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NUMERICAL SOLUTION OF THE FRACTIONAL DYNAMICS OF A SURFACE ENERGY BALANCE-MASS BALANCE MODEL OF CRYOSPHERE

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Abstract. In the present task, the dynamics of Cryosphere is represented based on the modified Caputo-Fabrizio fractional derivative. A numerical scheme based on the fifth-order fractional Adams-Bashforth approach is applied to the dynamical system of the proposed model. The nonlinear system of differential equations of arbitrary order in the model is described by the modified Caputo-Fabrizio fractional operator, and the model has been checked for non-zero outcomes under the non-singular fractional operator. Existence and uniqueness of the given fractional model is discussed. The scheme's stability analysis is checked. All model findings are sketched with the application of MATLAB R2016a mathematical software. The novelty of this work lies in using fractional order five-step Adams-Bashorth method in order to approximate the surface energy balance-mass balance fractional model of the Cryosphere. We represent the system using Caputo-Fabrizio fractional differential equations.

1. Introduction

Scientific models have played an important role in addressing numerous challenges across various scientific disciplines such as climate science, engineering, mathematics, and physics [1]. The models assist policymakers in making excellent decisions by offering deeper insights and advice [2]. In recent time, climate change has become one of the biggest issue of our planet. In every single day, the influences of climate change are now clearly seen by everyone all over the world.

With the help of mathematical models, more accurate future directions can be made about our planet phenomena. Creating and solving mathematical models are the primary responsibility of different researchers because of their massive importance. The representation of the full range of dynamical systems, involving memory effects, is a significant obstacle for mathematical modeling approaches. In order to get over this challenge, investigators have began using fractional operators [3, 4], which present a promising mathematical tool for strengthening these dynamical models. Nowadays, arbitrary order calculus has attracted the science community because of its huge significance in climate, biological, ecological, financial, and physical sciences. The use of arbitrary-order derivative and integration in mathematical modeling is relatively novel and has gained popularity in recent years. The non-integer order calculus is a generalized form of the old integer order calculus. The prime feature of the mathematical models with non-integer order differential equations is their non-local behavior, due to which they can represent physical processes and dynamical systems more precisely than the integer-order models.

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Currently, systems of fractional differential equations are in use to represent many true life issues. The nonlinear problems can be successfully modeled by a system of arbitrary order differential equations. Obtaining exact results of system of differential equations involving nonlinear terms can be highly difficult, so we need to search approximation techniques for finding numerical values. A lot of numerical schemes for obtaining approximate results of non-integer order differential equations have been created. Some of the methods are the multi-step method [5], the collocation method [6], the Adomian decomposition method [7], an exponential Galerkin method [8], an exponential collocation method [9], the Galerkin finite element method [10], Laguerre spectral collocation method [11] and so on. The methods are based on discretization of the independent variables and include modifications of the integer order methods. Owolabi and Atangana in [12], have presented a novel fractional third-order A-B scheme with the Caputo-Fabrizio non-integer order derivative. In this task, we applied the five-step fractional Adams-Bashforth scheme to solve Cryosphere model numerically.

Very recently, the mathematical models under non-integer order derivative operators were given big attention because they are more accurate, precise and realistic as compared to the integer order models [13,14]. It is clearly known that a non-integer order calculus is a generalization of an integer-order calculus. The extremely significant question that led to the birth of non-integer-order calculus in 1695 came from a letter that L'Hopital sent to Leibniz. The respected mathematicians Euler, Fourier, Riemann-Liouville, Caputo, Caputo-Fabrizio, Atangana-Baleanu and Yang-Abdel-Cattani made excellent contributions to the further progress of non-integer-order calculus. The first definition of a non-singular arbitrary order derivative by Caputo and Fabrizio made big contribution to the starting of the concept of non-singular kernel fractional calculus. Nonlocal operators are one of the key reasons why noninteger-order calculus is becoming increasingly popular. Very recently, the idea of arbitrary-order non local operators has become the main study subject in science and technology, attracting a huge number of authors. Few years ago, a lot of attempts on target were made to find more interesting, and fresh non singular arbitrary order derivatives based on kernels. A novel noninteger order Caputo–Fabrizio operator derived in 2015 [15] and this operator addressed several linear and nonlinear issues. In many branches of mathematics and engineering, the Caputo-Fabrizio fractional operator is frequently used.

In recent years, a lot of world issues have been studied with the use of arbitrary order calculus [16]. Motivated by the advancement of arbitrary order calculus, many authors have focused to study the results of dynamical systems of nonlinear differential equations with the fractional operator by developing quite a few exact or approximate techniques to produce numerical values [17]. The main reasons given for using non-integer order derivative models are that many systems show history, memory, or non local effects, which can be challenging to model using integer order derivatives. Because of the singularity in the kernel of the Caputo fractional derivative [18] at the end point of the interval of integration, the Caputo fractional derivative is not always a suitable kernel to effectively show the memory effect in a real system. Caputo and Fabrizio have derived a novel fractional derivative with none singularity in its kernel. The kernel of the new fractional derivative has the form of an exponential function. Losada and Nieto [19] have proposed the fractional integral associated with the new fractional Caputo–Fabrizio fractional derivative. The number of recent papers have been developed with the application of the new Caputo-Fabrizio fractional operator to tackle the world issues. For instance, Moore et al. [20] have modeled HIV/AIDS model with treatment using a Caputo-Fabrizio fractional differential equation. A third-order Adams–Bashforth predictor scheme was used for their numerical investigation. Atangana et al. [21] have compared the CF arbitrary positive order derivative and the ABC derivative for delay differential equations [22] and in model of chaotic systems [21]. The Caputo-Fabrizio fractional derivative in exponential decay kernel has high memory properties than the power law kernel while the Atangana–Baleanu fractional derivative provides a nice description. There are many modified form of arbitrary order calculus; the reason why we choose the modified Caputo-Fabrizio fractional operator is this operator is very suitable in the formulation of the fifth-step fractional Adams-Bashforth scheme. The present work focuses on the dynamics of Cryosphere model in frame of the modified Caputo-Fabrizio non-integer order derivative operator, which is a modified version of the Caputo-Fabrizio fractional derivative operator by Losada and Nieto.

