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DYNAMIC BEHAVIOR AND BIFURCATION ANALYSIS OF A DIFFERENCE EQUATION INCLUDING EXPONENTIAL TERMS

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ABSTRACT. In this paper, the local stability, the boundedness, rate of convergence and the conditions of Neimark-Sacker bifurcation concerning difference equation

$$x_{n+1} = \alpha_1 x_n + \alpha_2 x_{n-1} + \beta_1 x_n \exp(-x_{n-1}^2) + \beta_2 x_{n-1} \exp(-x_{n-1}^2), \quad n = 0, 1, ...,$$

are investigated. We focus on the Neimark-Sacker bifurcations of the discrete model. The center manifold theorem and bifurcation theory are explicitly applied to reach conclusions about the occurrence and stability of the Neimark-Sacker bifurcation. Many numerical simulations that confirm the existence of the Neimark-Sacker bifurcation are also provided.

1. INTRODUCTION

In the recent years, many authors have great interest in studying the behavior of difference equation including exponential terms especially the global stability because it is considered one of the main to topics in the theory of difference equations. Also, they are interested in investigating the bifurcation phenomenon which occur in parameter-dependent dynamic systems. When the parameters are varied, changes may occur in the qualitative structure of the solutions for certain parameter values. Bifurcation may lead to different dynamical behaviors of a model when parameters pass through a critical value. Difference equations have many applications in applied sciences. Several authors have published research articles on discrete models which discuss the local, global asymptotic stability and bifurcation see [[1], [3], [5]- [7]].

In Metwally et al. [9] investigated the boundedness, the asymptotic behavior, the periodic character and the stability of solutions of the difference equation

$$x_{n+1} = \alpha + \beta x_{n-1} \exp(-x_n),$$

where the parameters α and β are positive numbers and the initial conditions are arbitrary non-negative real numbers.

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Wenjie et al in [34] investigated the boundedness and the asymptotic behavior of the positive solutions for difference equation

$$x_{n+1} = a + bx_n \exp(-x_{n-1}),$$

where a, b are positive constants and the initial values x_{-1}, x_0 are non-negative numbers.

In [27] was shown the globally asymptotically stability of the difference equation

$$y_{n+1} = \frac{\alpha + \beta \exp(-y_n)}{\gamma + y_{n-1}},$$

where the parameters α, β and γ are positive numbers and the initial conditions are arbitrary non-negative numbers.

Bozkurat in [14] discussed the local and global behavior of the positive solutions for difference equation

$$y_{n+1} = \frac{\alpha \exp(-y_n) + \beta \exp(-y_{n-1})}{\gamma + \alpha y_n + \beta y_{n-1}},$$

where the parameters α, β and γ and the initial conditions x_{-1}, x_0 are positive numbers.

Moreover, Din [4] investigated the global asymptotic stability of the following discrete-time population model:

$$x_{n+1} = \alpha x_n \exp(-y_n) + \beta, \quad y_{n+1} = \alpha x_n (1 - \exp(-y_n)),$$

In [17] the authors obtained the global behavior of the positive solutions for difference equation

$$x_{n+1} = ax_n + bx_{n-1} \exp(-x_n),$$

where a, b are positive constants and the initial values x_{-1}, x_0 are positive numbers.

Hui Feng et al in [19] investigated the global stability and bounded nature of the positive solutions for difference equation

$$x_{n+1} = a + bx_{n-1} + cx_{n-1} \exp(-x_n),$$

where the parameters $a \in (0, \infty)$, $b \in (0, 1)$, $c \in (0, \infty)$ and the initial conditions are arbitrary non-negative numbers. For other papers related to the qualitative behavior of difference can be found in the following papers [[8], [10], [15], [16], [20]- [24], [28], [29], [31], [32], [33], [36]].

Our purpose in this paper is to investigate the global stability character, boundedness, the rate of convergence of solutions and Naimark-Sacker bifurcation of the recursive sequence

$$x_{n+1} = \alpha_1 x_n + \alpha_2 x_{n-1} + \beta_1 x_n \exp(-x_{n-1}^2) + \beta_2 x_{n-1} \exp(-x_{n-1}^2) \quad n = 0, 1, ..., \tag{1}$$

where the parameters $\alpha_1, \alpha_2, \beta_1$ and β_2 are nonnegative numbers and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers. Moreover, introducing $y_n = x_{n-1}$, we obtain the following discrete-time system equivalent to Eq.(1):

$$\begin{aligned} x_{n+1} &= \alpha_1 x_n + \alpha_2 y_n + \beta_1 x_n \exp(-y_n^2) + \beta_2 y_n \exp(-y_n^2) \\ y_{n+1} &= x_n. \end{aligned} \tag{2}$$

This work investigates the bifurcation analysis of a novel class of difference equations with exponential terms. Our primary focus is on determining how the four system parameters affect the system's dynamics. In addition, the method we employed to analyze the system is novel enough in that it may be applied to the investigation of a wide range of models in the field of theoretical ecology, including those with exponential terms. There are various difference equations in mathematical biology; see [[26], [2]]. and references therein. Our analysis can be used to precisely describe the dynamic behaviors of many difference equations, all of which are realistic models in population dynamics.

We organized of this paper as follows. The first section introduces the sample research articles on discrete models, which discuss the behavior of difference equations, including exponential terms, especially the local asymptotic stability and bifurcation at once. Section 2 is devoted to investigating and giving lemmas to the local dynamical behavior of the system at its equilibrium points. Section 3 gives the boundedness of the solutions of the equation. Section 4 investigates the rate of the solution's convergence to the unique positive equilibrium point. Section 5 shows the Naimark- Sacker bifurcation of a fixed points for a map associated with the Difference equation with exponential terms. In section 6, numerical examples are examined to validate theoretical findings in previous sections. Finally, in section 7, we present numerical simulations to prove theoretical analysis.

2. Local Stability of the Equilibrium Point

In this section we investigate the local dynamical behavior of the system (2).

First we solve the system

$$x = \alpha_1 x + \alpha_2 y + \beta_1 x \exp(-y^2) + \beta_2 y \exp(-y^2), \qquad y = x.$$

Solving the aforementioned system yields that the two equilibria for the system (2) are $E_0 = (0,0)$, $E_1 = (\sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}, \sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})})$, when $(\alpha_1+\alpha_2) < 1$, and $E_2 = (-\sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}, -\sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})})$, when $(\alpha_1+\alpha_2) < 1$.

Assume that $J(x,y)$ denotes Jacobian matrix of system (2) evaluated at (x,y),

$$J(x,y) = \begin{pmatrix} \alpha_1 x + \beta_1 \exp(-y^2) & \alpha_2 - 2\beta_1 xy \exp(-y^2) - 2\beta_2 y^2 \exp(-y^2) + \beta_2 \exp(-y^2) \\ 1 & 0 \end{pmatrix},$$

then it follows:

$$J(0,0) = \begin{pmatrix} \alpha_1+\beta_1 & \alpha_2+\beta_2 \\ 1 & 0 \end{pmatrix},$$

and

$$J(\sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}, \sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}) =$$

$$\begin{pmatrix} \alpha_1 + \beta_1 \left(\frac{1-(\alpha_1+\alpha_2)}{\beta_1+\beta_2}\right) & \alpha_2 + \beta_2 \left(\frac{1-(\alpha_1+\alpha_2)}{\beta_1+\beta_2}\right) - 2(1-(\alpha_1+\alpha_2))\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)}) \\ 1 & 0 \end{pmatrix}.$$

Moreover, characteristic polynomial for $J(0,0)$ is computed as follows:

$$P(\lambda) = \lambda^2 - (\alpha_1+\beta_1)\lambda - (\alpha_2+\beta_2),$$

and characteristic polynomial computed for $J\left(\sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}, \sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}\right)$ is given by:

$$P(\lambda) = \lambda^2 - (\alpha_1 + \beta_1 \left(\frac{1-(\alpha_1+\alpha_2)}{\beta_1+\beta_2}\right))\lambda$$

$$-(\alpha_2 + \beta_2 \left(\frac{1-(\alpha_1+\alpha_2)}{\beta_1+\beta_2}\right) - 2(1-(\alpha_1+\alpha_2))\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})).$$

Keeping in view the relation between roots and coefficients for a quadratic equation, the following Lemma gives local dynamical behavior of system (2) at its trivial steady-state $(0,0)$.

Lemma 1. *The following statements hold true related to equilibrium* $(0,0)$ *of* (2)*:*

1. $(0,0)$ *is locally asymptotically stable if and only if* $\alpha_1+\beta_1+\alpha_2+\beta_2 < 1$.
2. $(0,0)$ *cannot be a saddle point for the system* (2).
3. $(0,0)$ *cannot be a source (repeller) for the system* (2).

Similarly, for positive equilibrium point, we have the following Lemma:

Lemma 2. *For positive equilibrium of the system* (2)*, the following statements hold true:*

1. $\left(\sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}, \sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}\right)$ *is locally asymptotically stable iff*

$$0 < \alpha_1+\alpha_2 < 1, \ \ (1-(\alpha_1+\alpha_2))e^{\frac{-1}{2(1-(\alpha_1+\alpha_2))}} < \beta_1+\beta_2 < (1-(\alpha_1+\alpha_2))e^{\frac{1+\alpha_2}{2(1-(\alpha_1+\alpha_2))}}.$$

2. $\left(\sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}, \sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}\right)$ *is a saddle point iff*

$$0 < \alpha_1+\alpha_2 < 1, \ \ 0 < \beta_1+\beta_2 < (1-(\alpha_1+\alpha_2))e^{\frac{1}{2(1-(\alpha_1+\alpha_2))}}.$$

3. $\left(\sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}, \sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}\right)$ *is a (source) iff*

$$0 < \alpha_1+\alpha_2 < 1, \ \ \beta_1+\beta_2 < (1-(\alpha_1+\alpha_2))e^{\frac{\alpha_2-1}{2(1-(\alpha_1+\alpha_2))}}.$$

3. Boundedness of Solutions

In this section the boundedness of solutions of Eq.(1) is studied.

