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THE SELECTION THEORY FOR MONOTONE UNIFORMLY EQUI-BOBTAIL OPERATORS ON SEPARABLE HILBERT SPACES

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ABSTRACT. We establish that a uniformly bounded infinite system $\{A_\alpha\}$ of monotone, uniformly continuous, uniformly equi-bobtail operators $A_\alpha : K \to H$ on the separable Hilbert space admits a convergent subsequence $\{A_n\} \subset \{A_\alpha\}$ and show that limit of this subsequence is a monotone operator $A^M : K \to H$.

1. INTRODUCTION

In this article, we improve the selection principle for an infinite system $\{A_\alpha\}$ of uniformly bounded, uniformly equi-bobtailed, continuous monotone operators on a separable Hilbert space. Theorems of this type are interesting because of their deep applications in functional analysis to a problem of moments of distribution.

We recall some definitions and results of the classical Helly selection theory.

Definition 1.1. *Let X be a complete metric space with metric ρ. Let $\Pi = \{t_0,\ t_1,\ ...,\ t_n\}$ be a partition of an interval $[a,\ b]$ on the real line. A function $f\ :\ [a,\ b] \to X$ has a bounded variation if and only if*

$$\sup_{\Pi} \sum_{k=1,\ldots,n} \rho\left(f\left(t_{k-1}\right),\ f\left(t_k\right)\right) = \underset{[a,\ b]}{var}\left(f\right) < \infty.$$

Then, the simple generalization of the original Helly theorem can be formulated in the form: let $\{f_\alpha\}$ be an infinite system of uniformly bounded functions $f_\alpha\ :\ [a,\ b] \to X$ such that $\underset{[a,\ b]}{var}\left(f_\alpha\right) < c$ with the same constant c for all indexes α, then, there exists a subsequence $\{f_n\}$ converging elementwise $f\left(t\right) = \lim_{n\to\infty} f_n\left(t\right)$ in X, to a function $f\ :\ [a,\ b] \to X$ with $\underset{[a,\ b]}{var}\left(f\right) < \infty$.

The proof of the Helly theorem is based on the fact that the set of discontinuity points of a function with bounded variation cannot be more than countable. For the countable set, we can use the similar diagonal approach that will be applied in our theorem 2.5 to select a convergent subsequence.

The relevance of the condition of bounded variation can be illustrated by the example of the sequence $\{f_n\ :\ [a,\ b] \to R\}\ \{f_n\left(t\right) = \sin\left(nt\right)\}$.

Since the Helly selection theory is closely connected with the compactness concept we consider several conditions for the closed and bounded set was compact in a separable Hilbert space. Assume $\{e_n\}$ is an orthonormal basis of the separable Hilbert space H, then a closed and bounded subset $K \subset H$ is compact

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if and only if the subset K satisfies the uniformly equi-bobtail condition, namely, for all $\varepsilon > 0$ there is a number n_0 such that $\sum_{n>n_0} (u,\ e_n)^2 < \varepsilon^2$ holds for all $u \in K$. A compact set can be approximated by finite-dimensional sets with an arbitrary degree of precision, namely, a necessary and sufficient condition that a closed and bounded subset $K \subset H$ is compact is that for all $\varepsilon > 0$ there is a finite-dimensional subspace $F \subset H$ such that $\inf\limits_{v\in F} \|v - u\| < \varepsilon$ for all $u \in K$.

In this article, we establish that if $\{A_n\}$ is a sequence of uniformly bounded, uniformly equi-bobtailed operators $A_n : K \to H$, and assume A_n are monotone and continuous, then we can select an elementwise converging subsequence $\{A_{k(n)}\} \subset \{A_n\}$ such that the operator $A^M(u) = \lim\limits_{k\to\infty} A_{k(n)}(u)$ is monotone.

In the setup of the classical Helly theorem, we select the subsequence that converges on the dense countable subset and spread the limiting function $f(t) = \lim\limits_{n\to\infty} f_n(t)$ on the set of its continuity since the limit $\lim\limits_{n\to\infty} f_n(t) = f(t)$ can be discontinuous at most on a countable set, and, for the discontinuity set, we can apply previous arguments. We require the continuity of each A_α as an initial assumption since our monotony condition is not enough for continuity, and to extend the limit operator $A(u) = \lim\limits_{n\to\infty} A_n(u)$ from the countable dense set to the whole compact subset we need an additional condition. This continuity condition can be weakened.

