

## UNCERTAINTY PRINCIPLES AND CALDERÓN'S FORMULA FOR THE MULTIDIMENSIONAL HANKEL-GABOR TRANSFORM

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**ABSTRACT.** The main crux of this paper is to introduce a new integral transform called the multidimensional Hankel-Gabor transform and to give some new results related to this transform as Plancherel's, Parseval's, inversion and Calderón's reproducing formulas. Next, we analyse the concentration of this transform on sets of finite measure and we give uncertainty principle for orthonormal sequences. Last, we extend the Donoho-Stark's uncertainty principle to the multidimensional Hankel-Gabor setting.

### 1. INTRODUCTION

Time-frequency analysis [13] and uncertainty principles [9, 18] play a fundamental role in field of mathematics and physics, these principles appear in harmonic analysis and signal theory in a various different forms involving not only the signal  $f$  and its Fourier transform  $\hat{f}$ , but also every representation of a signal in the time-frequency space.

The uncertainty principles are mathematical results that gives limitations on the simultaneous concentration of a signal and its Fourier transform and they have implications in signal analysis and quantum physics. In signal analysis they tell us that if we observe a signal only for a finite period of time, we will lose information about the frequencies of signal consists of.

Timelimited and bound limited functions are basic tools in signal analysis and imaging processing. In quantum physics they tell us that a particule's speed and position cannot both of them be measured with infinite precision, the mathematical formulation of this principle is given by the following Heisenberg-Pauli-Weyl sharp inequality [22], which shows that for every integrable function  $f$  we have

$$\left( \int_{\mathbb{R}} x^2 |f(x)|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}} \lambda^2 |\hat{f}(\lambda)|^2 d\lambda \right)^{1/2} \geq \frac{1}{2} \left( \int_{\mathbb{R}} |f(x)|^2 dx \right),$$

with equality if and only if  $f(x) = de^{-bx^2}$  for some  $d \in \mathbb{C}$  and  $b > 0$ ; where

$$\hat{f}(\lambda) = 1/\sqrt{2\pi} \int_{\mathbb{R}} f(x)e^{-ix\lambda} dx,$$

is the Fourier transform of  $f$ . Other uncertainty relations have been investigated among them, we refer to the papers of Benedick's [1], Donoho-Stark's [5], Jaming's [13].

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The multidimensional Bessel operator is an elliptic partial differential operator denoted by  $\Delta_{\alpha,d}$  defined for  $x = (x_1, \dots, x_d) \in \mathbb{R}_+^d$ ,  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ ,  $\alpha_k > -\frac{1}{2}$ ;  $k = 1, \dots, d$ , by

$$(1.1) \quad \Delta_{\alpha,d} = \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2} + \frac{2\alpha_k + 1}{x_k} \frac{\partial}{\partial x_k}.$$

The multidimensional Bessel operator (1.1) has several applications in pure and applied mathematics, especially in fluid mechanics see [6, 21]. The eigenfunctions of the operator (1.1) are related to the Bessel functions and they satisfies a product formula which permits to develop a new harmonic analysis associated to this operator for more information we refer the reader to [2, 7, 17].

Uncertainty principles play a fundamental role in the field of mathematics, physics and some area of engineering such as signal processing, image processing, quantum theory and optics see [8, 13, 18], in this context using the Gabor transform introduced by Gabor, using translation, modulation and convolution operators of a single Gaussian, the authors in [4, 23] gives a new uncertainty principles for the Gabor transform. Uncertainty principles associated with the Gabor was studied in the one dimensional Hankel setting [1, 10, 11], Opdam-Cherednik setting [16] and in the two-sided quaternion setting [5], motivated by these works the main purpose of this work is to introduce the Gabor transform associated with the multidimensional Bessel operator (1.1) called the multidimensional Hankel-Gabor transform and to give some new results related to this transform as Plancherel's, Parseval's, inversion and Calderon's reproducing formulas. Next, we give some new uncertainty principles associated with this transform.

The remainder of this paper is arranged as follows, in section 2 we recall the main results concerning the harmonic analysis associated with the multidimensional Hankel transform, in section 3 we introduce the multidimensional Hankel-Gabor transform and we give some new results related to this transform, the last section is devoted to analyse the concentration of the multidimensional Hankel-Gabor transform on sets of finite measure and to give some new uncertainty principles related to this transform.

## 2. HARMONIC ANALYSIS ASSOCIATED WITH THE MULTIDIMENSIONAL HANKEL TRANSFORM

In this section we set some notations and we recall some results in harmonic analysis related to the multidimensional Hankel transform and the Schatten-von Neumann classes, for more details, we refer the reader to [2, 7, 17].

In the following we denote by

- $\mathbb{R}_+^d = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d, x_1 > 0, x_2 > 0, \dots, x_n > 0\}$ , equipped with the weighted Lebesgue measure  $\mu_\alpha$  given by

$$(2.1) \quad d\mu_\alpha(x) = \prod_{k=1}^d \frac{x_k^{2\alpha_k+1}}{2^{|\alpha|}\Gamma(\alpha_k+1)} dx_k, \quad \alpha_k > -1/2.$$

- $L_\alpha^p(\mathbb{R}_+^d)$ ,  $1 \leq p \leq \infty$ , the space of measurable functions  $f$  on  $\mathbb{R}_+^d$  such that

$$\|f\|_{p,\alpha} = \left( \int_{\mathbb{R}_+^d} |f(x)|^p d\mu_\alpha(x) \right)^{1/p} < \infty, \quad p \in [1, \infty),$$

$$\|f\|_{\infty,\alpha} = \text{ess sup}_{x \in \mathbb{R}_+^d} |f(x)| < \infty$$

in particular for  $p = 2$ ,  $L_\alpha^2(\mathbb{R}_+^d)$  is a Hilbert space with inner product defined for  $f, g \in L_\alpha^2(\mathbb{R}_+^d)$  by

$$\langle f | g \rangle_\alpha = \int_{\mathbb{R}_+^d} f(x) \overline{g(x)} d\mu_\alpha(x)$$

- $\mathcal{A}_\alpha(\mathbb{R}_+^d) = \{f \in L_\alpha^1(\mathbb{R}_+^d); \mathcal{F}_\alpha(f) \in L_\alpha^1(\mathbb{R}_+^d)\}$  the Wiener algebra space, where  $\mathcal{F}_\alpha(f)$  is the multidimensional Hankel transform of  $f$  given by (2.6).

**2.1. The Eigenfunctions of the multidimensional Bessel operator.** The main purpose of this subsection is to define the eigenfunctions of the multidimensional Bessel operator  $\Delta_{\alpha,d}$  which will be used to define the multidimensional Hankel transform.

**2.1.1. One dimensional case.** For  $\alpha > -1/2$ , the one dimensional Bessel operator is defined by

$$(2.2) \quad B_\alpha = \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x} \frac{\partial}{\partial x}$$

For  $d \in \mathbb{N}$  with  $d \geq 2$  we have

$$B_{\frac{d-2}{d}} = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r},$$

is the radial part of the Laplace operator  $\Delta_{\alpha,d} = \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}$  on  $\mathbb{R}^d$ .

