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ON THE φ-ORDER OF GROWTH OF SOLUTIONS OF COMPLEX LINEAR DIFFERENTIAL EQUATIONS NEAR AN ESSENTIAL SINGULAR POINT

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ABSTRACT. In this article, we will take care about an interesting topic which is the study of the φ-order of growth of solutions of some given linear differential equations with analytic coefficients in $\overline{\mathbb{C}} - \{z_0\}$, where $z_0 \in \mathbb{C}$ represents an essential singularity. What we'll do in this paper is a generalization of the work of Long and Zeng by introducing the concept of the φ–order of growth near an essential singularity.

1. INTRODUCTION AND MAIN RESULTS

For $k \geq 2$ a positive integer, consider the following complex linear differential equation

$$f^{(k)} + A_{k-1}(z)\, f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

where the coefficients are analytic in some complex domain. Since it's hard to find some general forms for the solutions of (1.1), many searchers are interested on the study of the behavior of such solutions and specially the notion of the growth. The strongest tool they used for establishing their results is the Nevanlinna theory which can be found in [8], [10], [17] and [24].

In [12,13], Juneja, Kapoor and Bajpai have investigated some properties of entire functions of $[p, q]$-order and obtained some results about their growth. In order to maintain accordance with general definitions of the entire function f of iterated p-order [16], Liu-Tu-Shi in [19] gave a minor modification of the original definition of the $[p, q]$-order given in [12,13]. With this new concept of $[p, q]$-order, Liu, Tu and Shi [19] have considered equation (1.1) with entire coefficients and obtained different results concerning the growth of their solutions. After that, several authors used this new concept to investigate the growth of solutions in the complex plane, in the punctured plane $\bar{\mathbb{C}} - z_0$ (here z_0 represent a singular point) and in the unit disc [1,2,6,18,20].

In [5], Chyzhykov and Semochko showed that both definitions of iterated order and of $[p, q]$-order have the disadvantage that they do not cover arbitrary growth, i.e., there exist entire or meromorphic functions of infinite $[p, q]$-order and $p-$th iterated order for arbitrary $p \in \mathbb{N}$, i.e., of infinite degree, see Example 1.4 in [5]. They used more general scale, called the φ-order (see [5,23]). In recent times, the concept of φ-order is used to study the growth of solutions of complex differential equations which extend and improve many previous results (see [3,4,14,21].

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As in [20] Long and Zeng gave a generalization for the work of Fettouch and Hamouda (see [7]) by introducing the $[p,q]-$ order near an essential singular point. So, we find it very interesting to generalize the work done in [20] by introducing the concept of the φ-order near an essential singular point, that is the coefficients of (1.1) are all analytic in $\overline{\mathbb{C}}-\{z_0\}$.
In order to develop our study, we firstly recall some related notations. Let f be a meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$, where $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ is the whole extended complex plane, $z_0\in\mathbb{C}$ is some essential singularity. The Nevanlinna fundamentals are the most important step here. For that reason, we dedicated this section to fully developing the theory for a function with singular point z_0. Define the counting function of f near z_0 by the following formula

$$N_{z_0}(r,f)=-\int_{\infty}^{r}\frac{n(t,f)-n(\infty,f)}{t}dt-n(\infty,f)\log r,$$

where $n\,(t,f)$ denote the number of poles of f in the region $\{z\in\mathbb{C}: t\leq |z-z_0|\}\cup\{\infty\}$ counting its multiplicities, we also define the proximity function near z_0 by

$$m_{z_0}(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(z_0-re^{i\phi})|d\phi.$$

Summing up together, the characteristic function of f near z_0 will be

$$T_{z_0}(r,f)=m_{z_0}(r,f)+N_{z_0}(r,f).$$

Definition 1.1. Let f be a nonconstant meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$. The function f is called a transcendental or admissible meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$ provided that

$$\lim_{r\to 0}\frac{T_{z_0}(r,f)}{\log\frac{1}{r}}=+\infty$$

and f is a rational function in $\overline{\mathbb{C}}-\{z_0\}$ provided that

$$\liminf_{r\to 0}\frac{T_{z_0}(r,f)}{\log\frac{1}{r}}<+\infty.$$

Recently, Chyzhykov and Semochko [5] have given general definition of growth for an entire function in the complex plane by introducing a new class of functions. So, as in [5], let ϕ be the class of positive and unbounded increasing functions φ on $[1,+\infty)$ such that $\varphi(e^t)$ is slowly growing, i.e.,

$$\forall c>0:\lim_{t\to+\infty}\frac{\varphi(e^{ct})}{\varphi(e^t)}=1.$$

Here some useful properties of a function $\varphi\in\phi$.

