $(X_1, X_2, X_3)$ be the component functions of $\boldsymbol{X}$. Thus we have

$$X_1 = c\cosh\frac{u}{c}cosv, \;\; X_2 = c\cosh\frac{u}{c}sinv, \;\; X_3 = u.$$

We also have

$$\mathfrak{b}_{11} =< \boldsymbol{X_{uu}}, \boldsymbol{G} >= -\frac{1}{c}, \quad \mathfrak{b}_{12} =< \boldsymbol{X_{uv}}, \boldsymbol{G} >= 0, \quad \mathfrak{b}_{22} =< \boldsymbol{X_{vv}}, \boldsymbol{G} >= c,$$

It is well-known that

$$\Delta^{II}\boldsymbol{X} = (\Delta^{II}X_1, \Delta^{II}X_2, \Delta^{II}X_3). \tag{2.13}$$

Hence, by (2.2) and taking into account (2.13), we obtain

$$\Delta^{II}X_1 = 2\cosh\frac{u}{c}\cos v, \quad \Delta^{II}X_2 = 2\cosh\frac{u}{c}\sin v, \quad \Delta^{II}X_3 = 0.$$

So $Q$ satisfies (2.12) with corresponding matrix

$$D = \begin{bmatrix} \frac{2}{c} & 0 & 0 \\ 0 & \frac{2}{c} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

## 3. Proof of main result

Let $\mathfrak{C}$ be a smooth curve lies on the $x_1x_3$-plane parametrized by

$$\boldsymbol{r}(u) = (p(u), 0, q(u)), \quad u \in (a,b),$$

where $p, q$ are smoothly defined and $p > 0$. A surface of revolution is the point set $Q$ that results When $\mathfrak{C}$ is revolved about the $x_3$-axis. So, the $x_3$-axis is called the axis of revolution of $Q$ and $\mathfrak{C}$ is called the profile curve of $Q$ (see [5,21]).

The position vector of $Q$ is defined as follows

$$\boldsymbol{X}(u,v) = \big(p(u)\cos v, p(u)\sin v, q(u)\big), \quad u \in (a,b), \quad 0 \le v < 2\pi. \tag{3.1}$$

Here, we may consider that $\mathfrak{C}$ has the arc-length parametrization, i.e., it satisfies

$$(p\prime)^2 + (q\prime)^2 = 1 \tag{3.2}$$

where $\prime := \frac{d}{du}$. On the other hand $p\prime q\prime \neq 0$, because if $p$ = const. or $q$ = const., then $Q$ is a circular cylinder or a plane, respectively. Hence the Gaussian curvature of $Q$ vanishes. A case that has been excluded.

We have

$$\boldsymbol{X}_u = (p\prime(u)\cos v, p\prime(u)\sin v, q\prime(u))$$

and

$$\boldsymbol{r_v} = (-p(u)\sin v, p(u)\cos v, 0)\,.$$

Then the components $g_{km}$ are computed as follows

$$\mathfrak{g}_{11} =< \boldsymbol{X}_u, \boldsymbol{X}_u >= 1, \quad \mathfrak{g}_{12} =< \boldsymbol{X}_u, \boldsymbol{X}_v >= 0, \quad \mathfrak{g}_{22} =< \boldsymbol{X}_v, \boldsymbol{X}_v >= p^2.$$

The mean and the Gaussian curvature of $Q$ are respectively

$$2H = \kappa + \frac{q\prime}{p}, \quad K = \frac{\kappa q\prime}{p} = -\frac{p\prime\prime}{p},$$

where $\kappa$ denotes the curvature of the curve $\mathfrak{C}$.

The components $b_{km}$ are given as follows

$$\mathfrak{b}_{11} = \kappa, \quad \mathfrak{b}_{12} = 0, \quad \mathfrak{b}_{22} = pq\prime. \tag{3.3}$$

From (2.2) and (3.3) the Beltrami operator $\Delta^{II}$ is

$$\Delta^{II} = -\frac{1}{\kappa}\frac{\partial^2}{\partial u^2} - \frac{1}{pq\prime}\frac{\partial^2}{\partial v^2} + \frac{1}{2}\left(\frac{\kappa\prime}{\kappa^2} - \frac{p\prime q\prime + \kappa pp\prime}{\kappa pq\prime}\right)\frac{\partial}{\partial u}. \tag{3.4}$$