One of the biggest phenomenon that has attracted the attention of a lot of investigators in the recent decades is a climate change. Every single activity in this planet depends on climate condition. Human activities have a vital role in climate change. The Cryosphere, which refers to the frozen water part of the earth system, includes ice sheets, glaciers, sea ice, and permafrost [23]. The connection between the Cryosphere's mass balance and surface energy balance in nonlinear model was suggested by the authors in [24]. Due to a number of importance of Cryosphere model, it has taken big attention from investigators, and many significant results on this popular model by taking into account various parameters have also been obtained. For example; Chakraborty et al. in [25] considered the Caputo fractional order system of Cryosphere model to observe the minute changes in the behavior of the system. They looked at the effect of climate change on the dynamics of a modified surface energy balance-mass balance model of Cryosphere under the frame of a non-local operator. In dimensionless excess variables, Nicolis in [26], expressed the Cryosphere dynamical model by the following system of two equations:

$$\begin{cases} \dfrac{ds}{dt} = \eta, \\[2ex] \dfrac{d\eta}{dt} = \alpha\eta + \beta s - s^3 - s^2\eta + \gamma\sin(\delta t). \end{cases} \tag{1.1}$$

Authors of [25], have observed the effect of global warming on model (1.1). They reformulated the system (1.1) by including the radiative forcing of CO_2 to capture the impact of global warming and the model equations are represented using Caputo fractional differential equations. The model is given next:

$$\begin{cases} {}_0^C\mathscr{D}_t^{\wp} s(t) = \eta, \\[2ex] {}_0^C\mathscr{D}_t^{\wp} \eta(t) = \alpha\eta + \beta s - s^3 - s^2\eta + \gamma\sin(\delta t) + F. \end{cases} \tag{1.2}$$

Very recently, the fractional derivative has become popular for several researchers in mathematics and technology areas. Due to excellent characteristics of non-singular fractional operator, a fractional class of Cryosphere model Eq. (1.2) is considered by replacing the singular Caputo time fractional derivative equations so that the Caputo-Fabrizio model takes the following form:

$$\begin{cases} {}_0^{CF}\mathscr{D}_t^{\wp} S(t) = \eta, \\[2ex] {}_0^{CF}\mathscr{D}_t^{\wp} \eta(t) = \alpha\eta + \beta S - S^3 - S^2\eta + \gamma\sin(\delta t) + F. \end{cases} \tag{1.3}$$

Authors of [25] stated that no significant work has yet been done on the Cryosphere model (1.1) and they focused on modification and solving the system. They used the fractional seventh order Runge-Kutta method to numerically solve their model. This motivated us to test the five-step fractional A-B numerical technique for the modified model. The primary goal of this manuscript is to discuss a fractional fifth order A-B approach to know about the numerical and graphical behavior of the nonlinear fractional mathematical model. The novelty of this work lies in using fractional order five-step A-B scheme in order to approximate the fractional model of the Cryosphere. We represent the system using Caputo-Fabrizio fractional differential equations. To the best of the our knowledge, the proposed scheme have not been used yet for the solution of the time-fractional Cryosphere model.

The following is an outline of this paper: In Section 2, the definitions of Caputo-Fabrizio fractional derivative and integration are written. Numerical approximation based on five-step Adams-Bashforth scheme for fractional Cryosphere model including the radiative forcing of CO_2 gas found in Section 3. Section 4 contains the discussion of the results. Finally, the paper ends in Section 5.

2. Preliminaries

In the 2015s, Caputo and Fabrizio introduced a fresh definition of non-singular fractional derivative and integration called Caputo-Fabrizio fractional calculus of exponential decay law kernel. This definition was able to overcome a prominent limitation of the singular kernel Caputo definition of fractional calculus [15].

Definition 1. *For $\wp \in (0,\ 1)$, and $m(t) \in \mathbb{H}^1(a,\ b)$, the C-F fractional derivative of order $\wp$ [15] is defined as:*

$$ {}^{CF}\mathscr{D}_t^{\wp} m(t) = \frac{\beta(\wp)}{1-\wp} \int_a^t m'(\chi) \exp\left(-\wp \frac{t-\chi}{1-\wp}\right) d\chi. \tag{2.1}$$

where $\beta(\wp)$ is the normalization function satisfied $\beta(0) = \beta(1) = 1$. However, if $m \notin \mathbb{H}^1(a,\ b)$, then the derivative is defined as [20]:

$$ {}^{CF}\mathscr{D}_t^{\wp} m(t) = \frac{\wp\beta(\wp)}{1-\wp} \int_a^t \{m(t) - m(\chi)\} \exp\left(-\wp \frac{t-\chi}{1-\wp}\right) d\chi. \tag{2.2}$$

Remark 1. *([20]) If we let $\varsigma = \dfrac{1-\wp}{\wp} \in (0,\ \infty)$, then $\wp = \dfrac{1}{1+\varsigma} \in (0,\ 1)$. In consequence, Eq. (2.2) can be reduced to:*

$$ {}^{CF}\mathscr{D}_t^{\varsigma} m(t) = \frac{\psi(\varsigma)}{\varsigma} \int_a^t m'(\chi) \exp\left(-\frac{t-\chi}{\varsigma}\right) d\chi. \tag{2.3}$$

where $\psi(\varsigma)$ is the normalization term corresponding to $\beta(\wp)$ such that $\psi(0) = \psi(\infty) = 1$.

Remark 2. *([20]) We have the following property:*

$$ \lim_{\varsigma \to 0} \frac{1}{\varsigma} \exp\left(-\frac{t-\chi}{\varsigma}\right) = \delta(\chi - t), \tag{2.4}$$

where $\delta(\chi - t)$ is the Dirac delta function.

The updated version of Caputo-Fabrizio fractional derivative and integration defined by Losada and Nieto [19] are given next.

Definition 2. *For $1 > \wp > 0$, and $m(t) \in \mathbb{H}^1(a,\ b)$, the C–F fractional derivative of order $\wp$ modified by Losada and Nieto is defined as*

$$ {}^{CF}\mathscr{D}_t^{\wp} m(t) = \frac{(2-\wp)\beta(\wp)}{2(1-\wp)} \int_a^t m'(\chi) \exp\left(-\wp \frac{t-\chi}{1-\wp}\right) d\chi. \tag{2.5}$$

The fractional integral corresponding to the derivative in Eq. (2.5) was defined by Losada and Nieto [19] as follows.

Definition 3. *Let $1 > \wp > 0$, The fractional integral of order $\wp$ of a function $m(t)$ is defined by*

$$ {}^{CF}\mathcal{I}_t^{\wp} m(t) = \frac{2(1-\wp)}{(2-\wp)\beta(\wp)} m(t) + \frac{2\wp}{(2-\wp)\beta(\wp)} \int_0^t m(\chi) d\chi. \tag{2.6}$$

where $\beta(\wp) = \dfrac{2}{2-\wp}$, $\wp \in (0,\ 1)$ (see [20], page 4).

Qualitative analysis. A very important part of mathematical analysis, that ensures that problems have clearly defined results, appropriate computations, and enables futures, is demonstrating the existence and oneness of a result. Here, with the help of Banach fixed-point theorem and Picard's operator, we need to show that the result of the system (1.3) exists and is one.