We recall that the normalized Bessel function of the first kind and order  $\alpha$  is defined on  $\mathbb{C}$  as follows

$$(2.3) \quad j_\alpha(z) = \Gamma(\alpha+1) \sum_{k=0}^{+\infty} \frac{(-1)^k (z)^{2k}}{2^{2k} k! \Gamma(\alpha+k+1)},$$

we will use this function to define the eigenfunctions of the multidimensional Bessel operator.

**2.1.2. The multidimensional case.** Now, we consider the n-orders differential operator  $\Delta_{\alpha,d}$  defined for  $x = (x_1, \dots, x_d) \in \mathbb{R}_+^d$ ,  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ ,  $\alpha_k > -\frac{1}{2}$ ;  $k = 1, \dots, d$ , by

$$\Delta_{\alpha,d} = \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2} + \frac{2\alpha_k+1}{x_k} \frac{\partial}{\partial x_k} = \sum_{k=1}^d B_{\alpha_k},$$

where  $B_{\alpha_k}$  is the one dimensional Bessel operator given by the relation (2.2).

One can remark that

- If  $\alpha_k = -\frac{1}{2}$  for  $i = 1, \dots, d$  then  $\Delta_{\alpha,d} = \Delta_\alpha = \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}$  is the Laplace operator on  $\mathbb{R}^d$ .
- If  $\alpha_k = -\frac{1}{2}$  for  $k = 1, \dots, d-1$  and  $\alpha_d > -\frac{1}{2}$ , then

$$\Delta_{\alpha,d} = \Delta_\alpha = \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2} + \frac{2\alpha_d+1}{x_d} \frac{\partial}{\partial x_d},$$

$\Delta_\alpha$  coincides with the Weinstein operator defined on  $\mathbb{R}_+^d$  see [6, 21].

For  $x, \lambda$ , we put

$$(2.4) \quad \psi_{\alpha,d}(\lambda x) = \prod_{k=1}^d j_{\alpha_k}(\lambda_k x_k),$$

where  $j_{\alpha_k}$  is the Bessel function given by (2.3), from [2, 7] we have the following results

**Proposition 2.1.** the function  $\psi_\alpha(\lambda.)$  is the unique solution of the following Cauchy problem

$$\begin{cases} \Delta_\alpha u = -\|\lambda\|^2 u, \\ u(0_{\mathbb{R}^d}) = 1, \\ \frac{\partial}{\partial x_i} u(x) \Big|_{x_i=0} = 0; i = 1 \dots d, \end{cases}$$

furthermore it is infinitely differentiable on  $\mathbb{R}_+^d$ , even with respect to each variable and satisfies the following important result, for all  $x, \lambda \in \mathbb{R}_+^d$  we have

$$(2.5) \quad |\psi_{\alpha,d}(\lambda x)| \leq 1.$$

We will use this function to define the multidimensional Hankel transform.

## 2.2. The multidimensional Hankel transform.

**Definition 2.1.** The multidimensional Hankel transform  $\mathcal{F}_\alpha$  is defined on  $L^1_\alpha(\mathbb{R}^d_+)$  by

$$(2.6) \quad \mathcal{F}_\alpha(f)(\lambda) = \int_{\mathbb{R}^d_+} f(x) \psi_{\alpha,d}(\lambda x) d\mu_\alpha(x), \quad \lambda \in \mathbb{R}^{d+1}_+,$$

where  $\mu_\alpha$  is the measure on  $\mathbb{R}^d_+$  given by the relation (2.1). Some basic properties of this transform are as follows, for the proofs, we refer the reader to [2, 7].

**Proposition 2.2.** (1) For all  $f \in L^1_\alpha(\mathbb{R}^d_+)$ , the function we have

$$(2.7) \quad \|\mathcal{F}_\alpha(f)\|_{\infty,\alpha} \leq \|f\|_{1,\alpha}.$$

(2) (Parseval's formula) For all  $f, g \in L^2_\alpha(\mathbb{R}^d_+)$ , we have

$$(2.8) \quad \int_{\mathbb{R}^d_+} f(x) \overline{g(x)} d\mu_\alpha(x) = \int_{\mathbb{R}^d_+} \mathcal{F}_\alpha(f)(\lambda) \overline{\mathcal{F}_\alpha(g)(\lambda)} d\mu_\alpha(\lambda)$$

(3) (Plancherel's theorem) The Weinstein transform  $\mathcal{F}_\alpha$  extends uniquely to an isometric isomorphism on  $L^2_\alpha(\mathbb{R}^d_+)$  and we have

$$(2.9) \quad \|\mathcal{F}_\alpha(f)\|_{\alpha,2} = \|f\|_{\alpha,2},$$

for all  $L^2_\alpha(\mathbb{R}^d_+)$ .

(4) (Inversion formula) Let  $f \in \mathcal{A}_\alpha(\mathbb{R}^d_+)$ , then we have

$$(2.10) \quad f(\lambda) = \int_{\mathbb{R}^d_+} \mathcal{F}_\alpha(f)(x) \psi_{\alpha,d}(\lambda x) d\mu_\alpha(x), \quad \text{a.e. } \lambda \in \mathbb{R}^d_+.$$

## 2.3. Generalized translation operator associated with the multidimensional Hankel transform.

**Definition 2.2.** The translation operator  $\tau_\alpha^x, x \in \mathbb{R}^d_+$  associated with the multidimensional Bessel operator  $\Delta_{\alpha,d}$ , is defined for a suitable function  $f$  by

$$\tau_\alpha^x f(y) = c'_\alpha \int_{[0,\pi]^d} f(X_1, \dots, X_n) \prod_{i=1}^d (\sin \theta_i)^{2\alpha_i} d\theta_1 \dots d\theta_d,$$

with  $c'_\alpha = \prod_{i=1}^d \frac{\Gamma(\alpha_i+1)}{\pi^{d/2} \Gamma(\alpha_i+1/2)}$  and  $X_i = \sqrt{x_i^2 + y_i^2 - 2y_i x_i \cos \theta_i}$ , for  $i = 1, \dots, d$ . The following proposition summarizes some properties of the generalized translation operator see [2, 7].

**Proposition 2.3.** . (1) For all  $x, y \in \mathbb{R}^{d+1}_+$ , we have

(1)

$$(2.11) \quad \tau_\alpha^x f(y) = \tau_\alpha^y f(x)$$

(2)

$$(2.12) \quad \int_{\mathbb{R}^d_+} \tau_\alpha^x(f)(y) d\mu_\alpha(y) = \int_{\mathbb{R}^d_+} f(y) d\mu_\alpha(y).$$

(3) for  $f \in L^p_\alpha(\mathbb{R}^d_+)$  with  $p \in [1; +\infty]$   $\tau_\alpha^x(f) \in L^p_\alpha(\mathbb{R}^d_+)$  and we have

$$(2.13) \quad \|\tau_\alpha^x(f)\|_{p,\alpha} \leq \|f\|_{p,\alpha}$$

(4)

$$(2.14) \quad \mathcal{F}_\alpha(\tau_\alpha^x(f))(\lambda) = \psi_{\alpha,d}(\lambda x) \mathcal{F}_\alpha(f)(\lambda)$$

By using the generalized translation, we define the generalized convolution product of  $f, g$  by

$$(f *_{\alpha} g)(x) = \int_{\mathbb{R}_+^{d+1}} \tau_{\alpha}^x(f)(y)g(y)d\mu_{\alpha}(y).$$

This convolution is commutative, associative and it satisfies the following properties.