Since the literature on the monotone operators and Hilbert spaces is extremely extensive, its reviewing is beyond the framework of the present work, and some of the recent works in the field and applications [1–11].

2. Monotone Bobtail Operators in Separable Hilbert Spaces

Let H be a separable Hilbert space and let $\{e_n\}$ be an orthonormal basis of H.

Definition 2.1. *An operator $A : H \to H$ is called monotone if and only if the inequality*

$$(A(u) - A(v),\ u - v) \geq 0$$

holds for all $u,\ v \in H$.

An operator $A : H \to H$ *is called strictly monotone if and only if the inequality*

$$(A(u) - A(v),\ u - v) > 0$$

holds for all $u,\ v \in H$ such that $u \neq v$.

Lemma 2.2. *An operator $A : H \to H$ is monotone if and only if the function $f : [0,\ 1] \to R$, given by*

$$f(t) \equiv (A(u + tz),\ z)$$

for all $u,\ z \in H$, is a monotone increasing function of t for all $u,\ z \in H$.

Proof. Assume $A : H \to H$ is monotone then the monotony of f follows from

$$f(t) - f(s) = \frac{(A(u + tz) - A(u + sz),\ (t - s)z)}{t - s} \geq 0$$

for all $0 \leq s < t \leq 1$.

Let $f(t) = (A(u + tz),\ z)$ be monotone then

$$(A(u + z) - A(u),\ z) = f(1) - f(0) \geq 0,$$

which proves the lemma.

Definition 2.3. *A subset K of H is said to be uniformly equi-bobtailed if for all $\varepsilon > 0$ there exists n_0 such that the estimate*

$$\sum_{n>n_0} (u,\ e_n)^2 < \varepsilon^2$$

holds for all $u \in K$.

Definition 2.4. *An operator $A\ :\ H \to H$ is called uniformly equi-bobtailed if and only if any $\varepsilon > 0$ there exists n_0 such that the inequality*

$$\sum_{n>n_0} (A(u)\,,\ e_n)^2 < \varepsilon^2$$

holds for all $u \in H$.

Now, we are going to study the properties of the operator sequences in separable Hilbert space.

The system $\{A_\alpha\}$ of operators $A_\alpha\ :\ H \to H$ is called uniformly bounded if there exists a positive constant M such that $\|A_\alpha(u)\| \le M$ for all $u \in H$.

Theorem 2.5. *Let K be a compact subset of H. Let $\{A_\alpha\}$ be an infinite system of uniformly bounded, uniformly equi-bobtailed operators $A_\alpha\ :\ H \to H$. Let E be not more than countable subset of K. Then, there exists a sequence $\{A_n(u)\} \subset \{A_\alpha(u)\}$ that converges at each $u \in E$.*

Proof. We assume $E = \{u_i\}$ is a countable system, the system $\{e_n\}$ is an orthogonal basis of H and consider the system $\{A_\alpha(u_1)\} \subset H$. Form the bobtail condition, we have that for each $\varepsilon > 0$ we take a number n_0 such that

$$\sum_{n>n_0} (A_\alpha(u_1)\,,\ e_n)^2 < \varepsilon^2.$$

We express $\{A_\alpha(u_1)\}$ as an expansion in terms of the orthonormal basis of H, and the Fourier coefficients $(A_\alpha(u_1)\,,\ e_n)$. Thus, there exists a positive constant M such that $\|A_\alpha(u_1)\| \le M$ for all α. So, the inequality

$$|(A_\alpha(u_1)\,,\ e_n)| \le \|A_\alpha(u_1)\|\,\|e_n\| \le M$$

holds for all α and all n. Therefore, for each n, the system $\{(A_\alpha(u_1)\,,\ e_n)\}_\alpha$ is a bounded infinite set of numbers. By the classical Bolzano-Weierstrass theorem, any infinite bounded set of real numbers has a convergent subsequence. Thus, for each n, there exist convergent sequences $\left\{\left(A_{\tilde{n}(1)}(u_1)\,,\ e_1\right)\right\}_{\tilde{n}(1)} \subset \{(A_\alpha(u_1)\,,\ e_1)\}_\alpha$, and $\left\{\left(A_{\tilde{n}(2)}(u_1)\,,\ e_2\right)\right\}_{\tilde{n}(2)} \subset \{(A_\alpha(u_1)\,,\ e_2)\}_\alpha$, we can repeat this process a countable number of times, and we obtain a sequence $\left\{A_{\tilde{n}(i)}(u_1)\right\}_{\tilde{n}(i)} \subset \{A_\alpha(u_1)\}_\alpha$ such that the subsequence $\left\{\left(A_{\tilde{n}(i)}(u_1)\,,\ e_i\right)\right\}_{\tilde{n}(i)} \subset \{(A_\alpha(u_1)\,,\ e_i)\}_\alpha$ converges in the first i entries.