Theorem 2.1. *In the region* $[0,\ \mathscr{T}]\times\mathcal{S}$, *where*

$$\mathcal{S}=\left\{(s,\ \eta)\in\mathbb{R}^2:\ \|s\|<\rho_1,\ \|\eta\|<\rho_2\right\},$$

and $\mathscr{T}<+\infty$, *the result of the system* (1.3) *exists and is one.*

Proof. We consider $S(t)=\{s(t),\ \eta(t)\}$ and $S_1(t)=\{s_1(t),\ \eta_1(t)\}$ and $C(t,\ S)=\{C_1(t,\ S),\ C_2(t,\ S)\}$, where

$$\begin{cases} C_1(t,\ S)=\eta, \\ C_2(t,\ S)=\alpha\eta+\beta s-s^3-s^2\eta+\gamma\sin(\delta t)+F. \end{cases}$$

Here, C is defined on $[0,\ \mathscr{T}]\times\mathcal{S}$.

Let $K=\sup_{\mathcal{S}}\|C(t,\ S)\|$ and consider the norm as $\|S(t)\|=\sup_{t\in[0,\ \mathscr{T}]}|S(t)|$. We show that there exists some constant φ such that

$$\|C(t,\ S)-C(t,\ S_1)\|\le\varphi\|S-S_1\|.$$

Now,

$$\|C(t,\ S)-C(t,\ S_1)\|=\|\eta-\eta_1+\alpha(\eta-\eta_1)+\beta(s-s_1)+s_1^3-s^3+s_1^2\eta_1-s^2\eta\|.$$

Applying the triangular inequality, we have:

$$\begin{aligned}&\le\|\eta-\eta_1\|+\alpha\|\eta-\eta_1\|+\beta\|s-s_1\|+\|s^3-s_1^3\|+\|s^2\eta-s_1^2\eta_1\|\\&\le\left(\beta+3\rho_1^2+2\rho_1\rho_2\right)\|s-s_1\|+\left(1+\alpha+\rho_1^2\right)\|\eta-\eta_1\|\\&=\varphi_1\|s-s_1\|+\varphi_2\|\eta-\eta_1\|,\end{aligned}$$

where $\varphi_1=\beta+3\rho_1^2+2\rho_1\rho_2,\ \ \varphi_2=1+\alpha+\rho_1^2$.

Let $\varphi=\max\{\varphi_1,\ \varphi_2\}$. Therefore, we can write

$$\|C(t,\ S)-C(t,\ S_1)\|\le\varphi\|S-S_1\|.$$

Using the function C and the Caputo-Fabrizio fractional integral, we can construct Picard's operator in the following way:

$$\Delta S=S(0)+{}^{CF}\mathcal{I}_t^{\wp}C(t,\ S),\quad \wp\in(0,\ 1). \tag{2.7}$$

Now, we need to show that this operator maps a non-empty complete metric space to itself and also it is a contraction map. Let

$$\|S-S(0)\|\le\lambda.$$

By taking norm on left and right sides of Eq. (2.7), we get

$$\|\Delta S-S(0)\|=\|C(t,\ S)\|^{CF}\mathcal{I}_t^{\wp}(1)$$

$$\le K\left(\frac{2(1-\upsilon)}{(2-\upsilon)\psi(\upsilon)}+\frac{2\upsilon}{(2-\upsilon)\psi(\upsilon)}t\right)<\lambda \tag{2.8}$$

where the last inequality holds if $\dfrac{2(1-\upsilon)}{(2-\upsilon)\psi(\upsilon)}+\dfrac{2\upsilon}{(2-\upsilon)\psi(\upsilon)}t<\dfrac{\lambda}{K}$.

Now, we derive a condition for the operator Δ to become a contraction map. For that, we start with

$$\|\Delta S-\Delta S_1\le\|^{CF}\mathcal{I}_t^{\wp}(C(t,\ S)-C(t,\ S_1))$$

$$\leq^{CF} \mathcal{I}_t^{\wp} \|C(t, S) - C(t, S_1)\|$$

$$\leq \|C(t, S) - C(t, S_1)\|^{CF}\mathcal{I}_t^{\wp}(1) \tag{2.9}$$

$$\leq \left(\frac{2(1-\upsilon)}{(2-\upsilon)\psi(\upsilon)} + \frac{2\upsilon}{(2-\upsilon)\psi(\upsilon)}t\right)\varphi\|S - S_1\|.$$

From inequality (2.9), we conclude that Δ is a contraction if

$$\frac{2(1-\upsilon)}{(2-\upsilon)\psi(\upsilon)} + \frac{2\upsilon}{(2-\upsilon)\psi(\upsilon)}t \leq \frac{1}{\varphi}.$$

So, the above condition ensures that Picard's operator Δ is a contraction. Hence, using the Banach Fixed-Point Theorem, we conclude that Δ has a unique fixed point. Therefore, the system (1.3) has solution and is unique when

$$\frac{2(1-\upsilon)}{(2-\upsilon)\psi(\upsilon)} + \frac{2\upsilon}{(2-\upsilon)\psi(\upsilon)}t \leq \min\left\{\frac{\lambda}{K}, \frac{1}{\varphi}\right\}.$$

This completes the proof of existence and oneness of result of the given system. □

3. Five-step Adams-Bashforth fractional approach

In recent times, there have been a lot of novel direct techniques developed for obtaining the wide range of nonlinear arbitrary order derivative systems that have been used as models of actual world issues. The exact approaches include the local fractional homotopy perturbation method coupled with Laplace transform [27], the homotopy analysis transform technique [28] and the homotopy analysis method together with Sumudu integral transform method. In contrast, different approximate schemes for calculating numerical values of non-integer order differential equations have been formulated. These techniques are typically depend on discretization of the independent variable and use modifications of the integer order approaches such as, the finite volume schemes [29], the finite element schemes [30], finite difference schemes [31] and the Moulton-Adams-Bashforth kind corrector-predictor techniques [32]. In this work, we will apply a five-step non-integer order A-B method to find approximate values for a mass balance-surface energy balance fractional dynamics of Cryosphere model (1.3) under the modified Caputo–Fabrizio fractional operator. This method is quite accurate and easy to computer programming.

