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STABILITY OF LEFT AND RIGHT WEYL ESSENTIAL SPECTRA

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ABSTRACT. In this work, we use the concept of demicompact linear operators to provide a characterization of the left and right Weyl essential spectra. Additionally, we study the stability of these spectra in Banach space. We illustrate our theoretical results by investigating this type of essential spectra in transport equations.

INTRODUCTION

Many papers, in the last years, were devoted to the study of demicompact operators. The Fredholm theory of this operators attracted many authors. This concept was first introduced by Petryshyn in [15]. Initially, demicompact operators were defined in the context of fixed point theory of the operators on Banach spaces. Following this, Akashi in [2] and Jeribi A. in [8], [9] used demicompact operators to establish a stability results of Fredholm operators. Building upon this research, our focus is on studying the right and left essential spectra in this article.

Before proceeding, we introduve some definitions, theorems, and properties Fredholm operators. $\mathcal{C}(X,Y)$ denotes the collection of densely defined closed linear operators, $\mathcal{L}(X,Y)$ represents the bounded linear operators, and $\mathcal{K}(X,Y)$ corresponds to the set of compact operators.
If X and Y are the same space, the pair (X,Y) is replaced by X. For instance, $\mathcal{C}(X,Y)$ becomes $\mathcal{C}(X)$. For $T \in \mathcal{C}(X)$, we defined the resolvent and the spectrum set as follows:

$$\begin{aligned} \rho(T) &:= \{\mu \in \mathbb{C},\ \mu - T \text{ is invertible and } (\mu - T)^{-1} \text{is bounded}\}, \\ \sigma(T) &:= \mathbb{C} \setminus \rho(T). \end{aligned}$$

Additionally, we denote $\mathcal{R}(T) \subseteq Y$ as the range of T, $\mathcal{N}(T)$ as the kernel of T, $\alpha(T)$ as the dimension of $\mathcal{N}(T)$, and $\beta(T)$ as the codimension of $\mathcal{R}(T)$. The index of T, denoted as $i(T) = \alpha(T) - \beta(T)$. The set of left invertible and right invertible operators is denoted as $\mathcal{GL}(Y,X)$ and $\mathcal{GR}(Y,X)$, respectively. We noted that T is invertible if and only if T is both left and right invertible, meaning there exist operators T_r and T_l such that $TT_r = I$ and $T_lT = I$.
Let M be a subspace of X. We say that M to be complemented, if there exists $N \subset X$, a closed subspace, such that $X = M \oplus N$. We will define the sets of left and right Fredholm operators, respectively:

$$\Phi_l(X,Y) = \{T \in \mathcal{C}(X,Y),\ \alpha(T) \text{is finite}, \mathcal{R}(T) \text{ is closed and complemented}\},$$

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and

$$\Phi_r(X,Y) = \{T \in \mathcal{C}(X,Y),\ \beta(T) \text{ is finite and } \mathcal{N}(T) \text{ is complemented}\},$$

The sets Fredholm operators are defined by:

$$\Phi(X,Y) = \Phi_+(X,Y)\ \cap\ \Phi_-(X,Y),$$

where, the upper semi-Fredholm and lower semi-Fredholm operators are defined respectively:

$$\Phi_+(X,Y) = \{T \in \mathcal{C}(X,Y),\ \alpha(T) \text{is finite and } \mathcal{R}(T) \text{ is closed in } Y\},$$

and

$$\Phi_-(X,Y) = \{T \in \mathcal{C}(X,Y),\ \beta(T) \text{ is finite and } \mathcal{R}(T) \text{ is closed in } Y\},$$

Hence, we have the following relations:

$$\Phi(X,Y) \subseteq \Phi_l(X,Y) \subseteq \Phi_+(X,Y),$$

and

$$\Phi(X,Y) \subseteq \Phi_r(X,Y) \subseteq \Phi_-(X,Y).$$

The left Weyl operators, denoted as $\mathcal{WL}(X)$, is defined:

$$\mathcal{WL}(X) = \{T \in \Phi_l(X),\ i(T) \text{ is negative}\}.$$

Similarly, the right Weyl operators, $\mathcal{WR}(X)$, is defined as:

$$\mathcal{WR}(X) = T \in \Phi_r(X) \ \text{ such that } \ i(T) \geq 0.$$

We say that $\mu \in \mathbb{C}$ is in Φ_{lT} (Φ_{rT}) if $\mu - T \in \Phi_l(X)$ ($\Phi_r(X)$).