Proposition 1.2. *([5]) If $\varphi\in\phi$, then the following hold:*

$$(i)\ \forall\delta>0:\lim_{x\to+\infty}\frac{\log\varphi^{-1}((1+\delta)x)}{\log\varphi^{-1}(x)}=+\infty,$$

$$(ii)\ \forall m>0,k\geq 0:\lim_{x\to+\infty}\frac{\varphi^{-1}(\log x^m)}{x^k}=+\infty,$$

$$(iii)\ \forall c>0,\ \varphi(ct)\leq\varphi(t^c)\leq(1+o(1))\varphi(t),\quad t\longrightarrow+\infty.$$

Definition 1.3. ([5]) Let φ be an increasing unbounded function on $[1,+\infty)$. The φ–orders of a meromorphic function f are defined by

$$\rho_\varphi^0(f):=\limsup_{r\to+\infty}\frac{\varphi\left(e^{T(r,f)}\right)}{\log r},\qquad \rho_\varphi^1(f):=\limsup_{r\to+\infty}\frac{\varphi(T(r,f))}{\log r}.$$

If f is an entire function, then the φ–orders are defined by

$$\tilde{\rho}^0_\varphi(f) := \limsup_{r\to+\infty} \frac{\varphi(M(r,f))}{\log r}, \qquad \tilde{\rho}^1_\varphi(f) := \limsup_{r\to+\infty} \frac{\varphi(\log M(r,f))}{\log r}.$$

Proposition 1.4. *([5]) Let $\varphi \in \Phi$ and f be an entire function. Then*

$$\rho^i_\varphi(f) = \tilde{\rho}^i_\varphi(f),\ i = 0, 1.$$

Below, we will define the φ–order of growth for a meromorphic function of f near a singular point.

Definition 1.5. Let φ be an increasing and unbounded function on $[1, +\infty)$. Then, the φ–orders of the growth of a meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ are given by

$$\rho^0_\varphi\left(f, z_0\right) = \limsup_{r\to 0} \frac{\varphi\left(e^{T_{z_0}(r,f)}\right)}{\log \frac{1}{r}},\ \rho^1_\varphi\left(f, z_0\right) = \limsup_{r\to 0} \frac{\varphi\left(T_{z_0}(r,f)\right)}{\log \frac{1}{r}}.$$

If f is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$, then the φ–orders are defined by

$$\widetilde{\rho}^0_\varphi(f, z_0) = \limsup_{r\to 0} \frac{\varphi(M_{z_0}(r,f))}{\log \frac{1}{r}},\ \widetilde{\rho}^1_\varphi(f, z_0) = \limsup_{r\to 0} \frac{\varphi(\log M_{z_0}(r,f))}{\log \frac{1}{r}},$$

here $M_{z_0}(r, f) = \max\{|f(z)| : |z - z_0| = r\}$.

Remark 1.6. A motivational observation of the above definition is that $\varphi(r) = \log\log r \in \phi$, it's also obvious that $\widetilde{\rho}^i_\varphi\left(f, z_0\right) = \rho^i_\varphi\left(f, z_0\right)$ $(i = 1, 2)$ for an analytic function in $\overline{\mathbb{C}} - \{z_0\}$. In indeed, by Lemma 2.2 in [7], we know that if f is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$ and $g(w) = f(z_0 - \frac{1}{w})$, then $g(w)$ is an entire function in $\mathbb{C}$ and we have $T(R, g) = T_{z_0}(r, f)$, where $R = \frac{1}{r}$. So, by using Proposition 1.4, we get the conclusion.