Now, we first consider the general fractional differential equation with the Caputo–Fabrizio fractional derivative. That is,

$${}^{CF}\mathscr{D}_t^{\wp} m(t) = A(t, m(t)), \tag{3.1}$$

or equivalently, we have

$$\frac{(2-\wp)\beta(\wp)}{2(1-\wp)}\int_0^t m'(\chi)\exp\left(-\wp\frac{t-\chi}{1-\wp}\right)d\chi = A(t, m(t)). \tag{3.2}$$

Using the fractional Caputo-Fabrizio fundamental theorem of calculus, we convert Eq. (3.2) to the next form:

$$m(t) = m(0) + \frac{2(1-\wp)}{(2-\wp)\beta(\wp)}A(t, m(t)) + \frac{2\wp}{(2-\wp)\beta(\wp)}\int_0^t A(\chi, m(\chi))d\chi. \tag{3.3}$$

so that

$$m(t_{i+1}) = m(0) + \frac{2(1-\wp)}{(2-\wp)\beta(\wp)}A(t_i, m(t_i)) + \frac{2\wp}{(2-\wp)\beta(\wp)}\int_0^{t_{i+1}} A(t, m(t))dt, \tag{3.4}$$

and

$$m(t_i) = m(0) + \frac{2(1-\wp)}{(2-\wp)\beta(\wp)} A(t_{i-1},\ m(t_{i-1})) + \frac{2\wp}{(2-\wp)\beta(\wp)} \int_0^{t_i} A(t,\ m(t))\,dt. \tag{3.5}$$

Thus,

$$m(t_{i+1}) - m(t_i) = \frac{2(1-\wp)}{(2-\wp)\beta(\wp)} \left[A(t_i,\ m(t_i)) - A(t_{i-1},\ m(t_{i-1}))\right] + \frac{2\wp}{(2-\wp)\beta(\wp)} \int_{t_i}^{t_{i+1}} A(t,\ m(t))\,dt. \tag{3.6}$$

We now derive a fifth-order A–B type predictor formula, by approximating the integral $\int_{t_i}^{t_{i+1}} A(t,\ m(t))\,dt$ in the above equation by the approximation $\int_{t_i}^{t_{i+1}} Q_4(t)\,dt$ where $Q_4(t)$ is the Lagrange interpolating polynomial of degree four passing through the following five points
$(t_{i-4},\ A(t_{i-4},\ m(t_{i-4})))$, $(t_{i-3},\ A(t_{i-3},\ m(t_{i-3})))$, $(t_{i-2},\ A(t_{i-2},\ m(t_{i-2})))$, $(t_{i-1},\ A(t_{i-1},\ m(t_{i-1})))$ and
$(t_i,\ A(t_i,\ m(t_i)))$. That is

$$Q_4(t) = \sum_{k=0}^{4} A(t_{i-k},\ m(t_{i-k}))\, L_k(t). \tag{3.7}$$

where the $L_k(t)$ are the Lagrange basis polynomials on the five points $(t_{i-4},\ t_{i-3},\ t_{i-2},\ t_{i-1},\ t_i)$. Make change of variable $u = \frac{t_{i+1}-t}{h}$, substituting for the Lagrange basis polynomials and integrating, we obtain

$$\begin{aligned}
\int_{t_i}^{t_{i+1}} A(t,\ m(t))\,dt &= \int_0^1 \frac{(u-2)(u-3)(u-4)(u-5)}{(1-2)(1-3)(1-4)(1-5)} hA(t_i,\ m(t_i))\,du \\
&+ \int_0^1 \frac{(u-1)(u-3)(u-4)(u-5)}{(2-1)(2-3)(2-4)(2-5)} hA(t_{i-1},\ m(t_{i-1}))\,du \\
&+ \int_0^1 \frac{(u-1)(u-2)(u-4)(u-5)}{(3-1)(3-2)(3-4)(3-5)} hA(t_{i-2},\ m(t_{i-2}))\,du \\
&+ \int_0^1 \frac{(u-1)(u-2)(u-3)(u-5)}{(4-1)(4-2)(4-3)(4-5)} hA(t_{i-3},\ m(t_{i-3}))\,du \\
&+ \int_0^1 \frac{(u-1)(u-2)(u-3)(u-4)}{(5-1)(5-2)(5-3)(5-4)} hA(t_{i-4},\ m(t_{i-4}))\,du \\
&= \frac{1901h}{720} A(t_i,\ m(t_i)) - \frac{1387h}{360} A(t_{i-1},\ m(t_{i-1})) + \frac{109h}{30} A(t_{i-2},\ m(t_{i-2})) \\
&- \frac{637h}{360} A(t_{i-3},\ m(t_{i-3})) + \frac{251h}{720} A(t_{i-4},\ m(t_{i-4})).
\end{aligned} \tag{3.8}$$

Inserting Eq. (3.8) into Eq. (3.6) and further simplification, we have the iterative formula as below:

$$\begin{aligned}
m(t_{i+1}) - m(t_i) &= \frac{1}{(2-\wp)\beta(\wp)} \left[2(1-\wp) + \frac{1901h\wp}{360}\right] A(t_i,\ m(t_i)) \\
&- \frac{1}{(2-\wp)\beta(\wp)} \left[2(1-\wp) + \frac{1387h\wp}{180}\right] A(t_{i-1},\ m(t_{i-1})) \\
&+ \frac{h\wp}{15(2-\wp)\beta(\wp)} \left[109A(t_{i-2},\ m(t_{i-2})) - \frac{637}{12} A(t_{i-3},\ m(t_{i-3})) + \frac{251}{24} A(t_{i-4},\ m(t_{i-4}))\right].
\end{aligned} \tag{3.9}$$

which is the fifth-step fractional A-B scheme for the modified Caputo-Fabrizio fractional derivative. The truncation error for the five-step formula can be estimated by using the error estimate for the Lagrange interpolating polynomial, namely,

$$A(t,\ m(t)) = Q_4(t) + E_4(t),$$

(3.10)
$$E_4(t) = \frac{A^{(5)}(\eta_i,\ m(\eta_i))}{5!}(t-t_i)(t-t_{i-1})(t-t_{i-2})(t-t_{i-3})(t-t_{i-4}), \quad \eta_i \in (t_{i-4},\ t_i).$$

Then we have

$$\int_{t_i}^{t_{i+1}} E_4(t)\,dt = \int_{t_i}^{t_{i+1}} \frac{A^{(5)}(\eta_i,\ m(\eta_i))}{5!}(t-t_i)(t-t_{i-1})(t-t_{i-2})(t-t_{i-3})(t-t_{i-4})\,dt.$$

Make change of variable $u = \frac{t_{i+1}-t}{h} \Rightarrow -h\,du = dt \Rightarrow t = t_{i+1} - uh$

$$= \int_0^1 \frac{A^{(5)}(\eta_i,\ m(\eta_i))}{5!}(h-uh)(2h-uh)(3h-uh)(4h-uh)(5h-uh)h\,du$$

$$= \frac{h^6 A^{(5)}(\gamma_i,\ m(\gamma_i))}{5!}\int_0^1 (1-u)(2-u)(3-u)(4-u)(5-u)\,du$$

$$= \frac{95h^6 A^{(5)}(\gamma_i,\ m(\gamma_i))}{12(4!)}.$$

where $\gamma_i \in (t_{i-4},\ t_{i+1})$ and we have used a mean value theorem to evaluate the approximate integral value. Denoting the entire right-hand side of Eq. (3.9) by $\tilde{m}_i$, then we have $m_{i+1} = \tilde{m}_i + \frac{95h^6 A^{(5)}(\gamma_i,\ m(\gamma_i))}{12(4!)}$. Therefore, the local truncation error of the use of formula (3.9) is determined by