**Proposition 2.4.** For  $f, g \in L_{\alpha}^2(\mathbb{R}_+^d)$  the function  $f *_{\alpha} g$  belongs to  $L_{\alpha}^2(\mathbb{R}_+^d)$  if and only if the function  $\mathcal{F}_{\alpha}(f)\mathcal{F}_{\alpha}(g)$  belongs to  $L_{\alpha}^2(\mathbb{R}_+^d)$  and in this case we have

$$(2.15) \quad \mathcal{F}_{\alpha}(f *_{\alpha} g) = \mathcal{F}_{\alpha}(f)\mathcal{F}_{\alpha}(g)$$

and we have

$$(2.16) \quad \int_{\mathbb{R}_+^d} |f *_{\alpha} g(x)|^2 d\mu_{\alpha}(x) = \int_{\mathbb{R}_+^d} |\mathcal{F}_{\alpha}(f)(\lambda)|^2 |\mathcal{F}_{\alpha}(g)(\lambda)|^2 d\mu_{\alpha}(\lambda),$$

where both integrals are simultaneously finite or infinite.

### 3. GABOR TRANSFORM ASSOCIATED WITH THE MULTIDIMENSIONAL HANKEL TRANSFORM

The main purpose of this section is to introduce the multidimensional Hankel-Gabor transform and to give some new results related to this transform, for one dimensional case one can see [1, 10, 11].

Notation : we denote by

- $L_{\alpha}^p(\mathbb{R}_+^{2d})$ ,  $1 \leq p \leq +\infty$  the space of measurable functions on  $\mathbb{R}_+^d \times \mathbb{R}_+^d$  satisfying

$$\|f\|_{p, \mu_{\alpha} \otimes \mu_{\alpha}} := \begin{cases} \left( \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} |f(x, y)|^p d\mu_{\alpha}(x) \otimes d\mu_{\alpha}(y) \right)^{\frac{1}{p}}, & \text{if } p \in [1, +\infty[; \\ \text{ess sup}_{(x, y) \in \mathbb{R}_+^d \times \mathbb{R}_+^d} |f(x, y)| & \text{if } p = +\infty. \end{cases}$$

Let  $u$  in  $L_{\alpha}^2(\mathbb{R}_+^d)$  and  $y \in \mathbb{R}_+^d$ , we recall that the modulation operator of  $u$  is given by

$$\mathcal{M}^y(u) := u^y := \mathcal{F}_{\alpha} \left( \sqrt{\tau_{\alpha}^y} |\mathcal{F}_{\alpha}(u)|^2 \right).$$

By using Plancherel's formula (2.9) and the relation (2.12) we have  $u^y \in L_{\alpha}^2(\mathbb{R}_+^d)$  and

$$(3.1) \quad \|u^y\|_{2, \alpha} = \|u\|_{2, \alpha}.$$

Furthermore by using inversion formula (2.10) we find the following important result

$$(3.2) \quad \mathcal{F}_{\alpha}(u^y)(\lambda) = \sqrt{\tau_{\alpha}^y} \left( |\mathcal{F}_{\alpha}(u)|^2 \right)(\lambda).$$

Now, for every non-zero window function  $u$  in  $L_{\alpha}^2(\mathbb{R}_+^d)$ , we consider the family  $u^{x, y}$  defined by

$$(3.3) \quad u^{x, y}(z) = \tau_{\alpha}^x(u^y)(z), \quad \forall (x, y) \in \mathbb{R}_+^d \times \mathbb{R}_+^d.$$

**Definition 3.1.** For every  $f$  and  $u$  in  $L_{\alpha}^2(\mathbb{R}_+^d)$  we define the multidimensional Hankel-Gabor transform by

$$(3.4) \quad \mathcal{W}_u(f)(x, y) := \int_{\mathbb{R}_+^d} f(z) \overline{u^{x, y}(z)} d\mu_{\alpha}(z),$$

**Remark 3.1.** the multidimensional Hankel-Gabor transform can be also expressed by

$$(3.5) \quad \mathcal{W}_u(f)(x, y) = (\overline{u^y} *_{\alpha} f)(x).$$

By using Hölder's inequality and the relations (2.13), (3.1), (3.3), (3.4) we find that  $\mathcal{W}_u(f) \in L_{\alpha}^{\infty}(\mathbb{R}_+^{2d})$  and we have

$$(3.6) \quad \|\mathcal{W}_u(f)\|_{\infty, \mu_{\alpha} \otimes \mu_{\alpha}} \leq \|f\|_{2, \alpha} \|u\|_{2, \alpha}$$

**Definition 3.2.** Let  $u, v$  be non-zero functions in  $L^2_{\alpha}(\mathbb{R}^d_+)$ , we say that the pair  $(u, v)$  is admissible if for almost all  $\lambda \in \mathbb{R}^d_+$  we have

$$(3.7) \quad C_{u,v} = \int_{\mathbb{R}^d_+} \sqrt{\tau_{\alpha}^{\lambda}(|\mathcal{F}_{\alpha}(u)|^2)}(y) \tau_{\alpha}^{\lambda}(|\mathcal{F}_{\alpha}(v)|^2)(y) d\mu_{\alpha}(y) < \infty.$$

We have the following generalized Parseval's formula for the multidimensional Hankel-Gabor transform.

**Theorem 3.1.** Let  $(u, v)$  be an admissible pair then for all  $f, h \in L^2_{\alpha}(\mathbb{R}^d_+)$  we have

$$(3.8) \quad \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \mathcal{W}_u(f)(x, y) \overline{\mathcal{W}_v(h)(x, y)} d\mu_{\alpha}(x) \otimes d\mu_{\alpha}(y) = C_{u,v} \int_{\mathbb{R}^d_+} f(x) \overline{h(x)} d\mu_{\alpha}(x)$$