An operator F is called a Fredholm perturbation if $F \in \mathcal{L}(X,Y)$, and $T + F \in \Phi(X,Y)$, for every $T \in \Phi(X,Y)$. The set of perturbations is denoted as $\mathcal{F}(X,Y)$.

Now, consider $T \in \mathcal{C}(X)$. The left and right Weyl essential spectra, denoted, respectively, as $\sigma_{ewl}(T)$ and $\sigma_{ewr}(T)$ and defined as follows:

$$\begin{aligned}\sigma_{ewl}(T) &:= \bigcap_{K \in \mathcal{K}(X)} \sigma_l(T+K),\\ \sigma_{ewr}(T) &:= \bigcap_{K \in \mathcal{K}(X)} \sigma_r(T+K),\end{aligned}$$

where

$$\begin{aligned}\sigma_l(T) &:= \{\mu \in \mathbb{C}, \mu - T \text{ is not left invertible}\}\\ \sigma_r(T) &:= \{\mu \in \mathbb{C},\ \mu - T \text{ is not right invertible}\}.\end{aligned}$$

The sets $\sigma_l(.)$ and $\sigma_r(.)$ represent the left and right spectra respectively.

This article is organized as follows: The next Section, we introduce some definitions and results related with Fredholm theory. In Section 3, we develop a new characterization of the left and right essential spectra of closed linear operators defined on X. This characterization will be based on the essential spectrum of each individual operator. In the last section, we investigate the results from Section 3 to obtain a description of this type of spectra of transport equation in L_p-spaces.

1. Basic Tools

In this section, we will establish some preliminary results regarding the essential spectra of closed densely defined linear operators. These results will be used in our results.

Theorem 1.1. [7, Theorem 5.1.1] *Let $T \in \mathcal{C}(X)$. Then,*
i) $\mu \notin \sigma_{ewl}(T)$ if, and only if, $\mu - T \in \Phi_l(X)$ and $i(\mu - T) \leq 0$.
ii) $\mu \notin \sigma_{ewr}(T)$ if, and only if, $\mu - T \in \Phi_r(X)$ and $i(\mu - T) \geq 0$.
iii) $\sigma_{ew}(T) = \sigma_{ewl}(T) \cup \sigma_{ewr}(T)$.

Lemma 1.2. [7, Lemma 5.1.1] *Let $T \in \mathcal{C}(X)$ such that $0 \in \rho(T)$. Then, for $\mu \neq 0$ we have $\mu \in \sigma_{ei}(T)$ if, and only if, $\frac{1}{\mu} \in \sigma_{ei}(T^{-1})$, $i = wl, wr$.*

Theorem 1.3. [6, Theorem 2.3] *Let $T \in \mathcal{C}(X, Y)$, then*
i) $T \in \Phi_l(X, Y)$ if, and only if, there exist $S \in \mathcal{L}(Y, X)$ et $K \in \mathcal{K}(X)$ such that

$$R(S) \cup R(K) \subset \mathcal{D}(T)$$

and

$$ST = I - K \text{ on } \mathcal{D}(T).$$

ii) $T \in \Phi_r(X, Y)$ if, and only if, there exist $S \in \mathcal{L}(Y, X)$ et $K \in \mathcal{K}(Y)$ such that $R(S) \subset \mathcal{D}(T)$, ST and KT are continuous, and

$$TS = I - K.$$

Theorem 1.4. [6, Theorem 2.5] *Let $A \in \mathcal{C}(Y, Z)$, $B \in \mathcal{C}(X, Y)$, then*
i) If $A \in \Phi_l(Y, Z)$, $B \in \Phi_l(X, Y)$ and $\overline{\mathcal{D}(AB)} = X$, then $AB \in \Phi_l(X, Z)$ and