Theorem 1. *Every solution of Eq.*(1) *is bounded if*

$$\alpha_1+\beta_1+\alpha_2+\beta_2 < 1.$$

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1)

$$\begin{aligned} x_{n+1} &= \alpha_1 x_n + \alpha_2 x_{n-1} + \beta_1 x_n \exp(-x_{n-1}^2) + \beta_2 x_{n-1}\exp(-x_{n-1}^2) \\ &= \alpha_1 x_n + \alpha_2 x_{n-1} + \frac{\beta_1 x_n + \beta_2 x_{n-1}}{\exp(x_{n-1}^2)} \\ &< \alpha_1 x_n + \alpha_2 x_{n-1} + \beta_1 x_n + \beta_2 x_{n-1} \\ &< (\alpha_1+\beta_1)x_n + (\alpha_2+\beta_2)x_{n-1}. \end{aligned}$$

By using a comparison, we can write the right hand side as follows

$$y_{n+1} = A_1 y_n + A_2 y_{n-1},$$

where $A_1 = \alpha_1+\beta_1$, $A_2 = \alpha_2+\beta_2$ and this equation is locally asymptotically stable if $A_1 + A_2 < 1$ and converges to the equilibrium point $\bar{y} = 0$.

Thus

$$\limsup_{n\to\infty} x_n = 0,$$

then the solution is bounded. □

4. Rate of Convergence

In this section we will investigate the rate of convergence of a solution that converges to the unique positive equilibrium point of Eq. (1). Let $f : (0,\infty)^2 \to (0,\infty)$ be a function defined by

$$f(u,v) = \alpha_1 u + \alpha_2 v + \beta_1 u\exp(-v^2) + \beta_2 v\exp(-v^2). \tag{3}$$

Therefore it lead to that

$$f_u(u,v) = \alpha_1 + \beta_1\exp(-v^2), \tag{4}$$

$$f_v(u,v) = \alpha_2 - 2\beta_1 uv\exp(-v^2) - 2\beta_2 v^2\exp(-v^2) + \beta_2\exp(-v^2). \tag{5}$$

We see that at $\bar{x} = \sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}$ in (4) and (5)

$$f_u(\bar{x},\bar{x}) = \frac{\alpha_1\beta_2+\beta_1-\alpha_2\beta_1}{\beta_1+\beta_2} = a,$$
$$f_v(\bar{x},\bar{x}) = \frac{\alpha_2\beta_1+\beta_2-\alpha_1\beta_2}{\beta_1+\beta_2} + 2((\alpha_1+\alpha_2)-1)\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)}) = b.$$

The linearized equation of Eq.(1) about is $\bar{x} = \sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}$ is

$$y_{n+1} - ay_n - by_{n-1} = 0, \tag{6}$$

The characteristic equation of the linearized equation (6) is

$$\lambda^2 - a\lambda + b = 0. \tag{7}$$

We have

$$\begin{aligned}
x_{n+1} - \bar{x} &= \alpha_1 x_n + \alpha_2 x_{n-1} + \beta_1 x_n \exp(-x_{n-1}^2) + \beta_2 x_{n-1}\exp(-x_{n-1}^2) \\
&\quad -(\alpha_1\bar{x} + \alpha_2\bar{x} + \beta_1\bar{x}\exp(-\bar{x}^2) + \beta_2\bar{x}\exp(-\bar{x}^2)) \\
&= \alpha_1(x_n-\bar{x}) + \alpha_2(x_{n-1}-\bar{x}) + \beta_1 x_n\exp(-x_{n-1}^2) \\
&\quad + \beta_2 x_{n-1}\exp(-x_{n-1}^2) - \beta_1\bar{x}\exp(-\bar{x}^2) - \beta_2\bar{x}\exp(-\bar{x}^2)) \\
&= (\alpha_1+\beta_1\exp(-x_{n-1}^2))(x_n-\bar{x}) + (\alpha_2+\beta_2\exp(-x_{n-1}^2))(x_{n-1}-\bar{x}) \\
&\quad +(\frac{(\beta_1+\beta_2)\bar{x}\exp(-x_{n-1}^2) - (\beta_1+\beta_2)\bar{x}\exp(-\bar{x}^2)}{x_{n-1}-\bar{x}})(x_{n-1}-\bar{x})
\end{aligned}$$

$$x_{n+1} - \bar{x} = (\alpha_1+\beta_1\exp(-x_{n-1}^2))(x_n-\bar{x}) \qquad .$$
$$+(\alpha_2+\beta_2\exp(-x_{n-1}^2) + \frac{(\beta_1+\beta_2)\bar{x}\exp(-x_{n-1}^2) - (\beta_1+\beta_2)\bar{x}\exp(-\bar{x}^2)}{x_{n-1}-\bar{x}})(x_{n-1}-\bar{x}).$$

Set

$$e_n = x_n - \bar{x}, \quad e_{n-1} = x_{n-1} - \bar{x},$$
$$e_{n+1} = a_n e_n + b_n e_{n-1},$$

where

$$a_n = \alpha_1 + \beta_1\exp(-x_{n-1}^2),$$
$$b_n = \alpha_2 + \beta_2\exp(-x_{n-1}^2) + \frac{(\beta_1+\beta_2)\bar{x}\exp(-x_{n-1}^2) - (\beta_1+\beta_2)\bar{x}\exp(-\bar{x}^2)}{x_{n-1}-\bar{x}}.$$

As the positive equilibrium is a global attractor, we obtain

$$\lim_{n\to\infty} a_n = \alpha_1 + \beta_1\exp(-\bar{x}^2) = \frac{\alpha_1\beta_2+\beta_1-\alpha_2\beta_1}{\beta_1+\beta_2},$$

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \alpha_2 + \beta_2\exp(-\bar{x}^2) + \lim_{n\to\infty}\left[\frac{(\beta_1+\beta_2)\bar{x}\exp(-x_{n-1}^2) - (\beta_1+\beta_2)\bar{x}\exp(-\bar{x}^2)}{x_{n-1}-\bar{x}}\right] \\
&= \alpha_2 + \beta_2\exp(-\bar{x}^2) - 2(\beta_1+\beta_2)\bar{x}^2\exp(-\bar{x}^2)
\end{aligned}$$

$$\lim_{n\to\infty} b_n = \frac{\alpha_2\beta_1+\beta_2-\alpha_1\beta_2}{\beta_1+\beta_2} + 2((\alpha_1+\alpha_2)-1)\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)}).$$

Thus the limiting equation of Eq. (1) is the linearized equation (6).

Theorem 2. *All solutions of Eq.* (1) *which are eventually differ from the equilibrium satisfy*

$$\lim_{n\to\infty} \frac{x_{n+1} - \bar{x}}{x_n - \bar{x}} = \lambda_+, \quad or \quad \lim_{n\to\infty} \frac{x_{n+1} - \bar{x}}{x_n - \bar{x}} = \lambda_-$$

and

$$\limsup_{n\to\infty} (\mid x_j(n) \mid)^{\frac{1}{n}} = \mid \lambda_\pm \mid,$$

where $\lambda_\pm$ are the complex roots of the characteristic equation (7).

5. Naimark-Sacker Bifurcation of Eq.(1).

In this section we will execute the Naimark-Sacker bifurcation analysis of Eq.(1). Now we study bifurcation of a fixed point of map linked to Eq.(1) whose Jacobian matrix has a pair of complex conjugate eigenvalues. First, we display Naimark-Sacker bifurcation theorem, known also as the Poincare-Andronov-Hopf bifurcation theorem for maps, see[[18], [25], [30], [35]].

Theorem 3. *Let*

$$F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2; \quad (\lambda, x) \to F(\lambda, x)$$

be a C^4 map depending on real parameter λ satisfying the following conditions:

- **(i):** $F(\lambda, 0) = 0$ *for λ near some fixed λ_0;*
- **(ii):** $DF(\lambda, 0)$ *has two non-real eigenvalues $\mu(\lambda)$ and $\bar{\mu}(\lambda)$ for λ near λ_0 with* $\mid \mu(\lambda_0) \mid = 1$;
- **(iii):** $\frac{d}{d\lambda} \mid \mu(\lambda) \mid = d(\lambda_0) \neq 0$ *at* $\lambda = \lambda_0$;
- **(iv):** $\mu^k(\lambda_0) \neq 1$ *for* $k = 1, 2, 3, 4$.
- **:** *Then there is a smooth $\alpha-$dependent change of coordinate bringing F into the form*

$$F(\lambda, x) = G(\lambda, x) + O(\|x\|^5).$$

Consider a general map $F(\lambda, x)$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda) = \alpha(\lambda) + i\beta(\lambda)$ and $\bar{\mu}(\lambda) = \alpha(\lambda) - i\beta(\lambda)$ satisfying $\alpha(\lambda)^2 + \beta(\lambda)^2 = 1$ and $\beta(\lambda) \neq 0$.