The sequence $\left\{A_{\tilde{n}(i)}(u_1)\right\}_{\tilde{n}(i)} \subset \{A_\alpha(u_1)\}_\alpha$ converges at the n basis element as i tends to infinity. Since Hilbert space H is complete, each fundamental sequence $\left\{A_{\tilde{n}(i)}(u_1)\right\}_{\tilde{n}(i)}$ converges to an element of H.

We will show that sequence $\left\{A_{\tilde{n}(i)}(u_1)\right\}_{\tilde{n}(i)}$ is fundamental. Let $\varepsilon > 0$ then there exists n_0 such that the estimate

$$\sum_{n>n_0} \left(A_{\tilde{n}(i)}(u_1)\,,\ e_n\right)^2 < 16^{-1}\varepsilon^2$$

holds for all natural numbers i. We take a number $\tilde{N}$ such that

$$\sum_{n=1,\ldots,n_0} \left(\left(A_{\tilde{n}(i)}(u_1)\,,\ e_n\right) - \left(A_{\tilde{n}(j)}(u_1)\,,\ e_n\right)\right)^2 < 4^{-1}\varepsilon^2$$

for all $i,\ j \geq \tilde{N}$. Simple calculations yield

$$\begin{aligned}
&\left\|A_{\tilde{n}(i)}\left(u_1\right)-A_{\tilde{n}(j)}\left(u_1\right)\right\|= \\
&=\sum_{n=1,\ldots,n_0}\left(\left(A_{\tilde{n}(i)}\left(u_1\right)-A_{\tilde{n}(j)}\left(u_1\right),\ e_n\right)^2+\right. \\
&\left.+\sum_{n>n_0}\left(A_{\tilde{n}(i)}\left(u_1\right)-A_{\tilde{n}(j)}\left(u_1\right),\ e_n\right)^2\right)^{\frac{1}{2}} \leq
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{n=1,\ldots,n_0}\left(\left(A_{\tilde{n}(i)}\left(u_1\right)-A_{\tilde{n}(j)}\left(u_1\right),\ e_n\right)^2\right)^{\frac{1}{2}}+ \\
&+\left(\sum_{n>n_0}\left(A_{\tilde{n}(i)}\left(u_1\right)-A_{\tilde{n}(j)}\left(u_1\right),\ e_n\right)^2\right)^{\frac{1}{2}} \leq \\
&\leq 2^{-1}\varepsilon+\left(\sum_{n>n_0}\left(A_{\tilde{n}(i)}\left(u_1\right),\ e_n\right)^2\right)^{\frac{1}{2}}+\left(\sum_{n>n_0}\left(A_{\tilde{n}(j)}\left(u_1\right),\ e_n\right)^2\right)^{\frac{1}{2}}<\varepsilon
\end{aligned}$$

for chosen n_0. Thus, the sequence $\left\{A_i\left(u_1\right)\right\}_i$ is fundamental so it converges to an element of H.

We have obtained an infinite sequence $\left\{A_i\left(u\right)\right\}_i$ that is correctly defined on the whole countable set E, we can repeat the same argument for the next $u_2 \in E$ and obtain the subsequence $\left\{A_{\tilde{i}(s)}\left(u_2\right)\right\}_s \subset \left\{A_{\tilde{i}(s)}\left(u_1\right)\right\}_s$. By repeating this process, a countable number of times, we obtain the countable set of convergent subsequences

$$\left\{A_i{}^1\left(u_1\right)\right\} \supset \left\{A_i{}^2\left(u_2\right)\right\} \supset \ldots \left\{A_i{}^n\left(u_n\right)\right\}, \ldots.,$$

the order of elements of the sequence is preserved for all subsequences.