(3.11)
$$\frac{m_{i+1}-\tilde{m}_i}{h} = \frac{95\wp h^5 A^{(5)}(\gamma_i,\ m(\gamma_i))}{12(4!)(2-\wp)\beta(\wp)}.$$

3.1. **Stability analysis of the scheme.** Next, we examine the stability analysis of the five-step fractional order Adams–Bashforth scheme (3.15), by considering equation

(3.12)
$${}^{CF}\mathscr{D}_t^{\wp} m(t) = A(t,\ m(t)).$$

where ${}^{CF}\mathscr{D}_t^{\wp}$ is the modified Caputo-Fabrizio fractional derivative operator of order $\wp$ modified by Losada and Nieto [19]. Recall that

$$m(t_{i+1}) = m(t_i) + Af(t_i,\ m(t_i)) - Bf(t_{i-1},\ m(t_{i-1})) + Cf(t_{i-2},\ m(t_{i-2}))$$
$$-Df(t_{i-3},\ m(t_{i-3})) + Ef(t_{i-4},\ m(t_{i-4})).$$

By using Eq. (3.12), the above equation becomes

$$m(t_{i+1}) = (1+A)m(t_i) - Bm(t_{i-1}) + Cm(t_{i-2}) - Dm(t_{i-3}) + Em(t_{i-4}).$$

Next, we adopt the Von-Neumann stability analysis for the terms in above equation as

$$m(t_{i+1}) = \tilde{m}(t_{i+1})e^{(i+1)s\Delta t},\ \ m(t_i) = \tilde{m}(t_i)e^{(i)s\Delta t}$$
$$m(t_{i-1}) = \tilde{m}(t_{i-1})e^{(i-1)s\Delta t},\ \ m(t_{i-2}) = \tilde{m}(t_{i-2})e^{(i-2)s\Delta t}$$
$$m(t_{i-3}) = \tilde{m}(t_{i-3})e^{(i-3)s\Delta t},\ \ m(t_{i-4}) = \tilde{m}(t_{i-4})e^{(i-4)s\Delta t}.$$

So that

$$\tilde{m}(t_{i+1})e^{(i+1)s\Delta t} = (1+A)\tilde{m}(t_i)e^{(i)s\Delta t} - B\tilde{m}(t_{i-1})e^{(i-1)s\Delta t}$$
$$+C\tilde{m}(t_{i-2})e^{(i-2)s\Delta t} - D\tilde{m}(t_{i-3})e^{(i-3)s\Delta t} + E\tilde{m}(t_{i-4})e^{(i-4)s\Delta t}.$$

which reduces to

$$\tilde{m}(t_{i+1})e^{s\Delta t} = (1+A)\tilde{m}(t_i) - B\tilde{m}(t_{i-1})e^{-s\Delta t}$$
$$+C\tilde{m}(t_{i-2})e^{-2s\Delta t} - D\tilde{m}(t_{i-3})e^{-3s\Delta t} + E\tilde{m}(t_{i-4})e^{-4s\Delta t}$$

Further simplification and following Von-Neumann stability analysis, we arrived to the required stability condition.

3.2. **Application.** Next, we use the five-step fractional A-B scheme in Eq. (3.15) to obtain numerical solutions of the fractional model (1.3). For $\wp \in (0,\ 1)$, we can then write the system in the vector form:

$$ {}^{CF}\mathscr{D}_t^{\wp} m(t) = A(t,\ m(t)), \tag{3.13}$$

where

$$ m(t) = \left[s(t)\quad \eta(t)\right]^T \quad \text{and} \quad A(t,\ m(t)) = \left[A_1(t,\ m(t))\quad A_2(t,\ m(t))\right]^T. \tag{3.14}$$

The scalar functions A_1 and A_2 are defined from the right hand sides of system (1.3), that is,

$$A_1(t, m(t)) = \eta(t),$$

and

$$A_2(t, m(t)) = \alpha\eta(t) + \beta s(t) - s^3(t) - s^2(t)\eta(t) + \gamma \sin(\delta t) + F.$$

Applying the fractional integral in Eq. (3.2) to both sides of Eq. (3.13), we obtain

$$\begin{aligned} m(t_{i+1}) - m(t_i) = {} & \frac{1}{(2-\wp)\beta(\wp)}\left[2(1-\wp) + \frac{1901h\wp}{360}\right]A(t_i,\ m(t_i)) \\ & - \frac{1}{(2-\wp)\beta(\wp)}\left[2(1-\wp) + \frac{1387h\wp}{180}\right]A(t_{i-1},\ m(t_{i-1})) \\ & + \frac{h\wp}{15(2-\wp)\beta(\wp)}\left[109A(t_{i-2},\ m(t_{i-2})) - \frac{637}{12}A(t_{i-3},\ m(t_{i-3})) + \frac{251}{24}A(t_{i-4},\ m(t_{i-4}))\right]. \end{aligned} \tag{3.15}$$

where $m(t_0) = \left[s(t_0)\quad \eta(t_0)\right]^T$.

4. Results and Discussion

The following parameter values and initial conditions have been used for our simulations:
Starting guesses: $s(0) = 1$ & 2, $\eta(0) = 2$ & 4.
System parameters: $\alpha = 1.2$, $\beta = 0.4$, $\gamma = 1.4$, $\delta = 0.7$ and $F = 0,\ 1.66,\ 1.89$ and different values of fractional order $\wp$ and $h = 0.001$.

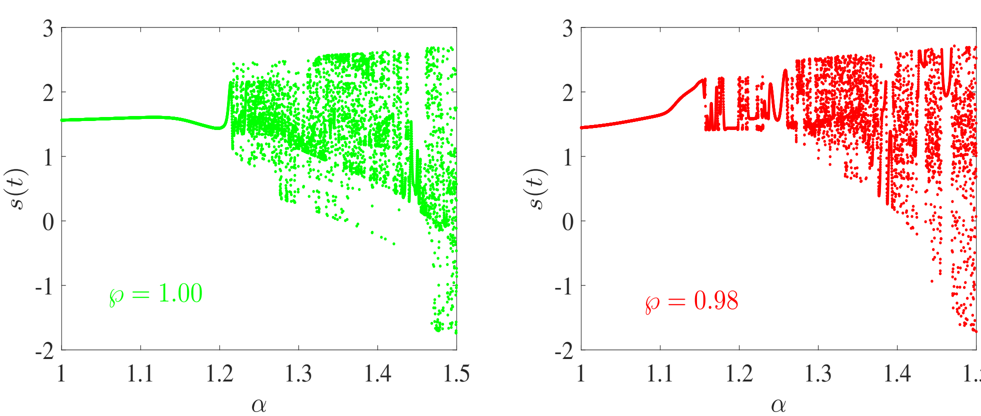


Figure 1. The bifurcation diagrams about α for $s(0) = 1$, $\eta(0) = 2$ and $\beta = 0.4$, $\gamma = 1.4$, $\delta = 0.7$, $F = 1.66$

For $1 \leq \alpha \leq 1.5$, Figure 1(a) reflects the bifurcation diagram in the $\alpha - s(t)$ plane when $\wp = 1.00$, while Figure 1(b) shows the bifurcation diagram for $\wp = 0.98$. We observe that the system (1.3) begins to exhibit chaotic behavior at $\alpha = 1.2$.