On account of (3.2) we can put

$$p\prime = \cos\xi, \quad q\prime = \sin\xi,$$

where $\xi = \xi(u)$. Then $\kappa = \xi\prime$ and relation (3.4) becomes

$$\Delta^{II} = -\frac{1}{\xi\prime}\frac{\partial^2}{\partial u^2} - \frac{1}{p\sin\xi}\frac{\partial^2}{\partial v^2} + \frac{1}{2}\left(\frac{\xi\prime\prime}{(\xi\prime)^2} - \frac{\cos\xi\sin\xi + p\xi\prime\cos\xi}{p\xi\prime\sin\xi}\right)\frac{\partial}{\partial u}, \tag{3.5}$$

while the mean and the Gaussian curvature become

$$2H = \xi\prime + \frac{\sin\xi}{p}, \tag{3.6}$$

$$K = \frac{\xi\prime\sin\xi}{p}. \tag{3.7}$$

Let $\boldsymbol{X}$ = $(X_1, X_2, X_3)$. Then from (2.13) and (3.5), we get

$$\Delta^{II}X_1 = \Delta^{II}(p\cos v) = \left(\sin\xi + \frac{1}{\sin\xi} - \frac{H\cos^2\xi}{\xi\prime\sin\xi} + \frac{\xi\prime\prime\cos\xi}{2\xi\prime^2}\right)\cos v, \tag{3.8}$$

$$\Delta^{II}X_2 = \Delta^{II}(p\sin v) = \left(\sin\xi + \frac{1}{\sin\xi} - \frac{H\cos^2\xi}{\xi\prime\sin\xi} + \frac{\xi\prime\prime\cos\xi}{2\xi\prime^2}\right)\sin v, \tag{3.9}$$

$$\Delta^{II}X_3 = \Delta^{III}(q) = -\frac{3}{2}\cos\xi + \frac{\xi\prime\prime\sin\xi}{2\xi\prime^2} - \frac{\sin\xi\cos\xi}{2p\xi\prime}. \tag{3.10}$$

Let $D = (d_{km}), k, m = 1, 2, 3, .$ By using (3.8), (3.9) and (3.10) then relation (2.12) analyzed to the following system

$$\left(\sin\xi + \frac{1}{\sin\xi} - \frac{H\cos^2\xi}{\xi\prime\sin\xi} + \frac{\xi\prime\prime\cos\xi}{2\xi\prime^2}\right)\cos v = d_{11}p\cos v + d_{12}p\sin v + d_{13}q, \tag{3.11}$$

$$\left(\sin\xi+\frac{1}{\sin\xi}-\frac{H\cos^2\xi}{\xi\prime\sin\xi}+\frac{\xi\prime\prime\cos\xi}{2\xi\prime^2}\right)\sin v=d_{21}p\cos v+d_{22}p\sin v+d_{23}q, \tag{3.12}$$

$$-\frac{3}{2}\cos\xi+\frac{\xi\prime\prime\sin\xi}{2\xi\prime^2}-\frac{\sin\xi\cos\xi}{2p\xi\prime}=d_{31}p\cos v+d_{32}p\sin v+d_{33}q. \tag{3.13}$$

From (3.13) it can be easily verified that $d_{31}=d_{32}=0$. Differentiating (3.11) and (3.12) twice with respect to $v$ we obtain

$$\left(\sin\xi+\frac{1}{\sin\xi}-\frac{H\cos^2\xi}{\xi\prime\sin\xi}+\frac{\xi\prime\prime\cos\xi}{2\xi\prime^2}\right)\cos v=d_{11}p\cos v+d_{12}p\sin v,$$
$$\left(\sin\xi+\frac{1}{\sin\xi}-\frac{H\cos^2\xi}{\xi\prime\sin\xi}+\frac{\xi\prime\prime\cos\xi}{2\xi\prime^2}\right)\sin v=d_{21}p\cos v+d_{22}p\sin v.$$

Thus $d_{13}q=d_{23}q=0$, so that $d_{13}$ and $d_{23}$ vanish. Equations (3.11), (3.12) and (3.13) are equivalent to the following

$$\left(\sin\xi+\frac{1}{\sin\xi}+\frac{\xi\prime\prime\cos\xi}{2\xi\prime^2}-\frac{H\cos^2\xi}{\xi\prime\sin\xi}\right)\cos v=d_{11}p\cos v+d_{12}p\sin v, \tag{3.14}$$

$$\left(\sin\xi+\frac{1}{\sin\xi}+\frac{\xi\prime\prime\cos\xi}{2\xi\prime^2}-\frac{H\cos^2\xi}{\xi\prime\sin\xi}\right)\sin v=d_{21}p\cos v+d_{22}p\sin v, \tag{3.15}$$

$$-\frac{3}{2}\cos\xi+\frac{\xi\prime\prime\sin\xi}{2\xi\prime^2}-\frac{\sin\xi\cos\xi}{2p\xi\prime}=d_{33}q. \tag{3.16}$$