Next, we compose the sequence $\left\{A_n{}^n\left(u\right)\right\}$ of diagonal elements for all natural numbers n. Thus, we obtain the sequence $\left\{A_n \equiv A_n{}^n\right\}$ of operators $A_n\ :\ E \to H$ that converge at each element of the set E, since for each fixed k the sequence $\left\{A_n{}^n\left(u_k\right)\right\},\quad n \geq k$ is a subsequence of convergent sequence $\left\{A_n{}^k\left(u_k\right)\right\}$ so that $\lim\limits_{n\to\infty} A_n\left(u\right) = A\left(u\right)$ for all $u \in E$ where E is countable.

3. The Helly Selection Principle for Monotone Bobtail Continuous Operators

We assume H is a separable Hilbert space and E is a countable subset of H. We extend the operator $A\ :\ E \to H$ to the whole H by taking$A^M\left(u\right) = z \in H$ at which the limit

$$\lim_{\varepsilon\to 0} \sup_{\substack{v \in E \subset H \\ (u,\ v) > 0 \\ z \in B(A(v),\ \varepsilon)}} \left(z - A\left(v\right),\ u - v\right) \equiv \inf_{\varepsilon>0} \sup_{\substack{v \in E \subset H \\ (u,\ v) > 0 \\ z \in B(A(v),\ \varepsilon)}} \left(z - A\left(v\right),\ u - v\right)$$

is achieved. Straightforwardly from this definition, we have that the operator $A^M\ :\ H \to H$ is a monotone.

Theorem 3.1. *Let H be a separable Hilbert space. Let K be a compact subset of H. Let $\{A_\alpha\}$ be an infinite system of uniformly bounded, uniformly equi-bobtailed operators $A_\alpha\ :\ K \to H$. Let all operators $A_\alpha\ :\ K \to H$ be monotone and uniformly continuous. Then, there exists an elementwise converging sequence $\{A_n\} \subset \{A_\alpha\}$ such that $\lim\limits_{n\to\infty} A_n\left(u\right) = A^M\left(u\right)$ for all $u \in K$ and the limit $A^M\left(u\right)$ satisfies the monotony condition.*

Proof. Since H is a separable Hilbert space, we can take a countable dense set $E \subset H$ such that there exists a sequence $\left\{A_n\left(v\right)\right\} \subset \left\{A_\alpha\left(v\right)\right\}$ that converges $\lim\limits_{n\to\infty} A_n\left(v\right) = A\left(v\right)$ in H for all $v \in E$. The operator $A\left(u\right)$ is defined only on the countable dense set E yet, and this operator is monotone. Next, we extend the operator $A : E \to H$ to all K. By employing our extension procedure, we spread operator $A(v) = \lim_{n\to\infty} A_n(v)$ over the whole K by

$$A^M(u) = \left\{ z \in K : \lim_{\varepsilon \to 0} \sup_{\substack{v \in E \subset H \\ (u,\ v) > 0 \\ z \in B(A(v),\ \varepsilon)}} (z - A(v),\ u - v) \quad is \quad achieved \right\}.$$

Now, we must show that the sequence $\{A_n(u)\}$ converges to $A^M(u)$ for all $u \in K$.

Let $\varepsilon > 0$ then there exists $v_k \in E$ such that

$$\left\| A^M(u) - A(v_k) \right\| < 3^{-1}\varepsilon$$

hence $A^M(u) \in B(A(v_k),\ \varepsilon)$. Since $\lim_{n\to\infty} A_n(v) = A(v)$ for all $v \in E$, there exist some numbers n_0 such that the estimate

$$\| A_n(v_k) - A(v_k) \| < 3^{-1}\varepsilon$$

holds for all $n \geq n_0$ and all $v_k \in E$. Once more applying the density of set E in H and the continuity of A_α, for each $\varepsilon > 0$ and each n, we have that for all $u \in K$ there is $v \in E$ such that

$$\| A_n(u) - A_n(v_k) \| < 3^{-1}\varepsilon.$$

Thus, for all $\varepsilon > 0$, there exists some n_0 such that for all $n \geq n_0$, we have

$$\begin{aligned} &\left\| A^M(u) - A_n(u) \right\| = \\ &= \left\| A^M(u) - A(v_k) + A(v_k) - A_n(v_k) + A_n(v_k) - A_n(u) \right\| \leq \\ &\leq \left\| A^M(u) - A(v_k) \right\| + \| A(v_k) - A_n(v_k) \| + \\ &+ \| A_n(v_k) - A_n(u) \| < 3^{-1}\varepsilon + 3^{-1}\varepsilon + 3^{-1}\varepsilon = \varepsilon \end{aligned}$$

for all $u \in K$. That proves theorem 3.1.

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