$$i(AB) = i(A) + i(B).$$

ii) If $A \in \Phi_r(Y, Z)$, $B \in \Phi_r(X, Y)$ and AB is closed, then $AB \in \Phi_r(X, Z)$ and

$$i(AB) = i(A) + i(B).$$

Theorem 1.5. [6, Theorem 2.7] *If $A \in \Phi_l(X)$ (resp.$\Phi_r(X)$) and $K \in \mathcal{K}(X)$, then $A + K \in \Phi_l(X)$ (resp. $\Phi_r(X)$) and*

$$i(A + K) = i(A).$$

Theorem 1.6. [16, Theorem 6] *Let X, Y and Z be a Banach spaces, $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$.*
(i) If $ST \in \Phi_+(X, Z)$, then $T \in \Phi_+(X, Y)$.
(ii) If $ST \in \Phi_-(X, Z)$, then $S \in \Phi_-(Y, Z)$.
(iii) If $ST \in \Phi(X, Z)$, then $T \in \Phi_+(X, Y)$ and $S \in \Phi_-(Y, Z)$.

Definition 1.7. [15] An operator $T \in \mathcal{C}(X)$, is said to be demicompact if for every bounded sequence $\{\varphi_n\}$ in $\mathcal{D}(T)$ such that $\varphi_n - T\varphi_n \to \varphi \in X$, there exist a subsequence of $\{\varphi_n\}$ which converge to x.

We denote $\mathcal{DC}(X)$ the concept of demicompact operators and we define the following sets:

$$\Lambda_X = \{T \in \mathcal{C}(X) \text{ such that } \upsilon T \in \mathcal{DC}(X) \text{ whenever } \upsilon \in [0, 1]\},$$

and

$$\Gamma_X = \{T \in \mathcal{C}(X) \text{ such that } AT \in \Lambda_X \text{for every } A \in \mathcal{L}(X)\}.$$

Remark 1.8. It is clear that all the sets $\mathcal{DC}(X)$, Λ_X and Γ_X contain $\mathcal{K}(X)$.

Theorem 1.9. [5] *Let $T \in \mathcal{C}(X)$, then T is demicompact if, and only if, $I - T \in \Phi_+(X)$.*

Theorem 1.10. [4, Theorem 3.2] *Let $T \in \mathcal{C}(X)$. If $T \in \Lambda_X$ then $I - T \in \Phi(X)$ of index zero.*

Proposition 1.1. [10, Proposition 2.1] Let $A \in \mathcal{C}(X), B \in \mathcal{C}(Y,X), C \in \mathcal{C}(X,Y)$ and $D \in \mathcal{C}(Y)$. We suppose that A and D are demicompact, B is compact and C is bounded. Then,
(i) $\mathcal{T} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathcal{DC}(X)$, and
(ii) $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{DC}(X)$.

Corollary 1.1. [10, Corollary 2.1] Let $A \in \mathcal{C}(X), B \in \mathcal{C}(Y,X), C \in \mathcal{C}(X,Y)$ and $D \in \mathcal{C}(Y)$. We suppose that A and D are demicompact operators, B is bounded and C is compact, then $\mathcal{A} \in \mathcal{DC}(X \times Y)$. In addition, if $A \in \Lambda_X$ and $D \in \Lambda_Y$, then $\mathcal{A} \in \Lambda_{X\times Y}$.

2. Stability of Left and Right Weyl Spectrum

Now, we will investigate the Weyl essential spectra of unbounded linear operators by using the concept of demicompactness.

Lemma 2.1. *Let $A \in \mathcal{C}(X)$. Then,*
$i)$ if $A \in \Phi_l(X)$ and $B \in \Gamma_X$, then $A + B \in \Phi_l(X)$ and,

$$i(A + B) = i(A).$$

$ii)$ if $A \in \Phi_r(X)$ and $B \in \Gamma_X$, then $A + B \in \Phi_r(X)$ and,

$$i(A + B) = i(A).$$

Proof 2.1. let $A \in \Phi_l(X)$, then by using Theorem 1.3, there exists $A_l \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that