By putting the linear part of such a map into Jordan Canonical form, we may assume F to have the following form near the origin

$$F(\lambda, x) = \begin{pmatrix} \alpha(\lambda) & -\beta(\lambda) \\ \beta(\lambda) & \alpha(\lambda) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1(\lambda, x_1, x_2) \\ g_2(\lambda, x_1, x_2) \end{pmatrix}.$$

Moreover, for all sufficiently small positive (negative) λ, F has an attracting (repelling) invariant circle if $a(\lambda_0) < 0$ ($a(\lambda_0) > 0$) respectively; and $a(\lambda_0)$ is given by the following formula:

$$a(\lambda_0) = Re\left[\frac{(1 - 2\mu(\lambda_0))\bar{\mu}^2(\lambda_0)}{1 - \mu(\lambda_0)}\gamma_{11}\gamma_{20}\right] + \frac{1}{2} \mid \gamma_{11} \mid^2 + \mid \gamma_{02} \mid^2 - Re(\bar{\mu}(\lambda_0)\gamma_{21}), \tag{8}$$

where

$$\gamma_{20} = \frac{1}{8}\left\{(g_1)_{x_1x_1} - (g_1)_{x_2x_2} + 2(g_2)_{x_1x_2} + i\left[(g_2)_{x_1x_1} - (g_2)_{x_2x_2} - 2(g_1)_{x_1x_2}\right]\right\};$$

$$\gamma_{11} = \frac{1}{4}\left\{(g_1)_{x_1x_1} + (g_1)_{x_2x_2} + i\left[(g_2)_{x_1x_1} + (g_2)_{x_2x_2}\right]\right\};$$

$$\gamma_{02} = \frac{1}{8}\left\{(g_1)_{x_1x_1} - (g_1)_{x_2x_2} - 2(g_2)_{x_1x_2} + i\left[(g_2)_{x_1x_1} - (g_2)_{x_2x_2} + 2(g_1)_{x_1x_2}\right]\right\};$$

$$\gamma_{21} = \frac{1}{8}\left\{(g_1)_{x_1x_1x_1} + (g_1)_{x_1x_2x_2} + (g_2)_{x_1x_1x_2} + (g_2)_{x_2x_2x_2} + i\left[(g_2)_{x_1x_1x_1} + (g_2)_{x_1x_2x_2} - (g_1)_{x_1x_1x_2} - (g_1)_{x_2x_2x_2}\right]\right\}.$$

We see that the equation

$$x_{n+1} = \alpha_1 x_n + \alpha_2 x_{n-1} + \beta_1 x_n \exp(-x_{n-1}^2) + \beta_2 x_{n-1} \exp(-x_{n-1}^2), \quad n = 0, 1, ...,$$

has the equilibrium points

$$\bar{x} = 0, \ \ \bar{x} = \sqrt{\ln(\frac{\beta_1 + \beta_2}{1 - (\alpha_1 + \alpha_2)})}, \ \ \bar{x} = -\sqrt{\ln(\frac{\beta_1 + \beta_2}{1 - (\alpha_1 + \alpha_2)})}.$$

5.1. **Naimark-Sacker bifurcation of fixed point** $\bar{x} = \sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}$**.** In order to apply Theorem 3 we make a change of variable $y_n = x_n - \bar{x}$. Then, transformed equation is given by

$$y_{n+1} = \alpha_1(y_n + \bar{x}) + \alpha_2(y_{n-1} + \bar{x}) + \beta_1(y_n + \bar{x}) \exp(-(y_{n-1} + \bar{x})^2)$$

$$+\beta_2(y_{n-1} + \bar{x}) \exp(-(y_{n-1} + \bar{x})^2) - \bar{x}. \tag{9}$$

By using the substitution $u_n = y_{n-1}$, $v_n = y_n$ we write Eq.(9) in the equivalent form:

$$\left\{ \begin{array}{c} u_{n+1} = v_n, \\ v_{n+1} = \alpha_1(v_n + \bar{x}) + \alpha_2(u_n + \bar{x}) + \beta_1(v_n + \bar{x}) \exp(-(u_n + \bar{x})^2) \\ +\beta_2(u_n + \bar{x}) \exp(-(u_n + \bar{x})^2) - \bar{x}. \end{array} \right. \tag{10}$$

Let F be the function defined by:

$$F(u, v) = \begin{pmatrix} v \\ \Big[\alpha_1(v + \bar{x}) + \alpha_2(u + \bar{x}) + \beta_1(v + \bar{x}) \exp(-(u + \bar{x})^2) \\ +\beta_2(u + \bar{x}) \exp(-(u + \bar{x})^2) - \bar{x}\Big]. \end{pmatrix}$$

Then F has the unique fixed point $(0, 0)$. The Jacobian matrix of F is given by

$$J_F(u, v) = \begin{pmatrix} 0 & 1 \\ p & q \end{pmatrix},$$

where

$$p = \alpha_2 - 2\beta_1(v + \bar{x})(u + \bar{x}) \exp(-(u + \bar{x})^2) + \beta_2[\exp(-(u + \bar{x})^2) - 2(u + \bar{x}) \exp(-(u + \bar{x})^2)],$$

$$q = \alpha_1 + \beta_1 \exp(-(u + \bar{x})^2).$$

At $(0, 0)$ $J_F(u, v)$ has the form

$$J_F(0, 0) = \begin{pmatrix} 0 & 1 \\ \alpha_2 + (\beta_2 - 2(\beta_1 + \beta_2)\bar{x}^2) \exp(-\bar{x}^2) & \alpha_1 + \beta_1 \exp(-\bar{x}^2) \end{pmatrix}. \tag{11}$$

The eigenvalues of (11) are $\mu_\pm(\beta_2)$ where

$$\mu_\pm(\beta_2) = \frac{r \pm i\sqrt{4s - r^2}}{2},$$

where

$$r = \alpha_1 + \beta_1 \exp(-\bar{x}^2),$$

$$s = [2(\beta_1 + \beta_2)\bar{x}^2 \exp(-\bar{x}^2)] - (\alpha_2 + \beta_2 \exp(-\bar{x}^2)).$$

Then we have that

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha_2 + (\beta_2 - 2(\beta_1 + \beta_2)\bar{x}^2) \exp(-\bar{x}^2) & \alpha_1 + \beta_1 \exp(-\bar{x}^2) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{12}$$

$$+\begin{pmatrix} f_1(\beta_2,u,v) \\ f_2(\beta_2,u,v) \end{pmatrix},$$

and

$$\begin{aligned} f_1(\beta_2,u,v) &= 0, \\ f_2(\beta_2,u,v) &= \alpha_1(v+\bar{x})+\alpha_2(u+\bar{x})+\beta_1(v+\bar{x})\exp(-(u+\bar{x})^2)+\beta_2(u+\bar{x})\exp(-(u+\bar{x})^2) \\ &\quad -[\alpha_2+(\beta_2-2(\beta_1+\beta_2)\bar{x}^2)\exp(-\bar{x}^2)]-(\alpha_1+\beta_1\exp(-\bar{x}^2))-\bar{x}. \end{aligned}$$

Let

$$2(\beta_1+\beta_2)\bar{x}^2\exp(-\bar{x}^2)=1+(\alpha_2+\beta_2\exp(-\bar{x}^2)),$$

we obtain

$$J_F(0,0)=\begin{pmatrix} 0 & 1 \\ -1 & \frac{\alpha_1\beta_2+\beta_1-\alpha_2\beta_1}{\beta_1+\beta_2} \end{pmatrix}.$$

The eigenvalues of $J_F(0,0)$ are $\mu(\beta_2^*)$ and $\bar{\mu}(\beta_2^*)$ where

$$\mu(\beta_2^*)=\frac{\alpha_1\beta_2+\beta_1-\alpha_2\beta_1+i\sqrt{4(\beta_1+\beta_2)^2-(\alpha_1\beta_2+\beta_1-\alpha_2\beta_1)^2}}{2(\beta_1+\beta_2)}.$$

The eigenvectors corresponding to $\mu(\beta_2^*)$ and $\bar{\mu}(\beta_2^*)$ are $v(\beta_2^*)$ and $\bar{v}(\beta_2^*)$ where

$$v(\beta_2^*)=(\frac{\alpha_1\beta_2+\beta_1-\alpha_2\beta_1-i\sqrt{4(\beta_1+\beta_2)^2-(\alpha_1\beta_2+\beta_1-\alpha_2\beta_1)^2}}{2(\beta_1+\beta_2)},1).$$

We have

$$\mu(\beta_2^*)=\frac{A+i\sqrt{B^2-A^2}}{B},$$

where

$$A=\alpha_1\beta_2+\beta_1-\alpha_2\beta_1,\quad B=2(\beta_1+\beta_2).$$

One can prove that

$$\begin{aligned} &\mid \mu(\beta_2^*)\mid=1, \\ \mu^2(\beta_2^*)&=-1+\frac{2iA\sqrt{4B^2-A^2}}{B^2}, \\ \mu^3(\beta_2^*)&=-\frac{AB+2(AB^2-A^3)}{B^3}+i\frac{(2A^2-B^2)\sqrt{B^2-A^2}}{B^3}, \\ \mu^4(\beta_2^*)&=\frac{B^4+4(A^4-A^2B^2)}{B^4}-i\frac{4A\sqrt{B^2-A^2}}{B^2}. \end{aligned}$$

From which follows that $\mu^k(\beta_2^*)\neq 1$ for $k=1,2,3,4$. Substituting $\beta_2=\beta_2^*$ and $\bar{x}$ into (12), one can get

$$F\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -1 & \frac{\alpha_1\beta_2+\beta_1-\alpha_2\beta_1}{\beta_1+\beta_2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}+\begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix},$$

and

$$\begin{aligned} h_1(u,v)&=f_1(\beta_2^*,u,v)=0 \\ h_2(u,v)&=f_2(\beta_2^*,u,v)=\alpha_1(v+\bar{x})+\alpha_2(u+\bar{x})+\beta_1(v+\bar{x})\exp(-(u+\bar{x})^2) \\ &\quad +\beta_2(u+\bar{x})\exp(-(u+\bar{x})^2)+u-\frac{\alpha_1\beta_2+\beta_1-\alpha_2\beta_1}{\beta_1+\beta_2}v-\bar{x}. \end{aligned}$$