*Proof.* By using Fubini's theorem and the relations (2.7)(2.10)(2.14)(3.2) (3.5) we find that

$$\begin{aligned} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \mathcal{W}_u(f)(x, y) \overline{\mathcal{W}_v(h)(x, y)} d\mu_{\alpha}(x) d\mu_{\alpha}(y) &= \int_{\mathbb{R}^d_+} \left[ \int_{\mathbb{R}^d_+} (\overline{u^y} *_{\alpha} f)(x) \overline{(\overline{v^y} *_{\alpha} h)(x)} d\mu_{\alpha}(x) \right] d\mu_{\alpha}(y) \\ &= \int_{\mathbb{R}^d_+} \left[ \int_{\mathbb{R}^d_+} \mathcal{F}_{\alpha}(\overline{u^y})(\lambda) \mathcal{F}_{\alpha}(f)(\lambda) \overline{\mathcal{F}_{\alpha}(\overline{v^y})(\lambda)} \mathcal{F}_{\alpha}(h)(\lambda) d\mu_{\alpha}(\lambda) \right] d\mu_{\alpha}(y) \\ &= \int_{\mathbb{R}^d_+} \left[ \int_{\mathbb{R}^d_+} \sqrt{\tau_{\alpha}^y(|\mathcal{F}_{\alpha}(u)|^2)}(\lambda) \sqrt{\tau_{\alpha}^y(|\mathcal{F}_{\alpha}(v)|^2)}(\lambda) \mathcal{F}_{\alpha}(f)(\lambda) \overline{\mathcal{F}_{\alpha}(h)(\lambda)} d\mu_{\alpha}(\lambda) \right] d\mu_{\alpha}(y) \\ &= C_{u,v} \int_{\mathbb{R}^d_+} f(x) \overline{h(x)} d\mu_{\alpha}(x). \end{aligned}$$

and the proof is complete. □

**corollary 3.1.** (1) If  $C_{u,v} = 0$  then the spaces  $\mathcal{W}_u(L^2_{\alpha}(\mathbb{R}^d_+))$  and  $\mathcal{W}_v(L^2_{\alpha}(\mathbb{R}^d_+))$  are orthogonal.

(2) (Parseval's formula for  $\mathcal{W}_g$ )

If  $u = v$  then  $C_{u,v} = \|u\|_{2,\alpha}^2$ , in this case we have

$$\int_{\mathbb{R}^{d+1}_+} \int_{\mathbb{R}^d_+} \mathcal{W}_u(f)(x, y) \overline{\mathcal{W}_v(h)(x, y)} d\mu_{\alpha}(x) \otimes d\mu_{\alpha}(y) = \|u\|_{2,\alpha}^2 \int_{\mathbb{R}^d_+} f(x) \overline{h(x)} d\mu_{\alpha}(x).$$

(3) (Plancherel's formula for  $\mathcal{W}_u$ )

If  $u = v$  and  $f = h$  we find that

$$(3.9) \quad \|\mathcal{W}_u(f)\|_{2,\mu_{\alpha} \otimes \mu_{\alpha}} = \|f\|_{2,\alpha} \|u\|_{2,\alpha}.$$

**Proposition 3.1.**  $u$  be non-zero window function in  $L^2_{\alpha}(\mathbb{R}^{d+1}_+)$ , for all  $f \in L^2_{\alpha}(\mathbb{R}^d_+)$ , the function  $\mathcal{W}_u(f)$  belongs to  $L^p_{\alpha}(\mathbb{R}^{2d}_+)$  for all  $p \in [2; +\infty]$  and we have

$$\|\mathcal{W}_u(f)\|_{p,\mu_{\alpha} \otimes \mu_{\alpha}} \leq \|f\|_{2,\alpha} \|u\|_{2,\alpha}.$$

*Proof.* Is a consequence of the relations (3.6),(3.9) and the Riesz-Thorin's interpolation theorem [20]. □

In the following, we establish an inversion formula for the multidimensional Hankel-Gabor transform.

**Theorem 3.2.** Let  $(u, v)$  be an admissible pair in  $L^2_{\alpha}(\mathbb{R}^d_+)$ , then for all

$f \in L^2_{\alpha}(\mathbb{R}^d_+)$  we have

$$f(\cdot) = \frac{1}{C_{u,v}} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \mathcal{W}_u(f)(x, y) v^{x,y}(\cdot) d\mu_{\alpha}(x) \otimes d\mu_{\alpha}(y),$$

weakly in  $L^2_{\alpha}(\mathbb{R}^d_+)$ .

*Proof.* By using the relations (3.4),(3.8) and Fubini's theorem we find that

$$\begin{aligned} \int_{\mathbb{R}_+^d} f(z) \overline{h(z)} d\mu_\alpha(z) &= \frac{1}{C_{u,v}} \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} \mathcal{W}_u(f)(x, y) \overline{\mathcal{W}_v(h)(x, y)} d\mu_\alpha(x) \otimes d\mu_\alpha(y) \\ &= \frac{1}{C_{u,v}} \int_{\mathbb{R}_+^{d+1}} \left( \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} \mathcal{W}_u(f)(x, y) v^{x,y}(z) d\mu_\alpha(x) \otimes d\mu_\alpha(y) \right) \overline{h(z)} d\mu_\alpha(z). \end{aligned}$$

which gives the result.  $\square$

The reproducing kernels for Hilbert space play an important role in harmonic analysis [19]. In this context, we have the following result.

**Theorem 3.3.** The space  $\mathcal{W}_u(L_\alpha^2(\mathbb{R}_+^d))$  is a reproducing kernel Hilbert space in  $L_\alpha^2(\mathbb{R}_+^{2d})$  with kernel function  $\mathcal{K}_u$  defined by

$$\mathcal{K}_u((x', y'); (x, y)) = \frac{1}{\|u\|_{2,\alpha}^2} (u^{x,y} *_\alpha \overline{u^{y'}})(x').$$

Furthermore, the kernel is pointwise bounded

$$|\mathcal{K}_u((x', y'); (x, y))| \leq 1, \quad \forall (x, y); (x', y') \in \mathbb{R}_+^{2d}.$$

*Proof.* From the relations (3.5) and (3.8) we find that

$$\begin{aligned} \mathcal{W}_u(f)(x, y) &= \frac{1}{\|u\|_{2,\alpha}^2} \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} \mathcal{W}_u(f)(x', y') \overline{\mathcal{W}_u(u^{x,y})(x', y')} d\mu_\alpha(x') d\mu_\alpha(y') \\ &= \langle \mathcal{W}_u(f) | \mathcal{K}_u((\cdot, \cdot); (x, y)) \rangle_{\mu_\alpha \otimes \mu_\alpha}, \end{aligned}$$

where

$$\mathcal{K}_u((x', y'); (x, y)) = \frac{1}{\|u\|_{2,\alpha}^2} (u^{x,y} *_\alpha \overline{u^{y'}})(x').$$

On the other hand, for every  $(x, y); (x', y') \in \mathbb{R}_+^{2d}$  and by a direct computation, we obtain

$$\|\mathcal{K}_u((\cdot, \cdot); (x, y))\|_{2,\mu_\alpha \otimes \mu_\alpha} \leq 1.$$

Finally by the Cauchy-Schwarz inequality, we get

$$|\mathcal{K}_u((x', y'); (x, y))| \leq \frac{1}{\|u\|_{2,\alpha}^2} \int_{\mathbb{R}_+^d} |u^{x,y}(z)| |u^{x',y'}(z)| d\mu_\alpha(z) \leq 1.$$

This shows that the kernel  $\mathcal{K}_u$  belongs to  $L_\alpha^2(\mathbb{R}_+^{2d})$  and is bounded.  $\square$

The rest of this section is devoted to give Calderón's type reproducing formula for the multidimensional Hankel-Gabor transform, to do this we need the help of the following result.