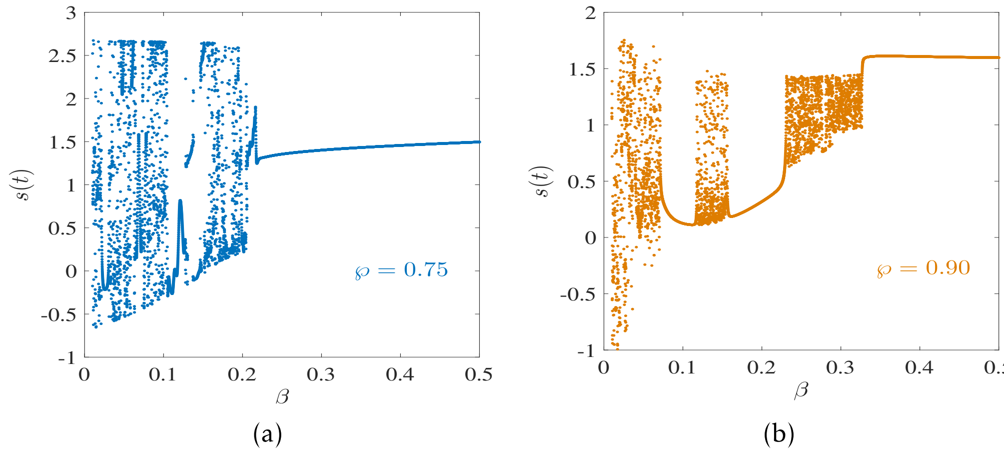


FIGURE 2. The bifurcation diagrams about β for $s(0) = 1$, $\eta(0) = 2$ and $\alpha = 1.2$, $\gamma = 1.4$, $\delta = 0.7$, $F = 1.66$

For $0.01 \leq \beta \leq 0.5$, Figure 2(a) reflects the bifurcation diagram in the $\beta - s(t)$ plane when $\wp = 0.75$, while Figure 2(b) shows the bifurcation diagram at $\wp = 0.90$. As shown in Figure 3(a), the bifurcation for the system (1.3) begins at $\gamma = 1.4$, while in Figure 3(b), the bifurcation starts at $\gamma = 1.45$.

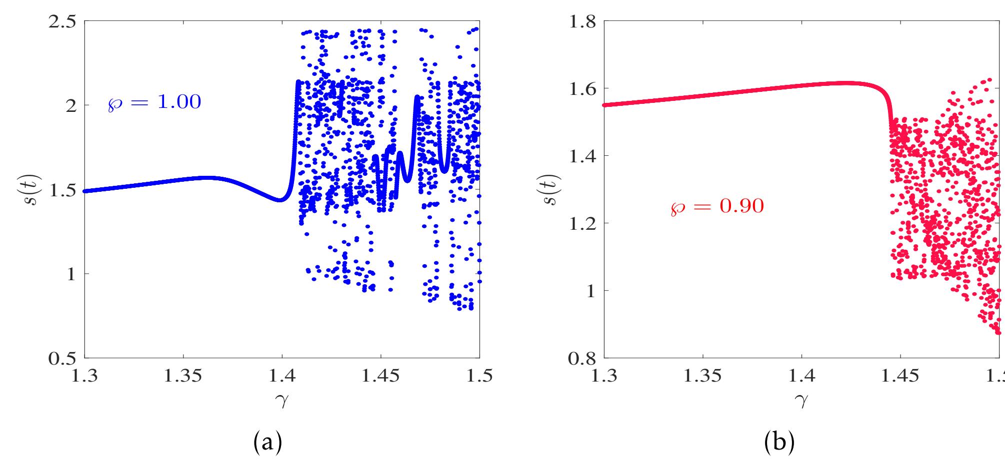
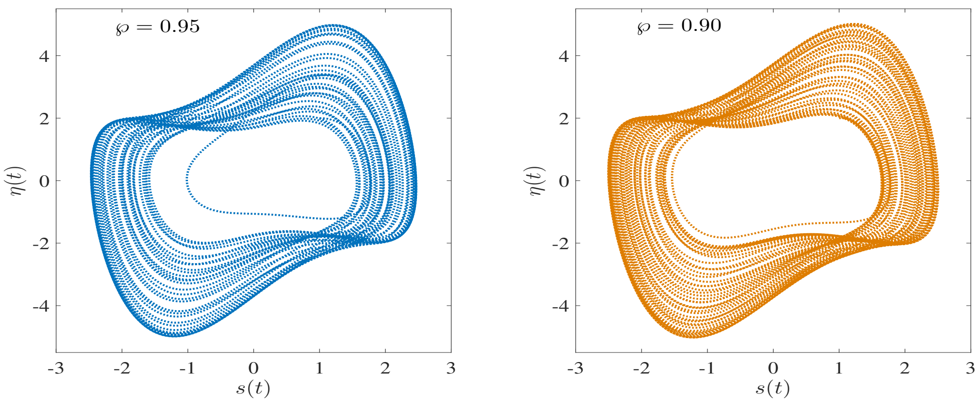


FIGURE 3. The bifurcation diagrams about γ for $s(0) = 1$, $\eta(0) = 2$ and $\alpha = 1.2$, $\beta = 0.4$, $\delta = 0.7$, $F = 1.66$

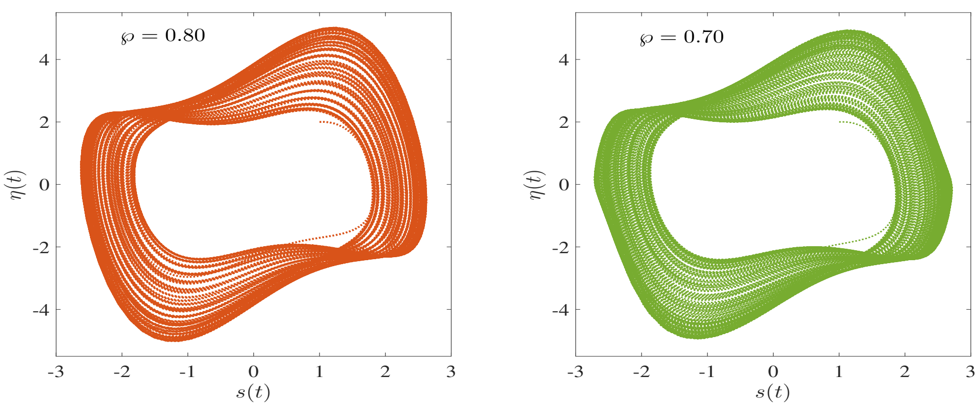
In our fractional system (1.3), the existence of oscillatory behavior of the cryosphere chaotic system for the above set of parametric values is observed in the following figures. The effects of changing the fractional order $\wp$ on each state variable can be observed more clearly in all figures of the system.