Hence, for $\beta_2=\beta_2^*$ system (12) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -1 & \frac{\alpha_1\beta_2+\beta_1-\alpha_2\beta_1}{\beta_1+\beta_2} \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix}+\begin{pmatrix} h_1(u_n,v_n) \\ h_2(u_n,v_n) \end{pmatrix}. \tag{13}$$

Let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix},$$

where

$$P = \begin{pmatrix} \frac{\alpha_1\beta_2+\beta_1-\alpha_2\beta_1}{2(\beta_1+\beta_2)} & \frac{\sqrt{4(\beta_1+\beta_2)^2-(\alpha_1\beta_2+\beta_1-\alpha_2\beta_1)^2}}{2(\beta_1+\beta_2)} \\ 1 & 0 \end{pmatrix},$$

$$P^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{2(\beta_1+\beta_2)}{\sqrt{4(\beta_1+\beta_2)^2-(\alpha_1\beta_2+\beta_1-\alpha_2\beta_1)^2}} & -\frac{\alpha_1\beta_2+\beta_1-\alpha_2\beta_1}{\sqrt{4(\beta_1+\beta_2)^2-(\alpha_1\beta_2+\beta_1-\alpha_2\beta_1)^2}} \end{pmatrix}.$$

Then system (12) is equivalent to its normal form

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1\beta_2+\beta_1-\alpha_2\beta_1}{2(\beta_1+\beta_2)} & -\frac{\sqrt{4(\beta_1+\beta_2)^2-(\alpha_1\beta_2+\beta_1-\alpha_2\beta_1)^2}}{2(\beta_1+\beta_2)} \\ \frac{\sqrt{4(\beta_1+\beta_2)^2-(\alpha_1\beta_2+\beta_1-\alpha_2\beta_1)^2}}{2(\beta_1+\beta_2)} & \frac{\alpha_1\beta_2+\beta_1-\alpha_2\beta_1}{2(\beta_1+\beta_2)} \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + G \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix},$$

where

$$H \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix}.$$

Let

$$G \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \end{pmatrix} = P^{-1} H \left(P \begin{pmatrix} u \\ v \end{pmatrix} \right).$$

$$P \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1\beta_2+\beta_1-\alpha_2\beta_1}{2(\beta_1+\beta_2)} & \frac{\sqrt{4(\beta_1+\beta_2)^2-(\alpha_1\beta_2+\beta_1-\alpha_2\beta_1)^2}}{2(\beta_1+\beta_2)} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\alpha_1\beta_2+\beta_1-\alpha_2\beta_1}{2(\beta_1+\beta_2)} u + \frac{\sqrt{4(\beta_1+\beta_2)^2-(\alpha_1\beta_2+\beta_1-\alpha_2\beta_1)^2}}{2(\beta_1+\beta_2)} v \\ u \end{pmatrix} = \begin{pmatrix} \frac{A}{B} u + \frac{\sqrt{B^2-A^2}}{B} v \\ u \end{pmatrix},$$

$$H \left(P \begin{pmatrix} u \\ v \end{pmatrix} \right) = \begin{pmatrix} h_1(\frac{A}{B} u + \frac{\sqrt{B^2-A^2}}{B} v, u) \\ h_2(\frac{A}{B} u + \frac{\sqrt{B^2-A^2}}{B} v, u) \end{pmatrix} = \begin{pmatrix} 0 \\ h_2(\frac{A}{B} u + \frac{\sqrt{B^2-A^2}}{B} v, u) \end{pmatrix},$$

$$P^{-1} H \left(P \begin{pmatrix} u \\ v \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ \frac{B}{\sqrt{B^2-A^2}} & \frac{A}{\sqrt{B^2-A^2}} \end{pmatrix} \begin{pmatrix} 0 \\ h_2(\frac{A}{B} u + \frac{\sqrt{B^2-A^2}}{B} v, u) \end{pmatrix}.$$

Then

$$\begin{aligned} g_1(u,v) &= h_2(\frac{A}{B} u + \frac{\sqrt{B^2-A^2}}{B} v, u), \\ &= (\alpha_1 + \alpha_2 \frac{A}{B}) u + \frac{\sqrt{B^2-A^2}}{B} (\alpha_2+1) v + \beta_1 (u + C) \exp(-(\frac{A}{B} u + \frac{\sqrt{B^2-A^2}}{B} v + C)^2) \\ &\quad + \beta_2 (\frac{A}{B} u + \frac{\sqrt{B^2-A^2}}{B} v + C) \exp(-(\frac{A}{B} u + \frac{\sqrt{B^2-A^2}}{B} v + C)^2) + (\alpha_1 + \alpha_2 - 1) C, \end{aligned}$$

where

$$\bar{x} = c = \sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})},$$

$$g_2(u,v) = -\frac{A}{\sqrt{B^2-A^2}} g_1(u,v).$$

Other calculation gives

$$\frac{\partial^2 g_1(0,0)}{\partial u^2} = -\frac{4A}{B} C \exp(-C^2) \beta_1 - 2\frac{A^2}{B^2} C \exp(-C^2) [(\beta_1 + 3\beta_2) - 2C^2(\beta_1+\beta_2)],$$

$$\frac{\partial^2 g_1(0,0)}{\partial v^2} = -2\frac{(B^2-A^2)}{B^2} C \exp(-C^2) [(\beta_1 + 3\beta_2) - 2C^2(\beta_1+\beta_2)],$$

$$\frac{\partial^2 g_1(0,0)}{\partial uv} = -2\frac{A\sqrt{B^2-A^2}}{B^2}C\exp(-C^2)[(\frac{B}{A}-2C^2+1)\beta_1+(C-2C^2+2)\beta_2],$$

$$\frac{\partial^2 g_1(0,0)}{\partial u^3} = 2\frac{A^2}{B^2}\exp(-C^2)(-3+6C^2+4\frac{A}{B}C-4\frac{A}{B}C^3)\beta_1$$
$$+2\frac{A^3}{B^3}\exp(-C^2)(-3-4C^4+12C^2)\beta_2,$$

$$\frac{\partial^2 g_1(0,0)}{\partial u\partial v^2} = 2\frac{(B^2-A^2)}{B^2}\exp(-C^2)(-1+8\frac{A}{B}C^2-4\frac{A}{B}C^4)\beta_1$$
$$-4\frac{A(B^2-A^2)}{B^3}\exp(-C^2)(1+C-5C^2-C^3+2C^4)\beta_2,$$

$$\frac{\partial^2 g_1(0,0)}{\partial u^2\partial v} = 4\frac{A\sqrt{B^2-A^2}}{B^2}\exp(-C^2)(-1+2C^2+3\frac{A}{B}C^2-2\frac{A}{B}C^4)\beta_1$$
$$+2\frac{A^2\sqrt{B^2-A^2}}{B^3}\exp(-C^2)(12C^2-4C^4-3)\beta_2,$$

$$\frac{\partial^2 g_1(0,0)}{\partial v^3} = -\frac{4(B^2-A^2)^{\frac{3}{2}}}{B^3}\exp(-C^2)(-3+6C^2+4\frac{A}{B}C-4\frac{A}{B}C^2)\beta_1$$
$$+2\frac{(B^2-A^2)^{\frac{3}{2}}}{B^3}\exp(-C^2)(12C^2-4C^4-3)\beta_2,$$

and

$$\gamma_{20}(0,0) = -\frac{A^2}{2B^2}C\exp(-C^2)[(\frac{B^2}{A^2}-2\frac{B^2}{A^2}C^2)\beta_1+(1-C-2\frac{B^2}{A^2}C^2+3\frac{B^2}{A^2})\beta_2]$$
$$+i\frac{A^3}{2B^2\sqrt{B^2-A^2}}C\exp(-C^2)[(-\frac{B^2}{2A^2}+\frac{B^2}{A^2}C^2-2\frac{B^3}{A^3})\beta_1+(1-C-\frac{B^2}{A^2}C^2+\frac{B^2}{A^2}C+\frac{B^2}{2A^2})\beta_2],$$

$$\gamma_{11}(0,0) = C\exp(-C^2)[-\frac{A}{B}\beta_2-\frac{[(\beta_1+3\beta_2)-2C^2(\beta_1+\beta_2)]}{2}][1-\frac{iA}{\sqrt{B^2-A^2}}],$$

$$\gamma_{02}(0,0) = \frac{C\exp(-C^2)}{4}[(\frac{4A}{B}+\frac{8A^2}{B^2}C^2-\frac{4A^2}{B^2}+1-2C^2)\beta_1+(\frac{-10A^2}{B^2}+\frac{8A^2C^2}{B^2}-\frac{2A^2C}{B^2}+3-2C^2)\beta_2]$$
$$+i\frac{AC\exp(-C^2)}{4\sqrt{B^2-A^2}}[(\frac{-4A}{B}-\frac{4A^2}{B^2}+\frac{8A^2C^2}{B^2}+\frac{2B}{A}+3-6C^2)\beta_1+(\frac{-10A^2}{B^2}+\frac{8A^2C^2}{B^2}-\frac{2A^2C}{B^2}+7-6C^2+2C)\beta_2],$$