**Proposition 3.2.** Let  $0 < \gamma < \delta < +\infty$  and  $(u, v)$  be an admissible pair such that  $\mathcal{F}_\alpha(u)$  and  $\mathcal{F}_\alpha(v)$  belongs to  $L_\alpha^\infty(\mathbb{R}_+^d)$ . We put

$$(3.10) \quad G_{\gamma,\delta}(x) := \frac{1}{C_{u,v}} \int_{D(\gamma,\delta)} (\overline{u^y} *_\alpha v^y)(x) d\mu_\alpha(y)$$

and

$$(3.11) \quad K_{\gamma,\delta}(\lambda) := \frac{1}{C_{u,v}} \int_{D(\gamma,\delta)} \sqrt{\tau_\alpha^\lambda(|\mathcal{F}_\alpha(u)|^2)(y) \tau_\alpha^\lambda(|\mathcal{F}_\alpha(v)|^2)(y)} d\mu_\alpha(y)$$

where

$$D(\gamma, \delta) = \{x \in \mathbb{R}_+^d : \gamma \leq x_k \leq \delta, 1 \leq k \leq d\}.$$

Then we have  $G_{\gamma,\delta}$  belongs to  $L^2_\alpha(\mathbb{R}^d_+)$  and

$$(3.12) \quad \mathcal{F}_\alpha(G_{\gamma,\delta})(\lambda) = K_{\gamma,\delta}(\lambda)$$

*Proof.* By using Hölder's inequality and the relations (2.8),(2.16),(3.1) we find that

$$|G_{\gamma,\delta}(x)|^2 \leq \frac{\mu_\alpha(D(\gamma,\delta))}{C_{u,v}^2} \int_{D(\gamma,\delta)} |(\overline{u^y} *_\alpha v^y)(x)|^2 d\mu_\alpha(y)$$

So

$$\begin{aligned} \|G_{\gamma,\delta}\|_{2,\alpha}^2 &\leq \frac{\mu_\alpha(D(\gamma,\delta))}{C_{u,v}^2} \int_{D(\gamma,\delta)} \left( \int_{\mathbb{R}^d_+} |\mathcal{F}_\alpha(\overline{u^y})(\lambda)|^2 |\mathcal{F}_\alpha(v^y)(\lambda)|^2 d\mu_\alpha(\lambda) \right) d\mu_\alpha(y) \\ &\leq \left( \frac{\mu_\alpha(D(\gamma,\delta))}{C_{u,v}} \right)^2 \|\mathcal{F}_\alpha(f)\|_{\infty,\alpha}^2 \|v\|_{2,\alpha}^2 < \infty. \end{aligned}$$

Which proves that  $G_{\gamma,\delta}$  belongs to  $L^2_\alpha(\mathbb{R}^d_+)$ , furthermore by using Parseval's relation (2.8), (2.14) we find that

$$\begin{aligned} (\overline{u^y} *_\alpha v^y)(x) &= \int_{\mathbb{R}^d_+} \tau_\alpha^x(\overline{u^y})(z) v^y(z) d\mu_\alpha(z) \\ &= \int_{\mathbb{R}^d_+} \mathcal{F}_\alpha(\overline{u^y})(\lambda) \psi_{\alpha,d}(\lambda x) \overline{\mathcal{F}_\alpha(v^y)(\lambda)} d\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}^d_+} \psi_{\alpha,d}(\lambda x) \sqrt{\tau_\alpha^y(|\mathcal{F}_\alpha(u)|^2)(\lambda) \tau_\alpha^y(|\mathcal{F}_\alpha(v)|^2)(\lambda)} d\mu_\alpha(\lambda) \end{aligned}$$

now, by using Fubini's theorem and the relation (2.11) we find that

$$\begin{aligned} G_{\gamma,\delta}(x) &= \frac{1}{C_{u,v}} \int_{\mathbb{R}^d_+} \psi_{\alpha,d}(\lambda x) \left( \int_{D(\gamma,\delta)} \sqrt{\tau_\alpha^\lambda(|\mathcal{F}_\alpha(u)|^2)(y) \tau_\alpha^\lambda(|\mathcal{F}_\alpha(v)|^2)(y)} d\mu_\alpha(y) \right) d\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}^d_+} \psi_{\alpha,d}(\lambda x) K_{\gamma,\delta}(\lambda) d\mu_\alpha(\lambda) \end{aligned}$$

inversion formula (2.10) gives the relation (3.11).  $\square$

In the following we establish reproducing inversion formula of Calderón's type for the multidimensional Hankel-Gabor transform  $\mathcal{W}_u$ .

**Theorem 3.4.** Let  $0 < \gamma < \delta < +\infty$  and  $(u, v)$  be an admissible pair such that  $\mathcal{F}_\alpha(u)$  and  $\mathcal{F}_\alpha(v)$  belongs to  $L^\infty_\alpha(\mathbb{R}^d_+)$ . For all  $f \in L^2_\alpha(\mathbb{R}^d_+)$ , the function  $f_{\gamma,\delta}$  defined for all  $z \in \mathbb{R}^d_+$  by:

$$(3.13) \quad f_{\gamma,\delta}(z) = \frac{1}{C_{u,v}} \int_{D(\gamma,\delta)} \int_{\mathbb{R}^d_+} \mathcal{W}_u(f)(x, y) \overline{\tau_\alpha^x(v^y)}(z) d\mu_\alpha(x) \otimes d\mu_\alpha(y),$$

belongs to  $L^2_\alpha(\mathbb{R}^d_+)$  and satisfies

$$(3.14) \quad \lim_{(\gamma,\delta) \rightarrow (0,+\infty)} \|f_{\gamma,\delta} - f\|_{2,\alpha} = 0.$$

*Proof.* It is easy to see that for all  $f \in L^2_\alpha(\mathbb{R}^d_+)$  we have  $f_{\gamma,\delta} = f *_\alpha G_{\gamma,\delta}$ , where  $G_{\gamma,\delta}$  is the function given by the relation (3.10), by using the relations (3.7),(3.11) we find that

$$\|f_{\gamma,\delta} - f\|_{2,\alpha}^2 = \int_{\mathbb{R}^d_+} |\mathcal{F}_\alpha(f)(\lambda)|^2 (1 - K_{\gamma,\delta}(\lambda))^2 d\mu_\alpha(\lambda)$$

by using the relations (3.7),(3.11), the relation (3.14) follows from the dominated convergence theorem.  $\square$



#### 4. UNCERTAINTY PRINCIPLES ASSOCIATED WITH THE MULTIDIMENSIONAL HANKEL-GABOR TRANSFORM

**4.1. Uncertainty principle for orthonormal sequences.** In this subsection, we estimate the concentration of  $\mathcal{W}_u$  on subset of  $\mathbb{R}_+^d \times \mathbb{R}_+^d$  of finite measure, similar results have been checked in [11, 16, 23] and we establish the uncertainty principle for orthonormal sequences associated with the multidimensional Hankel-Gabor transform, first we consider the following orthogonal projections

- (1) Let  $P_u$  be the orthogonal projection from  $L_\alpha^2(\mathbb{R}_+^{2d})$  onto  $\mathcal{W}_u(L_\alpha^2(\mathbb{R}_+^d))$  and  $\text{Im } P_u$  denotes the range of  $P_u$ .  
 (2) Let  $P_E$  be the orthogonal projection on  $L_\alpha^2(\mathbb{R}_+^{2d})$  defined by

$$(4.1) \quad P_E F = \chi_E F, \quad F \in L_\alpha^2(\mathbb{R}_+^{2d}),$$

where  $E \subset \mathbb{R}_+^d \times \mathbb{R}_+^d$  and  $\text{Im } P_E$  is the range of  $P_E$ . Also, we define

$$\|P_E P_u\| = \sup \left\{ \|P_E P_u(F)\|_{2, \mu_\alpha \otimes \mu_\alpha} : F \in L_\alpha^2(\mathbb{R}_+^{2d}), \|F\|_{2, \mu_\alpha \otimes \mu_\alpha} = 1 \right\}.$$

We first need the following result.