(a) (b)

Figure 4. Solution of (1.3) for $s(0) = 1$, $\eta(0) = 2$ and $\alpha = 1.2$, $\beta = 0.4$, $\gamma = 1.4$, $\delta = 0.7$, $F = 0$

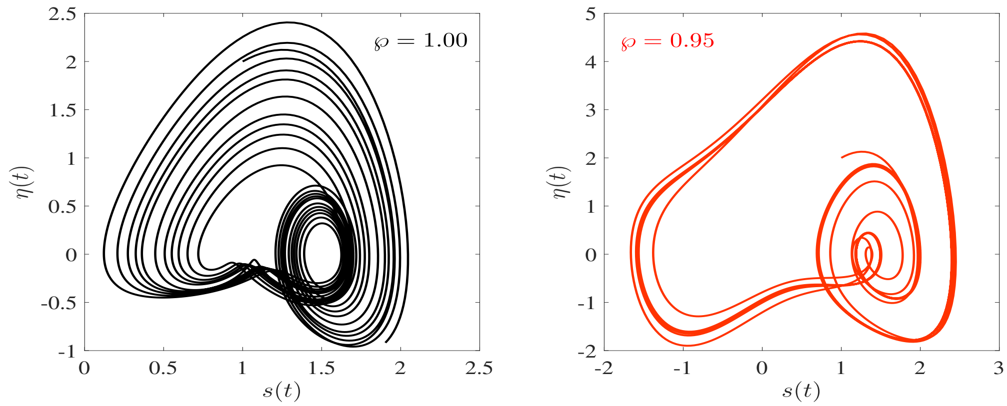
Figure 4 and 5 are plotted for different values of fractional order $\wp = 0.95,\ 0.9,\ 0.80,\ 0.70$.



(a) (b)

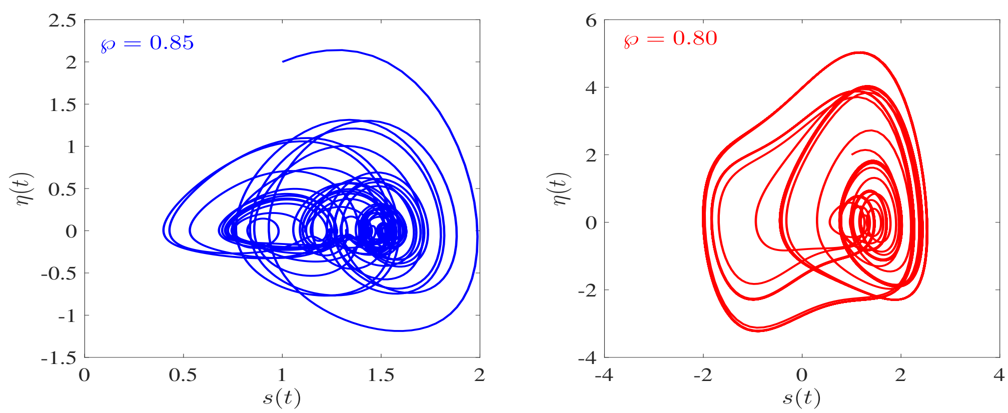
Figure 5. Solution of (1.3) for $s(0) = 1$, $\eta(0) = 2$ and $\alpha = 1.2$ $\beta = 0.4$, $\gamma = 1.4$, $\delta = 0.7$, $F = 0$

Figures (6) and (7) show graphical relations between $s(t)$ and $\eta(t)$ of the system (1.3) for different fractional orders $\wp = 1.00,\ 0.95,\ 0.85,\ 0.80$, respectively. We can confirm from these curves that the plots of each state variable have a significance effect when a fractional order $\wp$ value is changed.



(a) (b)

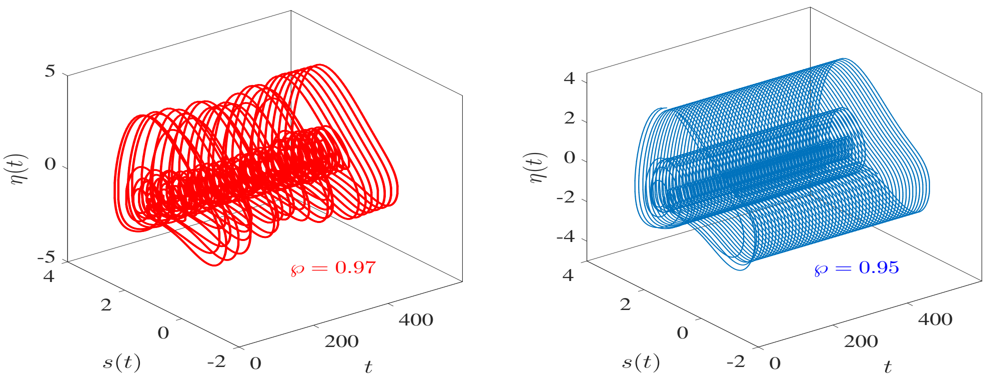
Figure 6. Solution of (1.3) for $s(0) = 1$, $\eta(0) = 2$ and $\alpha = 1.2$, $\beta = 0.4$, $\gamma = 1.4$, $\delta = 0.7$, $F = 1.66$



(a) (b)

Figure 7. Solution of (1.3) for $s(0) = 1$, $\eta(0) = 2$ and $\alpha = 1.2$, $\beta = 0.4$, $\gamma = 1.4$, $\delta = 0.7$, $F = 1.66$

Figure 8(a) and (b) show that the three dimensional view of the system solutions and the positive time series with respect to $\wp = 0.97$ and $\wp = 0.95$.



(a) (b)

Figure 8. 3D view of (1.3) for $s(0) = 1$, $\eta(0) = 2$ and $\alpha = 1.2$, $\beta = 0.4$, $\gamma = 1.4$, $\delta = 0.7$, $F = 1.66$

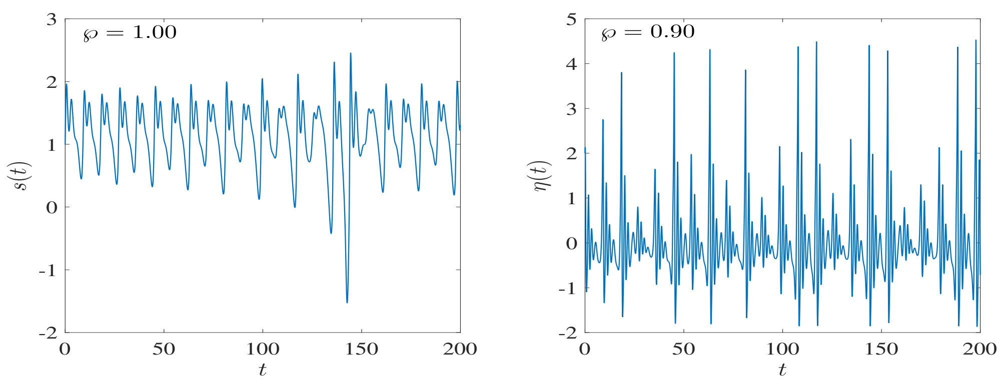


Figure 9. Time series plots of (1.3) for $s(0) = 1$, $\eta(0) = 2$ and $\alpha = 1.2$, $\beta = 0.4$, $\gamma = 1.4$, $\delta = 0.7$, $F = 1.66$