The functions $\sin v$, $\cos v$ are linearly independent of the variable $v$, so finally we get $d_{11}=d_{22}$ and $d_{12}=d_{21}=0$. Let $d_{11}=d_{22}=\lambda$ and $a_{33}=\mu$. Then the system of equations (3.14), (3.15) and (3.16) reduces to the following two equations

$$\sin\xi+\frac{1}{\sin\xi}+\frac{\xi\prime\prime\cos\xi}{2\xi\prime^2}-\frac{H\cos^2\xi}{\xi\prime\sin\xi}=\lambda p, \tag{3.17}$$

$$-\frac{3}{2}\cos\xi+\frac{\xi\prime\prime\sin\xi}{2\xi\prime^2}-\frac{\sin\xi\cos\xi}{2p\xi\prime}=\mu q. \tag{3.18}$$

Therefore the matrix $D$ for which condition (2.12) is satisfied becomes

$$D=\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mu\end{bmatrix}.$$

We distinguish the following cases:

*Case I.* $\lambda=\mu=0$. Equations (3.17) and (3.18) become

$$\sin\xi+\frac{1}{\sin\xi}+\frac{\xi\prime\prime\cos\xi}{2\xi\prime^2}-\frac{H\cos^2\xi}{\xi\prime\sin\xi}=0, \tag{3.19}$$

$$-\frac{3}{2}\cos\xi+\frac{\xi\prime\prime\sin\xi}{2\xi\prime^2}-\frac{\sin\xi\cos\xi}{2p\xi\prime}=0. \tag{3.20}$$

Multiplying (3.19) by $\sin\xi$ and (3.20) by $-\cos\xi$ and adding the resulting of these two equations it follows $\cos^2\xi+\sin^2\xi=-1$, which is a contradiction.

*Case II.* $\lambda=\mu\neq 0$. Equations (3.17) and (3.18) become

$$\sin\xi+\frac{1}{\sin\xi}+\frac{\xi\prime\prime\cos\xi}{2\xi\prime^2}-\frac{H\cos^2\xi}{\xi\prime\sin\xi}=\lambda p, \tag{3.21}$$

$$-\frac{3}{2}\cos\xi + \frac{\xi'' \sin\xi}{2\xi'^2} - \frac{\sin\xi\cos\xi}{2p\xi'} = \lambda q. \tag{3.22}$$

Similarly, multiplying (3.21) by $\sin\xi$ and (3.22) by $-\cos\xi$ and adding the resulting of these two equations whence it follows

$$\lambda p \sin\xi - \lambda q \cos\xi = 2.$$

On differentiating the last equation with respect to $u$ we find

$$\lambda\xi'(pp' + qq') = 0.$$

$\lambda\xi'$ cannot be equal 0 otherwise, from (3.7) the Gauss curvature vanishes. Hence, $pp' + qq' = 0$, i.e. $(p^2+q^2)' = 0$. Therefore $(p^2+q^2)' = const.$. Thus $\mathfrak{C}$ is part of a circle and $Q$ is obviously part of a sphere.

*Case III.* $\lambda \neq 0, \mu = 0$. Following the same procedure as in $Case\ I$ and $Case\ II$, we get

$$2 - \lambda p \sin\xi = 0. \tag{3.23}$$

Differentiating (3.23) with respect to $u$ we have

$$(\sin\xi + p\xi')\cos\xi = 0. \tag{3.24}$$

Taking into account relation (3.6), equation (3.24) becomes

$$2Hp\cos\xi = 0$$

which implies that the mean curvature $H$ vanishes identically. Therefore, the surface is minimal, that is, it is a catenoid. Furthermore, a catenoid satisfies the condition (2.12).

*Case IV.* $\lambda = 0, \mu \neq 0$. In this case (3.17) and (3.18) are given respectively by

$$\sin\xi + \frac{1}{\sin\xi} + \frac{\xi''\cos\xi}{2\xi'^2} - \frac{H\cos^2\xi}{\xi'\sin\xi} = 0, \tag{3.25}$$

$$-\frac{3}{2}\cos\xi + \frac{\xi''\sin\xi}{2\xi'^2} - \frac{\sin\xi\cos\xi}{2p\xi'} = \mu q.$$