**Theorem 4.1.** Let  $u \in L_\alpha^2(\mathbb{R}_+^d)$  be a non-zero window function. Then for any  $E \subset \mathbb{R}_+^d \times \mathbb{R}_+^d$  of finite measure  $\mu_\alpha \otimes \mu_\alpha(E) < \infty$ , the operator  $P_E P_u$  is a Hilbert-Schmidt operator. Moreover, we have the following estimation

$$(4.2) \quad \|P_E P_u\|^2 \leq \mu_\alpha \otimes \mu_\alpha(E).$$

*Proof.* since  $P_u$  is a projection onto a reproducing kernel Hilbert space, for any function  $F \in L_\alpha^2(\mathbb{R}_+^{2d})$ , the orthogonal projection  $P_u$  can be expressed as

$$P_u(F)(x, \xi) = \iint_{\mathbb{R}_+^{2d}} F(x', \xi') K_u((x', \xi'); (x, \xi)) d\mu_\alpha(x') \otimes d\mu_\alpha(\xi'),$$

where  $K_u((x', \xi'); (x, \xi))$  is given in theorem 3.3, using the relation (4.1), we find that

$$P_E P_u(F)(x, \xi) = \iint_{\mathbb{R}_+^{2d}} \chi_E(x, \xi) F(x', \xi') K_u((x', \xi'); (x, \xi)) d\mu_\alpha(x') \otimes d\mu_\alpha(\xi').$$

This shows that the operator  $P_E P_u$  is an integral operator with kernel  $K((x', \xi'); (x, \xi)) = \chi_E(x, \xi) K_u((x', \xi'); (x, \xi))$ . Using the relation (3.6) and Fubini's theorem, we find that

$$\begin{aligned} \|P_E P_u\|_{HS}^2 &= \iint_{\mathbb{R}_+^{2d}} \iint_{\mathbb{R}_+^{2d}} |K((x', \xi'); (x, \xi))|^2 d\mu_\alpha(x') \otimes d\mu_\alpha(\xi') d\mu_\alpha(x) \otimes d\mu_\alpha(\xi) \\ &= \iint_{\mathbb{R}_+^{2d}} \iint_{\mathbb{R}_+^{2d}} |\chi_\Sigma(x, \xi)|^2 |K_u((x', \xi'); (x, \xi))|^2 d\mu_\alpha(x') \otimes d\mu_\alpha(\xi') d\mu_\alpha(x) \otimes d\mu_\alpha(\xi) \\ (4.3) \quad &\leq \frac{\|u\|_{2, \mu_\alpha}^4}{\|u\|_{2, \mu_\alpha}^4} \iint_E d\mu_\alpha(x) \otimes d\mu_\alpha(\xi) = \mu_\alpha \otimes \mu_\alpha(E) < \infty. \end{aligned}$$

Thus, the operator  $P_E P_u$  is a Hilbert-Schmidt operator. Now, the proof follows from the fact that  $\|P_E P_u\| \leq \|P_E P_u\|_{HS}$ .  $\square$

In the following, we obtain the uncertainty principle for orthonormal sequences associated with the multidimensional Hankel-Gabor transform.

**Theorem 4.2.** Let  $u \in L_\alpha^2(\mathbb{R}_+^d)$  be a non-zero window function and  $\{\phi_n\}_{n \in \mathbb{N}}$  be an orthonormal sequence in  $L_\alpha^2(\mathbb{R}_+^d)$ . Then for any subset  $E \subset \mathbb{R}_+^d \times \mathbb{R}_+^d$  of finite measure  $\mu_\alpha \otimes \mu_\alpha(E) < \infty$ , we have

$$\sum_{n=1}^N \left( 1 - \|\chi_E \mathcal{W}_u(\phi_n)\|_{2, \mu_\alpha \otimes \mu_\alpha} \right) \leq \mu_\alpha \otimes \mu_\alpha(E),$$

for every  $N \in \mathbb{N}$ .

*Proof.* Proof. Let  $\{e_n\}_{n \in \mathbb{N}}$  be an orthonormal basis for  $L^2_\alpha(\mathbb{R}^{2d}_+)$ . Since  $P_E P_u$  is a Hilbert-Schmidt operator, and satisfied the relation (4.3) and we have

$$\sum_{n \in \mathbb{N}} \langle P_u P_E P_u e_n, e_n \rangle_{\mu_\alpha \otimes \mu_\alpha} = \|P_E P_u\|_{HS}^2 \leq \mu_\alpha \otimes \mu_\alpha(E) < \infty.$$

According to the paper [12], the positive operator  $P_u P_E P_u$  is a trace class operator and we have

$$\text{tr}(P_u P_E P_u) = \|P_E P_u\|_{HS}^2 \leq \mu_\alpha \otimes \mu_\alpha(E) < \infty$$

where  $\text{tr}(P_u P_E P_u)$  denotes the trace of the operator  $P_u P_E P_u$ . Since  $\{\phi_n\}_{n \in \mathbb{N}}$  be an orthonormal sequence in  $L^2_\alpha(\mathbb{R}^d_+)$ , from the orthogonality relation (3.9), we obtain that  $\{\mathcal{W}_u(\phi_n)\}_{n \in \mathbb{N}}$  is also an orthonormal sequence in  $L^2_\alpha(\mathbb{R}^{2d}_+)$  thus

$$\sum_{n=1}^N \langle P_E \mathcal{W}_u(\phi_n), \mathcal{W}_u(\phi_n) \rangle_{\mu_\alpha \otimes \mu_\alpha} = \sum_{n=1}^N \left\langle P_g P_\Sigma P_g \mathcal{W}_g^{(\alpha, \beta)}(\phi_n), \mathcal{W}_g^{(\alpha, \beta)}(\phi_n) \right\rangle_{\mu_\alpha \otimes \mu_\alpha} \leq \text{tr}(P_u P_E P_u)$$

Hence, we find that

$$\sum_{n=1}^N \langle P_E \mathcal{W}_u(\phi_n), \mathcal{W}_u(\phi_n) \rangle_{\mu_\alpha \otimes \mu_\alpha} \leq \mu_\alpha \otimes \mu_\alpha(E) < \infty$$

Moreover, for any  $n$  with  $1 \leq n \leq N$ , using the Cauchy-Schwarz inequality, we get

$$\langle P_E \mathcal{W}_u(\phi_n), \mathcal{W}_u(\phi_n) \rangle_{\mu_\alpha \otimes \mu_\alpha} = 1 - \langle P_{E^c} \mathcal{W}_u(\phi_n), \mathcal{W}_u(\phi_n) \rangle_{\mu_\alpha \otimes \mu_\alpha} \geq 1 - \|\chi_{E^c} \mathcal{W}_u(\phi_n)\|_{2, \mu_\alpha \otimes \mu_\alpha}.$$