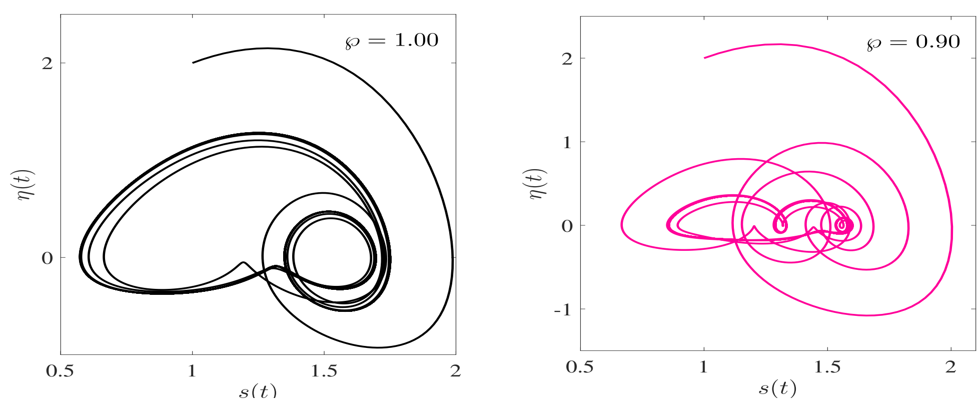


Figure 10. Solution of (1.3) for $s(0) = 1$, $\eta(0) = 2$ and $\alpha = 1.2$, $\beta = 0.4$, $\gamma = 1.4$, $\delta = 0.7$, $F = 1.89$

The effect of changing the fractional order $\wp$ on each state variable can be observed more clearly in all Figures. The numerical simulations of the system in Eq. (1.3) for the radiative forcing constant $F = 1.89$ [33] are shown in Figures (10)-(12).

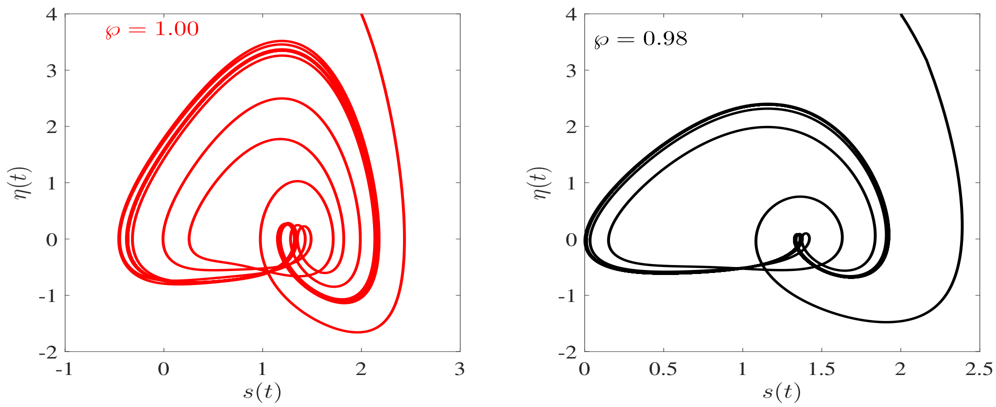


Figure 11. Solution of (1.3) for $s(0) = 2$, $\eta(0) = 4$ and $\alpha = 1.01$, $\beta = 0.1$, $\gamma = 1.4$, $\delta = 0.9$, $F = 1.89$

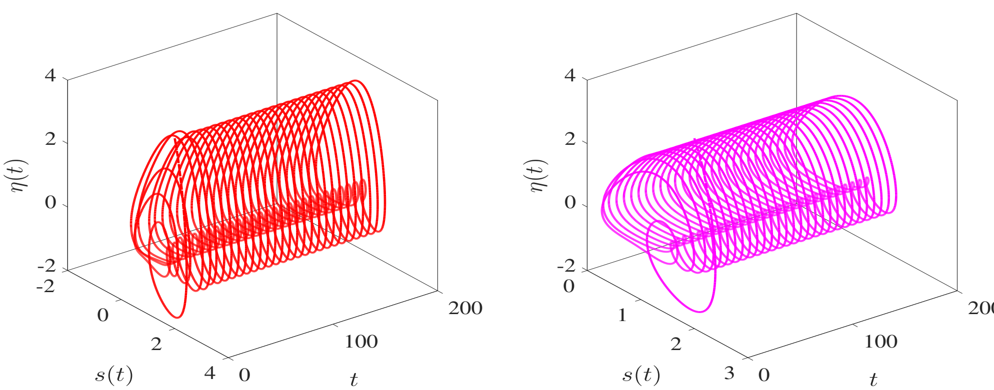


Figure 12. Solution of (1.3) for $s(0) = 2$, $\eta(0) = 4$ and $\alpha = 1.01$, $\beta = 0.1$, $\gamma = 1.4$, $\delta = 0.9$, $F = 1.89$

5. Conclusion

This article considers a nonlinear system of the Caputo-Fabrizio fractional dynamics model of Cryosphere with initial guesses. In the modified form of Caputo-Fabrizio fractional derivative sense, the Cryosphere model including the radiative forcing of CO_2 gas is presented as a system of differential equations, with the approximate results achieved using the fifth-order fractional Adams-Bashforth method. All computations in this paper are done using MATLAB R2016a version of mathematical software. From the above graphical results, the proposed scheme is highly effective and can be used to several dynamical mathematical models like climate system, biotherapy, HIV AIDS models, smoking models and drug targeting systems. In relation to the paper objectives set at the beginning of the paper, the study had successfully provide the following conclusions:

- The previously modified model of Cryosphere is represented using non-singular kernel fractional operator and we solved it using fractional five-step Adams-Bashforth scheme.
- The recommended method is easy to understand, and the numerical results obtained indicate that it is very effective for solving the aforementioned mathematical models approximately as well as for solving additional systems of differential equations.
- The scheme's stability analysis is successfully discussed.
- The graphical solutions support our theoretical procedures.

The future work will focus on modification of this model by including different parameters to investigate the effects of these parameters on climate change and investigating the model with the most recent fractional operators and approximating by the new numerical approaches.

Author contributions. The author contributed to the writing of this paper and approved the final manuscript.

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