$$\gamma_{21}(0,0) = \frac{\exp(-C^2)}{4}\{\frac{A}{B}[(\frac{-B}{A}+6C^2-4C^4+4C^3-4C)\beta_1+(1-2C-2C^2+2C^3)\beta_2]$$
$$+\frac{2A^2}{B^2}[(2C^2-\frac{16AC^2}{B}+\frac{8AC}{B}-\frac{4AC^3}{B}+\frac{8AC^4}{B})\beta_1+(-\frac{2A}{B}+\frac{10AC^4}{B}-\frac{2AC^3}{B})\beta_2]\}$$
$$+i\frac{\exp(-C^2)}{4\sqrt{B^2-A^2}}\{\frac{A^3}{B^2}[(-6+10C^2+\frac{4AC}{B}+\frac{10AC^2}{B}-\frac{8AC^4}{B})\beta_1-(\frac{2AC}{B}-\frac{4A}{B}+\frac{14AC^2}{B}-\frac{2AC^3}{B}-\frac{4AC^4}{B})\beta_2]$$
$$-\frac{A^2}{B}[(14C^2-8C^4+\frac{4B}{A}C^2-\frac{3B}{A})\beta_1-(5-2C-22C^2-2C^3+8C^4)\beta_2]$$
$$-(\frac{1}{B}+\frac{A^2}{B^2}-\frac{2A}{B^2})[(C^2+2C-3C^3)\beta_1+(12C^2-4C^4-3)\beta_2]\},$$

since

$$Re\left(\mu(\bar{\beta}_2^*)\gamma_{21}(\beta_2^*)\right) = \frac{\exp(-C^2)}{4}\{\frac{A^2}{B^2}[(-8C^2+4C^4-\frac{4B}{A}C^2+\frac{2B}{A}-4C+4C^3)\beta_1-(5-2C-22C^2-2C^3+8C^4)\beta_2]$$
$$+\frac{2A^4}{B^4}[(7\frac{B}{A}C^2-5C^2+10C-4C^3+4C^4)\beta_1+(12C^4-7C^2-C^3-C)\beta_2]$$
$$-(\frac{1}{B^2}+\frac{A^2}{B^3}-\frac{2A}{B^3})[(C^2+2C-3C^3)\beta_1+(12C^2-4C^4-3)\beta_2]\}, \tag{14}$$

$$\left[\frac{(1-2\mu(\beta_2^*))\mu(\bar{\beta_2^*})}{1-\mu(\beta_2^*)}^2\right] = \frac{1}{2B^3}(3B^2-8A^3+6AB-2A^2B)$$
$$+i\frac{\sqrt{B+A}}{2B^3\sqrt{B-A}}(-B^3-8A^3+2AB^2+6A^2B),$$
$$\gamma_{11}\gamma_{20} = a_1+a_2+i(a_3+a_4),$$

where

$$a_1 = \frac{-A^2C^2\exp(-2C^2)}{4B^2}[(-\frac{2A}{B}-1+2C^2)\beta_1+(-3+2C^2)\beta_2][(-1+2C^2)\beta_1-(\frac{A^2}{B^2}-\frac{A^2C}{B^2}+3-2C^2)\beta_2]$$
$$a_2 = \frac{C^2\exp(-2C^2)A^4}{4(B^2-A^2)B^2}[(-\frac{2A}{B}-1+2C^2)\beta_1+(-3+2C^2)\beta_2][(-1+2C^2-\frac{4A}{B})\beta_1+(\frac{2A^2}{B^2}-\frac{2A^2C}{B^2}-2C^2+1+2C)\beta_2]$$
$$a_3 = i\frac{A^3C^2\exp(-2C^2)}{4B^2\sqrt{B^2-A^2}}[(-\frac{2A}{B}-1+2C^2)\beta_1+(-3+2C^2)\beta_2][(-1+2C^2)\beta_1-(\frac{A^2}{B^2}-\frac{A^2C}{B^2}+3-2C^2)\beta_2]$$
$$a_4 = i\frac{A^3C^2\exp(-2C^2)}{4B^2\sqrt{B^2-A^2}}[(-\frac{2A}{B}-1+2C^2)\beta_1+(-3+2C^2)\beta_2][(-1+2C^2-\frac{4A}{B})\beta_1+(\frac{2A^2}{B^2}-\frac{2A^2C}{B^2}-2C^2+1+2C)\beta_2],$$
$$Re\left[\frac{(1-2\mu(\beta_2^*))\mu(\bar{\beta_2^*})}{1-\mu(\beta_2^*)}^2\gamma_{11}\gamma_{20}\right] = \frac{1}{2B^3}(3B^2-8A^3+6AB-2A^2B)a_1+\frac{1}{2B^3}(3B^2-8A^3+6AB-2A^2B)a_2$$

$$-\frac{\sqrt{B+A}}{2B^3\sqrt{B-A}}(-B^3-8A^3+2AB^2+6A^2B)a_3-\frac{\sqrt{B+A}}{2B^3\sqrt{B-A}}(-B^3-8A^3+2AB^2+6A^2B)a_4 \tag{15}$$

$$\gamma_{11}(\beta_2^*)\bar{\gamma_{11}}(\beta_2^*) = \frac{B^2C^2\exp(-2C^2)}{4(B^2-A^2)}((1+2C^2)\beta_1+(\frac{A}{B}+3-2C^2)\beta_2)^2, \tag{16}$$

$$\gamma_{02}(\beta_2^*)\bar{\gamma_{02}}(\beta_2^*) = \frac{A^2C^2\exp(-2C^2)}{B^2 16}[(\frac{4B}{A}+\frac{B^2}{A^2}-\frac{2B^2C^2}{A^2}+8C^2-4)\beta_1+(\frac{3B^2}{A^2}-\frac{2B^2C^2}{A^2}+8C^2-2C-10)\beta_2]^2$$

(17)

$$+\frac{A^2C^2\exp(-2C^2)}{16(B^2-A^2)}[(\frac{-4A}{B}-\frac{4A^2}{B^2}+\frac{8A^2C^2}{B^2}+\frac{2B}{A}+3-6C^2)\beta_1+(\frac{-10B^2}{A^2}+\frac{8A^2C^2}{B^2}-\frac{2A^2C}{B^2}+2C-6C^2+7)\beta_2]^2$$

Then by using (14), (15), (16) and (17), one can get

$$a(\beta_2^*) = \frac{1}{2B^3}(3B^2-8A^3+6AB-2A^2B)(a_1+a_2)-\frac{\sqrt{B+A}}{2B^3\sqrt{B-A}}(-B^3-8A^3+2AB^2+6A^2B)(a_3+a_4)$$
$$+\frac{A^2C^2\exp(-2C^2)}{B^2 16}[(\frac{4B}{A}+\frac{B^2}{A^2}-\frac{2B^2C^2}{A^2}+8C^2-4)\beta_1+(\frac{3B^2}{A^2}-\frac{2B^2C^2}{A^2}+8C^2-2C-10)\beta_2]^2$$
$$+\frac{A^2C^2\exp(-2C^2)}{16(B^2-A^2)}[(\frac{-4A}{B}-\frac{4A^2}{B^2}+\frac{8A^2C^2}{B^2}+\frac{2B}{A}+3-6C^2)\beta_1+(\frac{-10B^2}{A^2}+\frac{8A^2C^2}{B^2}-\frac{2A^2C}{B^2}+2C-6C^2+7)\beta_2]^2$$
$$-\frac{\exp(-C^2)}{4}\{\frac{A^2}{B^2}[(-8C^2+4C^4-\frac{4B}{A}C^2+\frac{2B}{A}-4C+4C^3)\beta_1-(5-2C-22C^2-2C^3+8C^4)\beta_2]$$
$$+\frac{2A^4}{B^4}[(7\frac{B}{A}C^2-5C^2+10C-4C^3+4C^4)\beta_1+(12C^4-7C^2-C^3-C)\beta_2]$$
$$-(\frac{1}{B^2}+\frac{A^2}{B^3}-\frac{2A}{B^3})[(C^2+2C-3C^3)\beta_1+(12C^2-4C^4-3)\beta_2]\}$$
$$+\frac{B^2C^4\exp(-2C^2)}{4(B^2-A^2)}((1+2C^2)\beta_1+(\frac{A}{B}+3-2C^2)\beta_2)^2>0.$$

One can see that

$$\mid\mu(\beta_2)\mid^2 = \mu(\beta_2)\mu(\bar{\beta}_2) = 2(\beta_1+\beta_2)\bar{x}^2\exp(-\bar{x}^2)-(\alpha_2+\beta_2\exp(-\bar{x}^2))$$

$$= (1-(\alpha_1+\alpha_2))[2\ln(\frac{\beta_1+\beta_2}{(1-(\alpha_1+\alpha_2))} - \frac{\beta_2}{(\beta_1+\beta_2)}] - \alpha_2,$$

from which we obtain

$$\frac{d}{d\beta_2} \mid \mu(\beta_2) \mid_{\beta_2=\beta_2^*} = \frac{(1-(\alpha_1+\alpha_2))[\frac{2}{\beta_1+\beta_2} - \frac{\beta_1}{(\beta_1+\beta_2)^2}]}{2\sqrt{(1-(\alpha_1+\alpha_2))[2\ln(\frac{\beta_1+\beta_2}{(1-(\alpha_1+\alpha_2))} - \frac{\beta_2}{(\beta_1+\beta_2)}] - \alpha_2}}$$

$$= (1-(\alpha_1+\alpha_2))\frac{2\beta_2^*+\beta_1}{2(\beta_1+\beta_2^*)^2} > 0.$$

From the above analysis, we have the following lemma:

Lemma 3. *If* $4(\beta_1+\beta_2)^2 > (\alpha_1\beta_2+\beta_1-\alpha_2\beta_1)^2$ *then the positive equilibrium* $\bar{x} = \sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}$ *of Eq.* (1) *undergoes a Naimark- Sacker bifurcation when* $\beta_2 = \beta_2^*$.