Thus, we obtain

$$\sum_{n=1}^N \left(1 - \|\chi_{E^c} \mathcal{W}_u(\phi_n)\|_{2, \mu_\alpha \otimes \mu_\alpha}\right) \leq \sum_{n=1}^N \langle P_E \mathcal{W}_u(\phi_n), \mathcal{W}_u(\phi_n) \rangle_{\mu_\alpha \otimes \mu_\alpha} \leq \mu_\alpha \otimes \mu_\alpha(E).$$

This completes the proof of the theorem.  $\square$

**4.2. Donoho-Stark's Uncertainty Principle for the multidimensional Hankel-Gabor transform.** *The main purpose of this subsection is to give an analogue of Donoho-Stark's uncertainty principle [8] for the multidimensional windowed Hankel transform. In particular, we investigate the case where  $f$  and  $\mathcal{W}_u(f)$  are close to zero outside measurable sets.*

*We start with the following result*

**Theorem 4.3.** *Let  $u \in L^2_\alpha(\mathbb{R}^d_+)$  be a non-zero window function and  $f \in L^2_\alpha(\mathbb{R}^d_+)$  such that  $f \neq 0$ . Then for any subset  $E \subset \mathbb{R}^d_+ \times \mathbb{R}^d_+$  of finite measure  $\mu_\alpha \otimes \mu_\alpha(E) < \infty$ , and  $\varepsilon \geq 0$  such that*

$$\iint_E |\mathcal{W}_u(f)(x, \xi)|^2 d\mu_\alpha(x) \otimes d\mu_\alpha(\xi) \geq (1 - \varepsilon) \|f\|_{2, \alpha}^2 \|u\|_{2, \alpha}^2$$

*then we have*

$$\mu_\alpha \otimes \mu_\alpha(E) \geq (1 - \varepsilon)$$

*Proof.* by using the relation (3.6) we find that

$$(1 - \varepsilon) \|f\|_{2, \mu_\alpha}^2 \|u\|_{2, \mu_\alpha}^2 \leq \iint_E |\mathcal{W}_u(f)(x, \xi)|^2 d\mu_\alpha(x) \otimes d\mu_\alpha(\xi) \leq \|f\|_{2, \alpha}^2 \|u\|_{2, \alpha}^2 \mu_\alpha \otimes \mu_\alpha(E).$$

Therefore we find that

$$\mu_\alpha \otimes \mu_\alpha(E) \geq (1 - \varepsilon).$$

$\square$

*The following proposition shows that the multidimensional Hankel-Gabor transform cannot be concentrated in any small set*

**Proposition 4.1.** Let  $u \in L^2_\alpha(\mathbb{R}^d_+)$  be a non-zero window function. Then for any function  $f \in L^2_\alpha(\mathbb{R}^{d+1}_+)$  and for any subset  $E \subset \mathbb{R}^d_+ \times \mathbb{R}^d_+$  such that  $\mu_\alpha \otimes \mu_\alpha(E) < 1$ , we have

$$\|\chi_{E^c} \mathcal{W}_u(f)\|_{2, \mu_\alpha \otimes \mu_\alpha} \geq \sqrt{1 - \mu_\alpha \otimes \mu_\alpha(E)} \|f\|_{2, \mu_\alpha} \|g\|_{2, \mu_\alpha}.$$

*Proof.* For any function  $f \in L^2_\alpha(\mathbb{R}^d_+)$ , by using the relation (3.6), we find that

$$\begin{aligned} \|\mathcal{W}_u(f)\|_{2, \mu_\alpha \otimes \mu_\alpha}^2 &= \|\chi_E \mathcal{W}_u(f) + \chi_{E^c} \mathcal{W}_u(f)\|_{2, \mu_\alpha \otimes \mu_\alpha}^2 \\ &= \|\chi_E \mathcal{W}_u(f)\|_{2, \mu_\alpha \otimes \mu_\alpha}^2 + \|\chi_{E^c} \mathcal{W}_u(f)\|_{2, \mu_\alpha \otimes \mu_\alpha}^2 \\ &\leq \mu_\alpha \otimes \mu_\alpha(E) \|\mathcal{W}_u(f)\|_{\infty, \mu_\alpha \otimes \mu_\alpha}^2 + \left\| \chi_{E^c} \mathcal{W}_g^{(\alpha, \beta)}(f) \right\|_{2, \mu_\alpha \otimes \mu_\alpha}^2 \\ &\leq \mu_\alpha \otimes \mu_\alpha(E) \|f\|_{L^2(\mathbb{R}, A_{\alpha, \beta})}^2 \|g\|_{2, \mu_\alpha \otimes \mu_\alpha}^2 + \left\| \chi_{E^c} \mathcal{W}_g^{(\alpha, \beta)}(f) \right\|_{2, \mu_\alpha \otimes \mu_\alpha}^2. \end{aligned}$$

Thus, using Plancherel's formula (3.9), we obtain

$$\|\chi_{E^c} \mathcal{W}_u(f)\|_{2, \mu_\alpha \otimes \mu_\alpha} \geq \sqrt{1 - \mu_\alpha \otimes \mu_\alpha(E)} \|f\|_{2, \mu_\alpha} \|g\|_{2, \mu_\alpha}.$$

□

**Definition 4.1.** Let  $S$  be a measurable subset of  $\mathbb{R}^d_+$  and  $0 \leq \varepsilon_S < 1$ . Then we say that a function  $f \in L^p_\alpha(\mathbb{R}^{d+1}_+)$ ,  $1 \leq p \leq 2$ , is  $\varepsilon_S$ -concentrated on  $S$  in  $L^p_\alpha(\mathbb{R}^d_+)$ -norm, if

$$\|\chi_{S^c} f\|_{p, \alpha} \leq \varepsilon_S \|f\|_{p, \alpha}.$$

If  $\varepsilon_S = 0$ , then  $S$  contains the support of  $f$ .

**Definition 4.2.** Let  $E$  be a measurable subset of  $\mathbb{R}^d_+ \times \mathbb{R}^d_+$  and  $0 \leq \varepsilon_E < 1$ . Let  $f, u \in L^2_\alpha(\mathbb{R}^d_+)$  be two non-zero functions. We say that  $\mathcal{W}_u(f)$  is  $\varepsilon_E$ -time-frequency concentrated on  $E$ , if

$$\|\chi_{E^c} \mathcal{W}_u(f)\|_{2, \mu_\alpha \otimes \mu_\alpha} \leq \varepsilon_E \|\mathcal{W}_u(f)\|_{2, \mu_\alpha \otimes \mu_\alpha}.$$

If  $\mathcal{W}_u(f)$  is  $\varepsilon_E$ -time-frequency concentrated on  $E$ , then in the following, we obtain an estimate for the size of the essential support of the multidimensional Hankel-Gabor transform.