Following the same procedure as in $Case\ I$ and $Case\ II$, we find

$$2 + \mu q \cos\xi = 0. \tag{3.26}$$

Differentiating this equation we have

$$q\xi' - \cos\xi = 0, \tag{3.27}$$

from which

$$\xi' = \frac{\cos\xi}{q}. \tag{3.28}$$

Another differentiation of (3.27), gives

$$2\xi'\sin\xi + q\xi'' = 0. \tag{3.29}$$

From (3.28) and (3.29), we have

$$\xi'' = -\frac{2\sin\xi\cos\xi}{q^2}. \tag{3.30}$$

Equation (3.25) can be written

$$1 + \sin^2\xi + \frac{\xi''\cos\xi\sin\xi}{2\xi'^2} - \frac{1}{2}\cos^2\xi - \frac{\cos^2\xi\sin\xi}{2p\xi'} = 0. \tag{3.31}$$

Consequently, from (3.26), (3.28) and (3.30), equation (3.31) becomes

$$2 - \cos^2 \xi + \frac{2 \sin \xi}{\mu p} = 0,$$

from which

$$\mu p = \frac{2 \sin \xi}{\cos^2 \xi - 2}. \tag{3.32}$$

Differentiating (3.32), we get

$$\mu = \frac{2\xi\prime}{\cos^2 \xi - 2} + \frac{4\xi\prime \sin^2 \xi}{(\cos^2 \xi - 2)^2}.$$

On using (3.28) and (3.26) after some computation, we can obtain that $\sin \xi = 0$, that is, $q = const.$, which implies that the Gauss curvature vanishes. A case that was excluded. Thus, there are no surfaces of revolution satisfying this case.

*Case V.* $\lambda \neq \mu$ and $\lambda \neq 0, \mu \neq 0$. Multiplying (3.21) by $\sin \xi$, and (3.22) by $-\cos \xi$ and adding the resulting equations, we obtain

$$\lambda p \sin \xi - \mu q \cos \xi = 2. \tag{3.33}$$

We put

$$\Omega := \lambda p \sin \xi + \mu q \cos \xi. \tag{3.34}$$

By using (3.33), the derivative of $\Omega$ is the following

$$\Omega\prime = \lambda \cos^2 \xi + \mu \sin^2 \xi - 2\xi\prime. \tag{3.35}$$

On differentiating (3.33) and using (3.34) we find

$$\Omega\xi\prime = (\mu - \lambda) \cos \xi \sin \xi. \tag{3.36}$$

It is easily verified that $\Omega \neq 0$, hence (3.36) can be written

$$\xi\prime = \frac{(\mu - \lambda) \cos \xi \sin \xi}{\Omega}. \tag{3.37}$$

Differentiating the last equation and using (3.35) and (3.36) we obtain

$$\xi\prime\prime = \frac{\big((\lambda - 2\mu) \sin^2 \xi + (\mu - 2\lambda) \cos^2 \xi\big)\xi\prime + 2\xi\prime^2}{\Omega}. \tag{3.38}$$

In view of (3.37) and (3.38) relation (3.18) takes the following form

$$\frac{\Omega \cos \xi}{p} + \frac{2(\lambda - \mu) \cos \xi \sin \xi}{\Omega} - 2\mu(\lambda - \mu) q \cos \xi - (\lambda - 2\mu) = 0,$$

which can be rewritten

$$\frac{\Omega \cos \xi}{p} + \frac{2(\lambda - \mu) \cos \xi \sin \xi}{\Omega} - 2\mu(\lambda - \mu) q \cos \xi - (\lambda - 2\mu) = 0.$$

Multiplying the last equation by $\Omega p \cos \xi$, we have

$$\begin{aligned} &2(\lambda - \mu) p \cos^2 \xi \sin \xi - 2\mu(\lambda - \mu) pq\Omega \cos^2 \xi \\ &-(\lambda - 2\mu)\Omega p \cos \xi + \Omega^2 \cos^2 \xi = 0. \end{aligned} \tag{3.39}$$

From (3.33) and (3.34), it can be is easily verified that

$$\Omega \cos \xi = \lambda p - 2 \sin \xi. \tag{3.40}$$

Therefore, on using (3.33) and (3.40), relation (3.39) becomes

$$a_1 p^3 + a_2 p^2 + a_3 p + a_4 = 0, \tag{3.41}$$

where here we put

$$a_1 = \lambda^2(\mu-\lambda)\sin\xi, \qquad a_2 = \lambda[(2\lambda-\mu) - 2(\mu-\lambda)\sin^2\xi],$$
$$a_3 = [(\mu-\lambda)\sin^2\xi - (\mu-4\lambda)]\sin\xi, \qquad a_4 = 2\sin^2\xi.$$

Taking the derivative of (3.41) and then by using (3.34), (3.37) and (3.40), we obtain