5.2. **Naimark-Sacker bifurcation of fixed point** $\bar{x} = -\sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)}}$**.** When $\bar{x} = -\sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})} = -c$

Then

$$\begin{aligned}
g_1(u,v) &= h_2(\frac{A}{B}u + \frac{\sqrt{B^2-A^2}}{B}v, u),\\
&= (\alpha_1+\alpha_2\frac{A}{B})u + \frac{\sqrt{B^2-A^2}}{B}(\alpha_2+1)v + \beta_1(u-C)\exp(-(\frac{A}{B}u + \frac{\sqrt{B^2-A^2}}{B}v - C)^2)\\
&\quad +\beta_2(\frac{A}{B}u + \frac{\sqrt{B^2-A^2}}{B}v - C)\exp(-(\frac{A}{B}u + \frac{\sqrt{B^2-A^2}}{B}v - C)^2) - (\alpha_1+\alpha_2-1)C,
\end{aligned}$$

and

$$g_2(u,v) = -\frac{A}{\sqrt{B^2-A^2}}g_1(u,v).$$

Other calculation gives

$$\begin{aligned}
\frac{\partial^2 g_1(0,0)}{\partial u^2} &= \frac{4A}{B}C\exp(-C^2)\beta_1 + 2\frac{A^2}{B^2}C\exp(-C^2)[(\beta_1+3\beta_2) - 2C^2(\beta_1+\beta_2)],\\
\frac{\partial^2 g_1(0,0)}{\partial v^2} &= 2\frac{(B^2-A^2)}{B^2}C\exp(-C^2)[(\beta_1+3\beta_2) - 2C^2(\beta_1+\beta_2)],\\
\frac{\partial^2 g_1(0,0)}{\partial uv} &= 2\frac{A\sqrt{B^2-A^2}}{B^2}C\exp(-C^2)[(\frac{B}{A} - 2C^2+1)\beta_1 + (C-2C^2+2)\beta_2],\\
\frac{\partial^2 g_1(0,0)}{\partial u^3} &= 2\frac{A^2}{B^2}\exp(-C^2)(-3+6C^2-4\frac{A}{B}C+4\frac{A}{B}C^3)\beta_1\\
&\quad +2\frac{A^3}{B^3}\exp(-C^2)(-3-4C^4+12C^2)\beta_2,\\
\frac{\partial^2 g_1(0,0)}{\partial u\partial v^2} &= 2\frac{A(B^2-A^2)}{B^3}\exp(-C^2)[(-\frac{B}{A}+8C^2-4C^4)\beta_1\\
&\quad -(2-2C-10C^2+2C^3+4C^4)\beta_2],\\
\frac{\partial^2 g_1(0,0)}{\partial u^2\partial v} &= 4\frac{A\sqrt{B^2-A^2}}{B^2}\exp(-C^2)(-1+2C^2+3\frac{A}{B}C^2-2\frac{A}{B}C^4)\beta_1\\
&\quad +2\frac{A^2\sqrt{B^2-A^2}}{B^3}\exp(-C^2)(12C^2-4C^4-3)\beta_2,\\
\frac{\partial^2 g_1(0,0)}{\partial v^3} &= \frac{4(B^2-A^2)^{\frac{3}{2}}}{B^3}\exp(-C^2)(C^2-2C+2C^3)\beta_1\\
&\quad +2\frac{(B^2-A^2)^{\frac{3}{2}}}{B^3}\exp(-C^2)(12C^2-4C^4-3)\beta_2,
\end{aligned}$$

and

$$\gamma_{20}(0,0) = -\frac{A^2}{2B^2}C\exp(-C^2)[(\frac{B^2}{A^2} - 2\frac{B^2}{A^2}C^2)\beta_1 + (-1-C-2\frac{B^2}{A^2}C^2+3\frac{B^2}{A^2})\beta_2]$$

$$+i\frac{A^3}{2B^2\sqrt{B^2-A^2}}C\exp(-C^2)[(-\frac{B^2}{2A^2}+\frac{B^2}{A^2}C^2-2\frac{B^3}{A^3})\beta_1+(-1-C+\frac{B^2}{A^2}C^2+\frac{B^2}{A^2}C+\frac{B^2}{2A^2})\beta_2],$$

$$\gamma_{11}(0,0)=C\frac{1}{2}\exp(-C^2)[(\frac{2A}{B}+1-2C^2)\beta_1+(3-2C^2)\beta_2][1-\frac{iA}{\sqrt{B^2-A^2}}],$$

$$\gamma_{02}(0,0)=\frac{C\exp(-C^2)}{4}[(\frac{4A}{B}-\frac{8A^2}{B^2}C^2+\frac{4A^2}{B^2}-1+2C^2)\beta_1+(\frac{10A^2}{B^2}-\frac{8A^2C^2}{B^2}-\frac{2A^2C}{B^2}-3+2C^2)\beta_2]$$

$$-i\frac{AC\exp(-C^2)}{4\sqrt{B^2-A^2}}[(\frac{4A}{B}+\frac{4A^2}{B^2}-\frac{8A^2C^2}{B^2}-\frac{2B}{A}-3+6C^2)\beta_1+(\frac{10A^2}{B^2}-\frac{8A^2C^2}{B^2}-\frac{2A^2C}{B^2}-7+6C^2+2C)\beta_2],$$

$$\gamma_{21}(0,0)=\frac{A\exp(-C^2)}{4B}\{\frac{A^2}{B^2}[(8C^4+\frac{2B}{A}-16C^2)\beta_1+(-1-2C+2C^2+2C^3)\beta_2]$$

$$+(\frac{-B}{A}+6C^2-4C^3-4C^4+4C)\beta_1+(1+2C-2C^2-2C^3)\beta_2\}$$

$$+\frac{i}{\sqrt{B^2-A^2}}\{\frac{A^3}{B^2}[(\frac{-6B}{A}+\frac{10C^2B}{A}+12C^2-8C^4)\beta_1+(28C^2-8C^4+2C^3-7)\beta_2]$$

$$+A[(\frac{-3B}{A}+10C^2-8C^4+\frac{4C^2B}{A}+8C-8C^3)\beta_1+(1+2C-2C^2-2C^3+4C^4)\beta_2$$

$$-\frac{B^2}{A}[(2C^2-4C+4C^3)\beta_1+(12C^2-4C^4-3)\beta_2]\}\},$$

since

$$Re\left(\mu(\bar{\beta_2^*})\gamma_{21}(\beta_2^*)\right)=\frac{A\exp(-C^2)}{4B^2}\{\frac{A^3}{B^3}[(-4C^2+\frac{12B}{A}C^2-6\frac{B}{A})\beta_1+(4C^3-8C^4+32C^2-2C-8)\beta_2]$$

$$+A[(16C^2+\frac{4B}{A}C^2-\frac{4B}{A}-12C^3-12C^4+12C)\beta_1+(-4C^3+8C^4-4C^2+4C+2)\beta_2]$$

$$-\frac{B^2}{A}[(2C^2-4C+4C^3)\beta_1+(12C^2-4C^4-3)\beta_2]\}\}, \tag{18}$$

$$\left[\frac{(1-2\mu(\beta_2^*))\mu(\bar{\beta_2^*})^2}{1-\mu(\beta_2^*)}\right]=\frac{1}{2B^3}(3B^2-8A^3+6AB-2A^2B)$$

$$+i\frac{\sqrt{B+A}}{2B^3\sqrt{B-A}}(-B^3-8A^3+2AB^2+6A^2B),$$

$$\gamma_{11}\gamma_{20}=(\frac{-A^2C^2\exp(-2C^2)}{4B^2}+i\frac{A^3C^2\exp(-2C^2)}{4B^2\sqrt{B^2-A^2}})b_1-(\frac{A^2C^2\exp(-2C^2)}{4B^2(B^2-A^2)}+i\frac{A^3C^2\exp(-2C^2)}{4B^2\sqrt{B^2-A^2}})b_2,$$

$$b_1=[(\frac{B^2}{A^2}-2\frac{B^2}{A^2}C^2)\beta_1+(-1-C-2\frac{B^2}{A^2}C^2+3\frac{B^2}{A^2})\beta_2][(\frac{2A}{B}+1-2C^2)\beta_1+(3-2C^2)\beta_2],$$

$$b_2=[(-\frac{B^2}{2A^2}+\frac{B^2}{A^2}C^2-2\frac{B^3}{A^3})\beta_1+(-1-C+\frac{B^2}{A^2}C^2+\frac{B^2}{A^2}C+\frac{B^2}{2A^2})\beta_2][(\frac{2A}{B}+1-2C^2)\beta_1+(3-2C^2)\beta_2]$$

$$Re\left[\frac{(1-2\mu(\beta_2^*))\mu(\bar{\beta_2^*})^2}{1-\mu(\beta_2^*)}\gamma_{11}\gamma_{20}\right]=-\frac{1}{2B^3}(3B^2-8A^3+6AB-2A^2B)\frac{A^2C^2\exp(-2C^2)}{4B^2}b_1$$

$$-\frac{1}{2B^3}(3B^2-8A^3+6AB-2A^2B)\frac{A^2C^2\exp(-2C^2)}{4B^2(B^2-A^2)}b_2$$

$$-\frac{\sqrt{B+A}}{2B^3\sqrt{B-A}}(-B^3-8A^3+2AB^2+6A^2B)\frac{A^3C^2\exp(-2C^2)}{4B^2\sqrt{B^2-A^2}}b_1$$

$$+\frac{\sqrt{B+A}}{2B^3\sqrt{B-A}}(-B^3-8A^3+2AB^2+6A^2B)\frac{A^2C^2\exp(-2C^2)}{4B^2(B^2-A^2)}b_2, \tag{19}$$