$$A_lA = I - K, \text{ on } \mathcal{D}(A).$$

Hence,

$$A_l(A + B) = I - K + A_lB, \text{ on } \mathcal{D}(A).$$

Since $B \in \Gamma_X$, it means that $A_lB \in \Lambda_X$, then by applying Theorem 1.10, we get $I + A_lB \in \Phi(X)$ and

$$i(I + A_lB) = 0.$$

Since K is compact, then Theorem 1.5 imply that $A_l(A+B) \in \Phi(X)$, and $\mathcal{R}(A_l) \cup \mathcal{R}(A_lB - K) \subset \mathcal{D}(A)$. By referring to Theorem 1.3, we obtain $(A + B) \in \Phi_l(X)$.
Let $X_A = (\mathcal{D}(A), \|\cdot\|)$ be a Banach space, where

$$\|x\|_A = \|x\| + \|Ax\|.$$

We consider A from X_A into X. We denoted $\widehat{A}$. Clearly $\widehat{A} + \widehat{B}$ and $\widehat{B}$ are bounded operator from X_A into X. Accordingly, we obtain the following equalities :

$$\begin{cases} \mathcal{R}(\hat{B}) = \mathcal{R}(B), \\ and\ \mathcal{R}(\hat{A} + \hat{B}) = \mathcal{R}(A + B). \\ \alpha(\hat{A}) = \alpha(A), \\ \beta(\hat{B}) = \beta(B), \\ \alpha(\hat{A} + \hat{A}) = \alpha(A + B), \beta(\hat{A} + \hat{B}) = \beta(A + B), \end{cases}$$

Obviously, $\widehat{K}$ and $\widehat{A_l}$ are bounded on X_A.We can write

$$\widehat{A_l}\widehat{A} = I_{X_A} - \widehat{K}.$$

$\widehat{A_l}\widehat{A} \in \Phi(X_A)$ and $i(\widehat{A_l}\widehat{A}) = 0$. On the other hand,

$$i(\widehat{A_l}(\widehat{A} + \widehat{B})) = i(A_l(A + B)).$$

Hence, $\widehat{A_l}$ is Fredholm if, and only if, $\widehat{A}$ is Fredholm if, and only if, $\widehat{A} + \widehat{B}$ is Fredholm. We conclude that

$$i(A + B) = i(\widehat{A} + \widehat{B}) = i(\widehat{A}) = i(A).$$

(ii) This assertion is checked in the same way as in (i).

Theorem 2.2. *Let $A \in \mathcal{C}(X)$. If $B \in \Gamma_X$, then*
(i) $\sigma_{ewl}(A + B) = \sigma_{ewl}(A)$.
(ii) $\sigma_{ewr}(A + B) = \sigma_{ewr}(A)$.

Proof 2.2. Let $\mu \notin \sigma_{ewl}(A)$, then by Theorem 1.1, we get $\mu - A \in \Phi_l(X)$ and $i(\mu - A) \leq 0$. Since $B \in \Gamma_X$, hence By Lemma 2.1, we obtain $\mu - A - B \in \Phi_l(X)$ and $i(\mu - A - B) \leq 0$. The opposite inclusion follows by symmetry and, we conclude that

$$\sigma_{ewl}(A + B) = \sigma_{ewl}(A).$$

(ii) The proof of (ii) may be checked in a similar way to that (i).

Theorem 2.3. *Let A and $B \in \mathcal{C}(X)$ such that the resolvent of A and B are not empty. If, for every $\mu \in \rho(A) \cap \rho(B)$, $(\mu - A)^{-1} - (\mu - B)^{-1} \in \Gamma(X)$, then*

$$\sigma_{ewl}(A) = \sigma_{ewl}(B)$$

and

$$\sigma_{ewr}(A) = \sigma_{ewr}(B).$$

Proof 2.3. Let $\mu \in \rho(A) \cap \rho(B)$. For $\lambda \neq 0$, We have the following equality