(3.42) $$b_1p^3 + b_2p^2 + b_3p + b_4 = 0,$$

where

$$b_1 = \lambda^2(\mu-\lambda)\sin\xi[(2\lambda+\mu) - (\mu-\lambda)\sin^2\xi],$$
$$b_2 = 2\lambda[\lambda(2\lambda-\mu) - (\mu-\lambda)(3\lambda+2\mu)\sin^2\xi + 2(\mu-\lambda)^2\sin^4\xi],$$
$$b_3 = [(8\lambda\mu - 8\lambda^2 - \mu^2) + 2(\mu-\lambda)(2\mu+\lambda)\sin^2\xi - 3(\mu-\lambda)^2\sin^4\xi]\sin\xi,$$
$$b_4 = 6(\mu-2\lambda)\sin^2\xi - 6(\mu-\lambda)\sin^4\xi.$$

Combining (3.41) and (3.42) we conclude that

(3.43) $$c_1p^2 + c_2p + c_3 = 0,$$

where

$$c_1 = a_1b_2 - a_2b_1 = \lambda[2(\mu-\lambda)^2\sin^4\xi - 3\mu(\mu-\lambda)\sin^2\xi - \mu(2\lambda-\mu)],$$
$$\begin{aligned} c_2 &= a_1b_3 - a_3b_1 = 2[-(\mu-\lambda)^2\sin^4\xi \\ &\quad +(\mu+2\lambda)(\mu-\lambda)\sin^2\xi + \lambda(3\mu-8\lambda)]\sin\xi, \end{aligned}$$
$$c_3 = a_1b_4 - a_4b_1 = [-4(\mu-\lambda)\sin^4\xi + 4(\mu-4\lambda)\sin^2\xi].$$

Taking the derivative of (3.43) and then by using (3.34), (3.37) and (3.40), we obtain

(3.44) $$d_1p^2 + d_2p + d_3 = 0,$$

where

$$d_1 = -4\lambda(\mu-\lambda)^3\sin^6\xi + \sum_{i=0}^{2} D_{1i}(\lambda,\mu)\sin^{2i}\xi,$$
$$d_2 = 5(\mu-\lambda)^3\sin^7\xi + \sum_{i=0}^{2} D_{2i}(\lambda,\mu)\sin^{2i+1}\xi,$$
$$d_3 = 10(\mu-\lambda)^2\sin^6\xi + \sum_{i=1}^{2} D_{3i}(\lambda,\mu)\sin^{2i}\xi,$$

and $D_{ji}(\lambda,\mu)$, $(j=1,2,3)$ are polynomials in $\lambda$ and $\mu$. Combining (3.43) and (3.44) we find that

(3.45) $$e_1p + e_2 = 0,$$

where

$$e_1 = c_1d_2 - c_2d_1 = 2(\mu-\lambda)^5\sin^{10}\xi + \sum_{i=0}^{4} E_{1i}(\lambda,\mu)\sin^{2i}\xi,$$
$$e_2 = c_1d_3 - c_3d_1 = 20(\mu-\lambda)^4\sin^9\xi + \sum_{i=0}^{3} E_{2i}(\lambda,\mu)\sin^{2i+1}\xi,$$

and $E_{ji}(\lambda,\mu)$, $(j=1,2)$ are some polynomials in $\lambda$ and $\mu$. Following the same procedure by taking the derivative of (3.45) and taking into account (3.34), (3.37) and (3.40), we find

(3.46) $$h_1p + h_2 = 0,$$

where

$$h_1 = -20(\mu-\lambda)^6 \sin^{12}\xi + \sum_{i=0}^{5} H_{1i}(\lambda,\mu)\sin^{2i}\xi,$$

$$h_2 = -184(\mu-\lambda)^5 \sin^{11}\xi + \sum_{i=0}^{4} H_{2i}(\lambda,\mu)\sin^{2i+1}\xi,$$

and $H_{ji}(\lambda,\mu),\ (j=1,2)$ are polynomials in $\lambda$ and $\mu$. Combining (3.45) and (3.46) we finally find

$$32(\mu-\lambda)^{10}\sin^{20}\xi + \sum_{i=0}^{9} P_i(\lambda,\mu)\sin^{2i}\xi = 0. \tag{3.47}$$

where $P_i(\lambda,\mu),\ (i=1,2,...,9)$ are polynomials in $\lambda$ and $\mu$. Equation (3.47) is equal to zero for every $\xi$, hence all its coefficients must be zero. A contradiction, since we must have $\mu-\lambda=0$. Consequently, there are no surfaces of revolution in this case. Thus our proof is completed.

**Competing interests.** The author declares no competing interests.

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