(20) $$\gamma_{11}(\beta_2^*)\bar{\gamma_{11}}(\beta_2^*)=\frac{B^2C^2\exp(-2C^2)}{4(B^2-A^2)}[(\frac{2A}{B}+1-2C^2)\beta_1+(3-2C^2)\beta_2]^2,$$

$$\gamma_{02}(\beta_2^*)\bar{\gamma_{02}}(\beta_2^*)=\frac{C^2\exp(-2C^2)}{16}[(\frac{4A}{B}-\frac{8A^2}{B^2}C^2+\frac{4A^2}{B^2}-1+2C^2)\beta_1+(\frac{10A^2}{B^2}-\frac{8A^2C^2}{B^2}-\frac{2A^2C}{B^2}-3+2C^2)\beta_2]^2$$

(21) $$+\frac{A^2C^2\exp(-2C^2)}{16(B^2-A^2)}[(\frac{4A}{B}+\frac{4A^2}{B^2}-\frac{8A^2C^2}{B^2}-\frac{2B}{A}-3+6C^2)\beta_1+(\frac{10A^2}{B^2}-\frac{8A^2C^2}{B^2}-\frac{2A^2C}{B^2}-7+6C^2+2C)\beta_2]^2,$$

Then by using (18), (19), (20) and (21), one can get

$$\begin{aligned}
a(\beta_2^*)=&-\frac{1}{2B^3}(3B^2-8A^3+6AB-2A^2B)\frac{A^2C^2\exp(-2C^2)}{4B^2}b_1\\
&-\frac{1}{2B^3}(3B^2-8A^3+6AB-2A^2B)\frac{A^2C^2\exp(-2C^2)}{4B^2(B^2-A^2)}b_2\\
&-\frac{\sqrt{B+A}}{2B^3\sqrt{B-A}}(-B^3-8A^3+2AB^2+6A^2B)\frac{A^3C^2\exp(-2C^2)}{4B^2\sqrt{B^2-A^2}}b_1\\
&+\frac{\sqrt{B+A}}{2B^3\sqrt{B-A}}(-B^3-8A^3+2AB^2+6A^2B)\frac{A^2C^2\exp(-2C^2)}{4B^2(B^2-A^2)}b_2\\
&+\frac{B^2C^2\exp(-2C^2)}{8(B^2-A^2)}[(\frac{2A}{B}+1-2C^2)\beta_1+(3-2C^2)\beta_2]^2\\
&+\frac{C^2\exp(-2C^2)}{16}[(\frac{4A}{B}-\frac{8A^2}{B^2}C^2+\frac{4A^2}{B^2}-1+2C^2)\beta_1+(\frac{10A^2}{B^2}-\frac{8A^2C^2}{B^2}-\frac{2A^2C}{B^2}-3+2C^2)\beta_2]^2\\
&+\frac{A^2C^2\exp(-2C^2)}{16(B^2-A^2)}[(\frac{4A}{B}+\frac{4A^2}{B^2}-\frac{8A^2C^2}{B^2}-\frac{2B}{A}-3+6C^2)\beta_1+(\frac{10A^2}{B^2}-\frac{8A^2C^2}{B^2}-\frac{2A^2C}{B^2}-7+6C^2+2C)\beta_2]^2,\\
&-\frac{A\exp(-C^2)}{4B^2}\{\frac{A^3}{B^3}[(-4C^2+\frac{12B}{A}C^2-6\frac{B}{A})\beta_1+(4C^3-8C^4+32C^2-2C-8)\beta_2]\\
&+A[(16C^2+\frac{4B}{A}C^2-\frac{4B}{A}-12C^3-12C^4+12C)\beta_1+(-4C^3+8C^4-4C^2+4C+2)\beta_2]\\
&-\frac{B^2}{A}[(2C^2-4C+4C^3)\beta_1+(12C^2-4C^4-3)\beta_2]\}>0.
\end{aligned}$$

One can see that

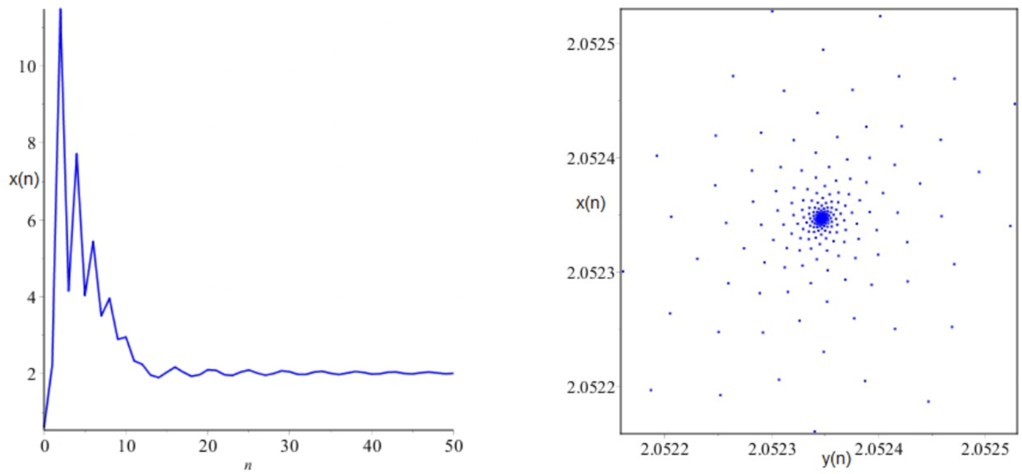
$$\begin{aligned}
\mid\mu(\beta_2)\mid^2=\mu(\beta_2)\mu(\bar{\beta}_2)&=2(\beta_1+\beta_2)\bar{x}^2\exp(-\bar{x}^2)-(\alpha_2+\beta_2\exp(-\bar{x}^2))\\
&=(1-(\alpha_1+\alpha_2))[2\ln(\frac{\beta_1+\beta_2}{(1-(\alpha_1+\alpha_2))}-\frac{\beta_2}{(\beta_1+\beta_2)}]-\alpha_2,
\end{aligned}$$

from that one can get

$$\begin{aligned}
\frac{d}{d\beta_2}\mid\mu(\beta_2)\mid_{\beta_2=\beta_2^*}&=\frac{(1-(\alpha_1+\alpha_2))[\frac{2}{\beta_1+\beta_2}-\frac{\beta_1}{(\beta_1+\beta_2)^2}]}{2\sqrt{(1-(\alpha_1+\alpha_2))[2\ln(\frac{\beta_1+\beta_2}{(1-(\alpha_1+\alpha_2))}-\frac{\beta_2}{(\beta_1+\beta_2)}]-\alpha_2}}\\
&=(1-(\alpha_1+\alpha_2))\frac{2\beta_2^*+\beta_1}{2(\beta_1+\beta_2^*)^2}>0.
\end{aligned}$$

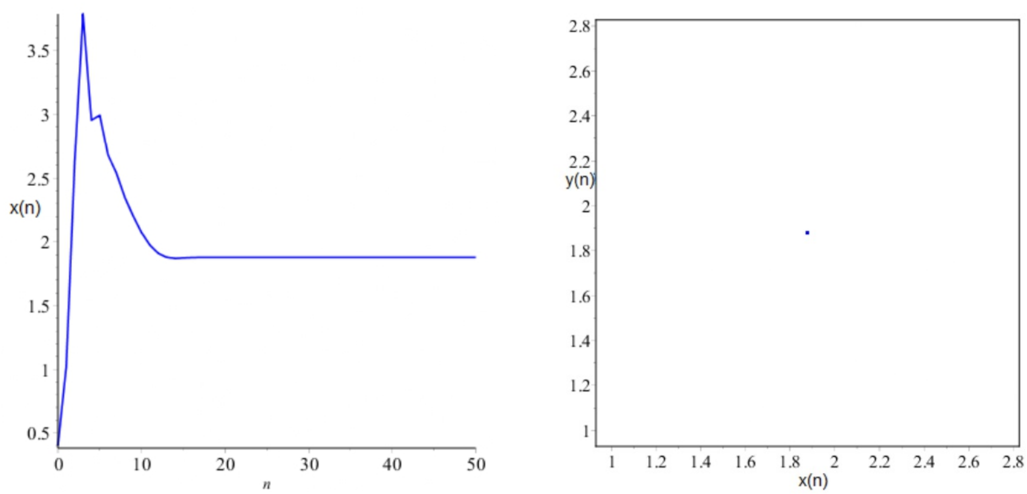
From the above analysis, we have the following lemma:

Lemma 4. *If* $4(\beta_1+\beta_2)^2>(\alpha_1\beta_2+\beta_1-\alpha_2\beta_1)^2$ *then the negative equilibrium* $\bar{x}=-\sqrt{\ln(\frac{\beta_1+\beta_2}{1-(\alpha_1+\alpha_2)})}$ *of Eq.* (1) *undergoes a Naimark- Sacker bifurcation when* $\beta_2=\beta_2^*$.



(a) (b)

FIG 1. The evolution of system (1) (a) The time series solution for local stability solution (b) The phase portrait of local stability solution.