$$\lambda - \mu + A = -\lambda(\lambda^{-1} - (\mu - A)^{-1})(\mu - A).$$

Since $(\mu - A)$ is one to one, then

$$\alpha(\lambda - \mu + A) = \alpha(\lambda^{-1} - (\mu - A)^{-1})$$

and

$$\mathcal{R}(\lambda - \mu + A) = \mathcal{R}(\lambda^{-1} - (\mu - A)^{-1}).$$

This show that $\lambda - \mu + A \in \Phi_l(X)$ if, and only if, $(\lambda^{-1} - (\mu - A)^{-1}) \in \Phi_l(X)$. Then $\lambda \in \Phi_{l(\mu-A)}$ if, and only if, $\lambda^{-1} \in \Phi_{l(\mu-A)^{-1}}$, and $i(\lambda - \mu + A) = i(\lambda^{-1} - (\mu - A)^{-1})$.
Let $\lambda \in \Phi_{l(\mu-A)}$, it means that $\lambda^{-1} \in \Phi_{l(\mu-A)^{-1}}$. Since $(\mu - A)^{-1} - (\mu - B)^{-1} \in \Gamma(X)$, then by combining Lemma 2.2 with Lemma 1.2, we obtain $\lambda^{-1} \in \Phi_{l(\mu-B)^{-1}}$, which equivalent to $\lambda \in \Phi_{l(\mu-B)}$, and $i(\lambda-\mu+A) = i(\lambda - \mu + B)$. Hence

$$\sigma_{ewl}(A) = \sigma_{ewl}(B).$$

(ii)The proof of (ii) may be checked in a similar way to that (i).

Definition 2.4. Let $T \in \mathcal{C}(X)$. T is called power demicompact if there exists $n \in \mathbb{N}^*$ satisfying $T^n \in \mathcal{DC}X$.

Theorem 2.5. *Let $\lambda A \in \mathcal{C}(X)$ be a power demicompact for every $\lambda \in [0, 1]$, then $I - A \in \Phi(X)$ and $i(I - A) = 0$.*

Proof 2.4. Since λA is power demiompact for every $\lambda \in [0,1]$, then there is a $n \in \mathbb{N}^*$ satisfying $(\lambda A)^n \in \mathcal{DC}X$. It follows by Lemma 1.10, that $I-(\lambda A)^n$ is Fredholm with index zero. On other hand, we have the following equations:

$$\begin{aligned} I-(\lambda A)^n &= (I-\lambda A)(I+\lambda A+...+\lambda^{n-1}A^{n-1}) \\ &= (I+\lambda A+...+\lambda^{n-1}A^{n-1})(I-\lambda A). \end{aligned}$$

Hence, from Lemma 1.6, we obtain that $I-(\lambda A)$ is Fredholm and $i(I-\lambda A)=0$. Now, by stability results of Kato [11] , the index of $I-(\lambda A)$ is continuous in λ. Since $i(I-\lambda A)\in \mathbb{N}$,then it is constant fore very $\lambda \in [0,1]$. Obviously, $i(I-\lambda A)=i((I-A)=i(I)=0$, hence $i(I-\lambda A)=0$.

Corollary 2.1. Let $A \in \mathcal{C}(X)$ be a power demicompact, then $I-A \in \Phi_+(X)$.

Proof 2.5. We have

$$I-A^n = (I+A+...+A^{n-1})(I-A).$$

then, by applying Theorem 1.6 (i), we conclude that $I-A\in\Phi_+(X)$.

3. An Application to a Two-Group Transport Operators

Assume that $D\subset\mathbb{R}^N$, and let $d\mu$ be a measure on $\mathbb{R}^N$ satisfying $d\mu(0)=0$. The support of $d\mu$ is denoted as V and is referred to as the velocity space. For any $(x,v)\in\overline{D}\times\overline{V}$, we examine the following neutron transport equations:

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}+\sigma(x,v)f(x,v,t)=Kf \tag{3.1}$$

subject to boundary and initial conditions

$$f(t,\cdot,\cdot)_{|\Gamma_-}=0, f(0,x,v)=f_0(x,v)$$

where

$$\Gamma_-=\{(x,v)\in\partial D\times V, vn(x)<0\}$$

and $n(x)$ is the outward normal at $x\in\partial D$.
The collision frequency at position x for neutrons traveling at velocity v is represented as $\sigma(x,v)$. The collision operator, referred to as K, is known for its property of being localized in relation to the spatial variable x within the domain D. We can write eq. 3.1 as an abstract Cauchy problem

$$\frac{df}{dt}=Tf+Kf, f(0)=f_0 \tag{3.2}$$

in the space

$$X=L_p(D\times V;dxd\mu(v))\ (1\le p<\infty)$$

The operator T is called streaming operator and defined by:

$$Tf=-v\frac{\partial f}{\partial x}-\sigma(x,v)f(x,v)$$

in domain

$$\mathcal{D}(T)=\{f\in L_p(D\times V); v\frac{\partial f}{\partial x}\in L_p(D\times V); f_{|\Gamma_-}=0\}.$$

We will suppose that the collision frequency is positive and bounded

$$\sigma(\cdot,\cdot)\in L^\infty_+(D\times V)$$

It is known that T generates an explicit c_0-semigroup

$$\theta(t)\varphi=e^{-\int_0^t\sigma(x-\tau v,v)}\varphi(x-tv,v)\chi(t<s(x,v)).$$

Where

$$s(x,v) = \inf\{s > 0; x - sv \notin D\}.$$

We recall that the essential type of $\theta(t); t > 0)$ is given by

$$\eta = -\lim_{t\to\infty} ess. \inf_{t<s(x,v} t^{-1} \int_0^t \sigma(x - sv, v)ds.$$

If D is bounded,

$$\begin{cases} \sigma(\theta(t)) = \{\mu; |\mu| \leq e^{\eta t}\} \\ \sigma(\theta(t)) = \{\mu; Re\mu \leq \eta\}. \end{cases}$$

We observe that if $0 \notin D$ (i.e. the velocities are bounded away from zero) and if D is bounded, then $U(t); t \geq 0$ is nilpotent (it vanishes for $t > \frac{d}{v_{min}}$ where d is the diameter of D and v_{min} is the minimwn speed) and therefore $\eta = -\infty$. If $0 \in D$, D is bounded, and if the collision frequency is homogeneous, then

$$\eta = -\lim \inf_{v\to 0} \sigma(v).$$

For more explication concerning the following two-group transport operators, the reader may refer to [7]. Let $1 < p < \infty$, we introduce the space

$$X_p := L_p[(-a,a) \times (-1,1); dxdv] \; (0 \leq a < \infty).$$

Let the following two-group transport operator defined on $X_p \times X_p$,

$$\mathcal{A}^H = \mathcal{T}^H + \mathcal{K},$$

where

$$\begin{aligned} \mathcal{T}^H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \begin{pmatrix} -v\dfrac{\partial\psi_1(x,v)}{\partial x} - \sigma_1(v)\psi_1(x,v) & 0 \\ 0 & -v\dfrac{\partial\psi_2(x,v)}{\partial x} - \sigma_2(v)\psi_2(x,v) \end{pmatrix} \\ &= \begin{pmatrix} T_1^H & 0 \\ 0 & T_2^H \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \end{aligned}$$

and

$$\mathcal{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix},$$

where K_{ij} are bounded on X_p and defined by:

$$\begin{cases} K_{ij} : X_p \longrightarrow X_p \\ \psi \longrightarrow \displaystyle\int_{-1}^{1} k_{ij}(x,v,v')\psi(x,v')dv'. \end{cases} \tag{3.3}$$

We define the streaming operator T_j^H, $j = 1,2$, by:

$$\begin{cases} T_j^H\psi(x,v) = -v\dfrac{\partial\psi_j(x,v)}{\partial x} - \sigma_j(v)\psi(x,v) \\ \mathcal{D}(T_j^H) = \{\psi \in W_p \text{ such that } \psi^i = H(\psi^o)\}, \end{cases}$$

The scattering kernels $k_{ij} : (-a,a) \times (-1,1) \times (-1,1) \longrightarrow \mathbb{R}$ are assumed to be measurable. We have the following definition.

Definition 3.1. Let K_{ij} be the integral operator defined by (3.3). Then, K_{ij} is said to be regular if $\{k_{ij}(x,\cdot,v')$ such that $(x,v') \in (-a,a) \times (-1,1)\}$ is a relatively compact subset of $L_p(-1,1)$, $p > 1$.