**Theorem 4.4.** Let  $u \in L^2_\alpha(\mathbb{R}^d_+)$  be a non-zero window function and  $f \in L^2_\alpha(\mathbb{R}^d_+)$  such that  $f \neq 0$ .  $E \subset \mathbb{R}^d_+ \times \mathbb{R}^d_+$  such that  $\mu_\alpha \otimes \mu_\alpha(E) < \infty$  and  $\varepsilon_E \geq 0$ . If  $\mathcal{W}_u(f)$  is  $\varepsilon_E$ -timefrequency concentrated on  $E$ , then

$$\mu_\alpha \otimes \mu_\alpha(E) \geq (1 - \varepsilon_E^2).$$

*Proof.* Since  $\mathcal{W}_u(f)$  is  $\varepsilon_E$ -time-frequency concentrated on  $E$ , using Plancherel's formula (3.9), we deduce that

$$\begin{aligned} \|f\|_{2, \mu_\alpha}^2 \|u\|_{2, \mu_\alpha}^2 &= \|\mathcal{W}_u(f)\|_{2, \mu_\alpha \otimes \mu_\alpha}^2 \\ &= \|\chi_{E^c} \mathcal{W}_u(f)\|_{2, \mu_\alpha \otimes \mu_\alpha}^2 + \|\chi_E \mathcal{W}_u(f)\|_{2, \mu_\alpha \otimes \mu_\alpha}^2 \\ &\leq \varepsilon_E^2 \|\mathcal{W}_u(f)\|_{2, \mu_\alpha \otimes \mu_\alpha}^2 + \|\chi_E \mathcal{W}_u(f)\|_{2, \mu_\alpha \otimes \mu_\alpha}^2. \end{aligned}$$

by using the relation (3.5) we find that

$$\begin{aligned} (4.4) \quad (1 - \varepsilon_E^2) \|f\|_{2, \mu_\alpha}^2 \|u\|_{2, \mu_\alpha}^2 &\leq \|\chi_E \mathcal{W}_u(f)\|_{2, \mu_\alpha \otimes \mu_\alpha}^2 \\ &\leq \mu_\alpha \otimes \mu_\alpha(E) \|\mathcal{W}_u(f)\|_{\infty, \mu_\alpha \otimes \mu_\alpha}^2 \leq \|f\|_{2, \mu_\alpha}^2 \|u\|_{2, \mu_\alpha}^2 \mu_\alpha \otimes \mu_\alpha(E) \end{aligned}$$

which completes the proof. □

**corollary 4.1.** Let  $E \subset \mathbb{R}_+^d \times \mathbb{R}_+^d$  such that  $\mu_\alpha \otimes \mu_\alpha(E) < \infty$ ,  $\varepsilon_E \geq 0$ ,  $u \in L_\alpha^2(\mathbb{R}_+^d)$  be a nonzero window function, and  $f \in L_\alpha^2(\mathbb{R}_+^d)$  such that  $f \neq 0$ . If  $\mathcal{W}_u(f)$  is  $\varepsilon_E$ -time-frequency concentrated on  $E$ , then for every  $p > 2$ , we have

$$\mu_\alpha \otimes \mu_\alpha(E) \geq (1 - \varepsilon_E^2)^{\frac{p}{p-2}}.$$

*Proof.* Is a consequence of (4.4) and Hölder's inequality for the conjugate exponent  $\frac{p}{2}$  and  $\frac{p}{p-2}$   $\square$

**Theorem 4.5.** Let  $E \subset \mathbb{R}_+^d \times \mathbb{R}_+^d$  such that  $\mu_\alpha \otimes \mu_\alpha(E) < \infty$ ,  $u \in L_\alpha^2(\mathbb{R}_+^d)$  and  $f \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^2(\mathbb{R}_+^d)$  such that  $\|\mathcal{W}_u(f)\|_{2,\mu_\alpha \otimes \mu_\alpha} = 1$  Let  $S$  be a measurable subset of  $\mathbb{R}_+^d$  such that  $\mu_\alpha(S) < \infty$ , if  $f$  is  $\varepsilon_S$ -concentrated on  $S$  in  $L_\alpha^1(\mathbb{R}_+^d)$ -norm and  $\mathcal{W}_u(f)$  is  $\varepsilon_E$ -time-frequency concentrated on  $E$  then we have

$$\mu_\alpha(S) \geq (1 - \varepsilon_S)^2 \|f\|_{1,\mu_\alpha}^2 \|u\|_{2,\mu_\alpha}^2,$$

and

$$\|f\|_{2,\mu_\alpha}^2 \|u\|_{2,\mu_\alpha}^2 \mu_\alpha \otimes \mu_\alpha(E) \geq (1 - \varepsilon_E^2)$$

in partucular we have

$$\mu_\alpha(S) \|f\|_{2,\mu_\alpha}^2 \mu_\alpha \otimes \mu_\alpha(E) \geq (1 - \varepsilon_E^2) (1 - \varepsilon_S)^2 \|f\|_{1,\alpha}^2$$

*Proof.* Since  $\mathcal{W}_u(f)$  is  $\varepsilon_E$ -time-frequency concentrated on  $E$  by using the relation (4.4), we find that

$$(1 - \varepsilon_E^2) \|f\|_{2,\alpha}^2 \|u\|_{2,\alpha}^2 \leq \|\chi_E \mathcal{W}_u(f)\|_{2,\mu_\alpha \otimes \mu_\alpha}^2$$

since  $\|\mathcal{W}_u(f)\|_{2,\mu_\alpha \otimes \mu_\alpha} = 1$ , using the relations (3.5),(3.8) we find that

$$(4.5) \quad (1 - \varepsilon_E^2) \leq \|\chi_E \mathcal{W}_u(f)\|_{2,\mu_\alpha \otimes \mu_\alpha}^2 \leq \mu_\alpha \otimes \mu_\alpha(E) \|\mathcal{W}_u(f)\|_{\infty,\mu_\alpha \otimes \mu_\alpha}^2 \leq \|f\|_{2,\mu_\alpha}^2 \|u\|_{2,\mu_\alpha}^2 \mu_\alpha \otimes \mu_\alpha(E)$$

similarly, since  $f$  is  $\varepsilon_S$ -concentrated on  $S$  in  $L_\alpha^1(\mathbb{R}_+^d)$ -norm, using the Cauchy-Schwartz inequality and the fact that  $\|f\|_{2,\mu_\alpha} \|u\|_{2,\mu_\alpha} = 1$  we get

$$(4.6) \quad (1 - \varepsilon_S) \|f\|_{1,\mu_\alpha} \leq \|\chi_S f\|_{1,\mu_\alpha} \leq \|f\|_{2,\mu_\alpha} (\mu_\alpha(S))^{\frac{1}{2}} = \frac{(\mu_\alpha(S))^{\frac{1}{2}}}{\|u\|_{2,\mu_\alpha}}$$

by using the relations (4.5),(4.6) we find that

$$\mu_\alpha(S) \|f\|_{2,\mu_\alpha}^2 \mu_\alpha \otimes \mu_\alpha(E) \geq (1 - \varepsilon_E^2) (1 - \varepsilon_S)^2 \|f\|_{2,\alpha}^2$$

this completes the proof of the theorem.  $\square$

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