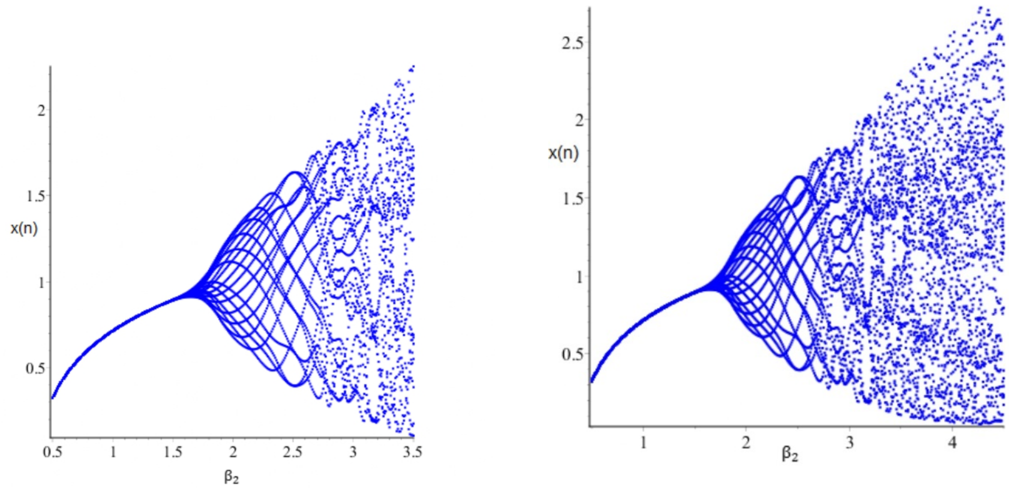
(a) (b)

FIG 2. The evolution of system (1) (a) The time series solution for local asymptotically stability (b) The phase portrait of local asymptotically stability.

6. NUMERICAL EXAMLPLES

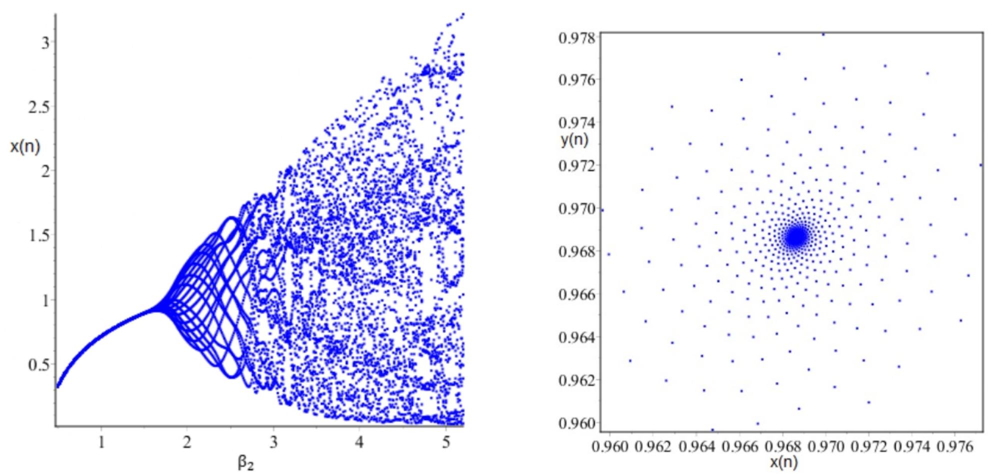
For confirming the above results, we present some numerical examples which represent the local stability diagrams, phase portraits and bifurcation diagrams which allow one to see where qualitative changes in the asymptotic solution occur. Such changes are termed bifurcations for a parameter range where the Naimark-Sacker bifurcation takes the place. All figures are drawn with maple.

Example 1. *Consider a special case of Eq.* (1) *by choosing* $\alpha_1 = 0.2$, $\alpha_2 = 0.6$, $\beta_1 = 5$ *and* $\beta_2 = 8.5$ *with initial conditions* $x_0 = 0.6, y_0 = 3.5$. *Hence Eq.* (1) *has a unique positive equilibrium which is locally asymptotic stable*



(a) **(b)**

FIG 3. (a) Bifurcation diagram of x_n with respect to β_2 (b) Bifurcation diagram of y_n with respect to β_2.

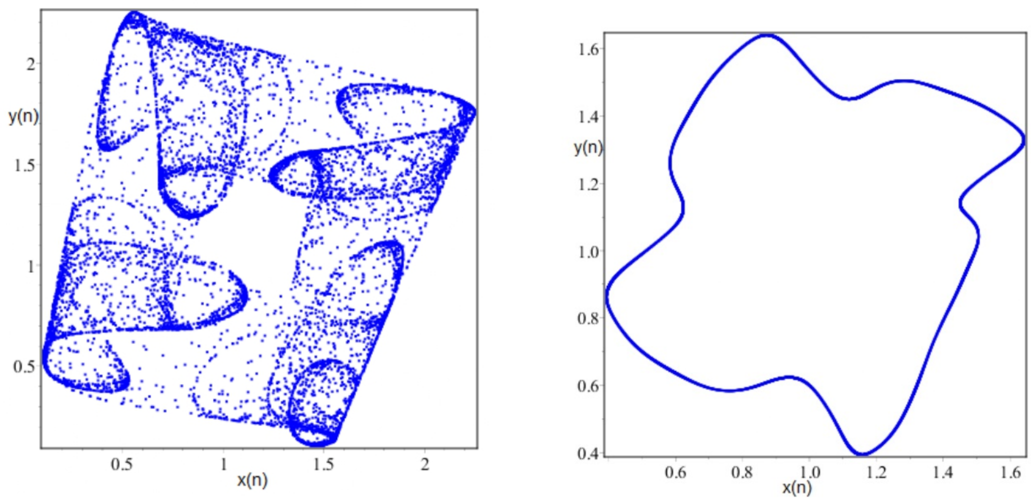


(a) **(b)**

FIG 4. (a) Bifurcation diagram of system (1) with respect to β_2(b) The phase portrait of system (1).

which is shown in Figure 1*a. Furthermore, the necessary and sufficient conditions of Lemma (2) for local asymptotic stability is satisfied and a phase portrait of system* (1) *is shown in Figure* 1*b.*

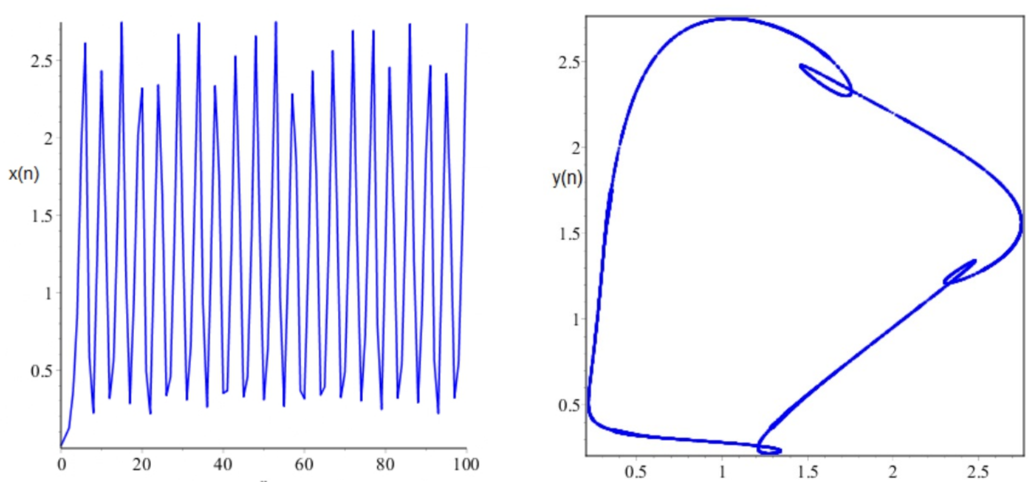
Example 2. *Let* $\alpha_1 = 0.5$, $\alpha_2 = 0.4$, $\beta_1 = 1.5$ *and* $\beta_2 = 1.9$ *with initial conditions* $x_0 = 0.4, y_0 = 0.1$. *Hence Eq.* (1) *has a unique positive equilibrium which is locally asymptotic stable which is presented in Figure* 2*a. Furthermore, the conditions of Lemma (2) for local asymptotic stability is satisfied and in Figure* 2*b a phase portrait of system* (1) *is shown .*



(a) **(b)**

FIG 5. (a) The strange attractor of system (2) (b) The strange attractor of system (2).

Example 3. *Let* $\alpha_1 = 0.1$, $\alpha_2 = 0$, $\beta_1 = 0.5$ *and* $0.5 \leq \beta_2 \leq 7$ *with initial conditions* $x_0 = 0.51, y_0 = 0.32$, *then system* (1) *undergoes Neimark-Sacker bifurcation and the bifurcation diagrams with respect to* β_2 *are presented in* (*Figure* 3*a*, *Figure* 3*b and Figure* 4*a*. *By fixing parametric values* $\alpha_1 = 0.1$, $\alpha_2 = 0$, $\beta_1 = 0.5$ *the phase portraits of system (2) while* β_2 *with different values* ($\beta_2 = 1.8$, 3.5 *and* 2.5) *are shown in* (*Figure* 4*b*, *Figure* 5*a and Figure* 5*b recpectivly, we see that Figure* 5*a and Figure* 5*b are chaotic attractors* (*strange attractors*) *of system (2).*



(a) **(b)**

FIG 6. (a) The time series solution for system (2) (b) The phase portrait of system (2).

Example 4. *Suppose that* $\alpha_1 = 0.1$, $\alpha_2 = 0.06$, $\beta_1 = 1.4$ *and* $\beta_2 = 2.5$ *with initial conditions* $x_0 = 0.01, y_0 = 0.02$, *then the unique positive equilibrium point of system (2) is unstable which is shown in Figure* 6*a and in Figure* 6*b we see phase portrait* (*chaotic attractor*)*of system (2).*

7. Conclusion

We have derived several kinds of results with important consequences for characterizing the dynamic behaviors of (2). The interesting system dynamics, including local stability, instability, bounded solutions, and the occurrence of a Neimark-Sacker bifurcation have been obtained. Our analysis can be used to precisely characterize the dynamic behaviors of numerous difference equations, all of which are realistic models of population dynamics.

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