Theorem 3.2. [13, Theorem 4.1] *Let $1 < p < \infty$ and let D be bounded. We assume that $d\mu$ is such that the hyperplanes have zero $d\mu$-measure and that the collision operator is regular. Then $K_{ij}(\lambda - T_i^H)^{-1}$ and $(\lambda - T_i^H)^{-1}K_{ij},\ i, j \in \{1, 2\}$, are compact in $L_p(D \times V; dxd\mu(v))$.*

We introduce the following spaces:

$$X_p^o = L_p[\{-a\} \times (-1, 0); |v|dv] \times L_p[\{a\} \times (0, 1); |v|dv] = X_{1,p}^o \times X_{2,p}^o,$$

$$X_p^i = L_p[\{-a\} \times (0, 1); |v|dv] \times L_p[\{a\} \times (-1, 0); |v|dv] = X_{1,p}^i \times X_{2,p}^i,$$

and

$$W_p = \left\{ \psi \in X_p \text{ such that } - v\frac{\partial \psi}{\partial x} \in X_p \right\}.$$

The boundary operator H is defined by:

$$\begin{cases} H : X_{1,p}^o \times X_{2,p}^o \longrightarrow X_{1,p}^i \times X_{2,p}^i \\ H \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \end{cases}$$

with $k, l \in \{1, 2\}$, and $H_{kl} \in \mathcal{L}(X_{l,p}^i, X_{k,p}^o)$.
The function $\psi(x, v)$ with position x and the cosine of direction of propagation v represents the density of gas particles; that is, the x-direction of the particle and v is the cosine of the angle between the velocity vector. The collision frequency satisfies

$$\sigma_j(\cdot) \in L^\infty(-1, 1),$$

and ψ^o, ψ^i represent the outgoing and the incoming fluxes related by the boundary operator H ($'o'$ for the outgoing and $'i'$ for the incoming). By Remark 4.1 in [12], we determine the expression of the resolvent of the operator T_{H_1}. Let

$$\lambda_j^* = \liminf_{v \to 0} \sigma_j(v),\ j = 1, 2,$$

and

$$\eta^j = \begin{cases} -\lambda_j^*,\ \text{if } \|H\| \le 1 \\ -\lambda_j^* + \dfrac{1}{2a}\log(\|H\|) \text{ if } \|H\| > 1. \end{cases}$$

Remark 3.3. If the operator $H \neq 0$ is strictly singular on X_p, that is for every infinite-dimensional subspace F, the restriction of H to F is not an homeomorphism, then by using Theorem 3.3 in [1] we obtain:

$$\sigma_{e_i}(T_j^H) = \{\lambda \in \mathbb{C} \text{ such that Re } \lambda \le -\lambda_j^*\},\ \text{for every } i \in \{l, r, wl, wr\}.$$

We will suppose that the operator $H \neq 0$ is strictly singular on X_p.

Theorem 3.4. If $K_{11}, K_{22} \in \Gamma(X_p)$, and K_{21} or $K_{12} \in \mathcal{K}(X_p)$, then

$$\sigma_i(\mathcal{A}^H) = \{\lambda \in \mathbb{C} \text{ such that Re } \lambda \le \min(-\lambda_1^*, -\lambda_2^*)\},\ \text{for every } i = \{ewl, ewr\}.$$

Proof 3.1. let $\lambda \notin \sigma_{ewl}(\mathcal{T}^H)$, then $\lambda - \mathcal{T}^H \in \phi_l(X_p \times X_p)$, it means that $\lambda - T_1^H \in \phi_l(X_p)$ and $\lambda - T_2^H \in \phi_l(X_p)$. Using Theorem 1.3, there exist $T_1^l, T_2^l \in \mathcal{L}(X_p)$, and $K_1, K_2 \in \mathcal{K}(X_p)$, respectively, such that

$$T_1^l(T_1^H - \lambda) = I - K_1,\ \text{on } \mathcal{D}(T_1^H),$$

and

$$T_2^l(T_2^H - \lambda) = I - K_2,\ \text{on } \mathcal{D}(T_2^H).$$

Now, we can define $\mathcal{T}_l$, and $\mathbf{K}$, respectively, on the following form:

$$\mathcal{T}_l = \begin{pmatrix} T_1^l & 0 \\ 0 & T_2^l \end{pmatrix}, \; \mathbf{K} = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}.$$

Hence, we have the relation

$$\mathcal{T}_l(\lambda - \mathcal{T}^H) = I - \mathbf{K} \text{ on } \mathcal{D}(\mathcal{T}^H).$$

We calculate

$$\mathcal{T}_l\mathcal{K} = \begin{pmatrix} T_1^l K_{11} & T_2^l K_{12} \\ T_1^l K_{21} & T_2^l K_{22} \end{pmatrix}.$$

Since $K_{11}, K_{22} \in \Gamma(X_p)$, then $T_1^l K_{11}, T_2^l K_{22} \in \Lambda(X_p)$. In addition, K_{21} or $K_{12} \in \mathcal{K}(X_p)$, then $T_1^l K_{21}$ or $T_2^l K_{12} \in \mathcal{K}(X_p)$. It follows from Corollary 1.1, that

$$\mathcal{T}_l\mathcal{K} \in \Lambda(X_p \times X_p),$$

which imply that $\mathcal{K} \in \Gamma(X_p \times X_p)$. Now, by applying Theorem 2.2, we obtain

$$\sigma_i(\mathcal{A}^H) = \sigma_i(\mathcal{T}^H) = \{\lambda \in \mathbb{C} \text{ such that Re } \lambda \leq \min(-\lambda_1^*, -\lambda_2^*)\}.$$

Theorem 3.5. *If K_{21} and K_{12} are regular, then*

$$\sigma_{ei}(\mathcal{A}^H) \subseteq \{\lambda \in \mathbb{C} \textit{ such that Re } \lambda \leq \min(-\lambda_1^*, -\lambda_2^*)\}, \textit{ for every } i = \{ewl, ewr\}.$$

Proof 3.2. let $\lambda \notin \sigma_l(\mathcal{T}^H)$, then $\lambda - \mathcal{T}^H \in \mathcal{GL}(X_p \times X_p)$, it means that $\lambda - T_1^H \in \mathcal{GL}(X_p)$ and $\lambda - T_2^H \in \mathcal{GL}(X_p)$. Now, We can define $\mathcal{T}_l$ on the following form:

$$\mathcal{T}_l = \begin{pmatrix} (\lambda - T_1^l)^{-1} & 0 \\ 0 & (\lambda - T_2^l)^{-1} \end{pmatrix}.$$

Hence, we have the relation $\mathcal{T}_l(\lambda - \mathcal{T}^H) = I$. Now, we calculate

$$\mathcal{T}_l\mathcal{K} = \begin{pmatrix} \lambda - T_1^l)^{-1} K_{11} & (\lambda - T_1^l)^{-1} K_{12} \\ (\lambda - T_1^2)^{-1} K_{21} & (\lambda - T_1^2)^{-1} K_{22} \end{pmatrix}.$$

Since K_{21} and K_{12} are regular, then by using Theorem 3.2 we obtain that Then $K_{21}(\lambda - T)^{-l}$ and $K_{12}(\lambda - T)^{-l}$ are compact in $L_p(D \times V; dxd\mu(v))$ which imply that $\mathcal{K} \in \Gamma(X_p \times X_p)$. In the rest of the proof, we continue by the same way as Theorem 3.4, and we get the relation

$$\sigma_{ewl}(\mathcal{A}^H) \subseteq \sigma_l(\mathcal{T}^H) = \{\lambda \in \mathbb{C} \text{ such that Re } \lambda \leq \min(-\lambda_1^*, -\lambda_2^*)\}.$$

$ii)$ By the same manner we prove

$$\sigma_{ewr}(\mathcal{A}^H) \subseteq \sigma_r(\mathcal{T}^H) = \{\lambda \in \mathbb{C} \text{ such that Re } \lambda \leq \min(-\lambda_1^*, -\lambda_2^*)\}.$$

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