Definition 1.7. Let φ be an increasing and unbounded function on $[1, +\infty)$. Then, the φ–types of the growth of an analytic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\widetilde{\rho}^0_\varphi(f, z_0) \in (0, +\infty)$ and $\widetilde{\rho}^1_\varphi(f, z_0) \in (0, +\infty)$ are defined by

$$\widetilde{\tau}^0_\varphi\left(f, z_0\right) = \limsup_{r\to 0} \frac{\exp\left\{\varphi\left(M_{z_0}\left(r, f\right)\right)\right\}}{\frac{1}{r^{\widetilde{\rho}^0_\varphi(f,z_0)}}},$$

$$\widetilde{\tau}^1_\varphi\left(f, z_0\right) = \limsup_{r\to 0} \frac{\exp\left\{\varphi\left(\log M_{z_0}\left(r, f\right)\right)\right\}}{\frac{1}{r^{\widetilde{\rho}^1_\varphi(f,z_0)}}}.$$

Recently, Long and Zeng have investigated the $[p, q]$-order of growth of solutions of equation (1.1) and obtained some estimations of $[p, q]$-order of growth of solutions of such equation which is a generalization of previous results from Fettouch and Hamouda. Before stating the results of Long and Zeng, we give here the definitions of the $[p, q]$-order and the $[p, q]$-type of a meromorphic function near a singular point.

Definition 1.8. ([20]) Let p, q be two integers such that $p \geq q \geq 1$, and let f be a meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$. Then, the $[p, q]$ –order of growth is defined by

$$\rho_{[p,q],T}(f, z_0) = \limsup_{r\longrightarrow 0} \frac{\log^+_p T_{z_0}(r,f)}{\log_q \frac{1}{r}}.$$

If f is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$, then the $[p, q]$ –order of growth is defined by

$$\rho_{[p,q],M}(f, z_0) = \limsup_{r\longrightarrow 0} \frac{\log^+_{p+1} M_{z_0}(r,f)}{\log_q \frac{1}{r}},$$

where $M_{z_0}(r, f) = \max\{|f(z)| : |z - z_0| = r\}$.

Remark 1.9. ([20]) Suppose that f is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$. Then, by using Lemma 2.2 in [7], we get $\rho_{[p,q],M}(f, z_0) = \rho_{[p,q],T}(f, z_0)$. Therefore, in the sequel, we denote $\rho_{[p,q]}(f, z_0) = \rho_{[p,q],M}(f, z_0) = \rho_{[p,q],T}(f, z_0)$.

Definition 1.10. ([20]) Let p,q be two integers such that $p \geq q \geq 1$, and let f be a meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\rho = \rho_{[p,q]}(f, z_0) \in (0, \infty)$. Then, the $[p, q]-$type of f is defined by

$$\tau_{[p,q]}(f, z_0) = \limsup_{r \to 0} \frac{\log^{+}_{p-1} T_{z_0}(r, f)}{(\log_{q-1} \frac{1}{r})^{\rho}}.$$

If f is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\rho = \rho_{[p,q]}(f, z_0) \in (0, \infty)$, then the $[p, q]-$type of f is defined by

$$\tau_{[p,q],M}(f, z_0) = \limsup_{r \to 0} \frac{\log^{+}_{p} M_{z_0}(r, f)}{(\log_{q-1} \frac{1}{r})^{\rho}}.$$

Theorem 1.11. *([20]) Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying $\max\left\{\rho_{[p,q]}\left(A_j, z_0\right) : j \neq 0\right\} < \rho_{[p,q]}\left(A_0, z_0\right) < \infty$. Then, every nontrivial solution f of (1.1) that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies $\rho_{[p+1,q]}(f, z_0) = \rho_{[p,q]}(A_0, z_0)$.*

Theorem 1.12. *([20]) Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying the following conditions*

$$\max\left\{\rho_{[p,q]}(A_j, z_0) : j \neq 0\right\} \leq \rho_{[p,q]}(A_0, z_0) < \infty,$$

$$\max\left\{\tau_{[p,q],M}(A_j, z_0) : \rho_{[p,q],M}(A_j, z_0) = \rho_{[p,q]}(A_0, z_0) > 0\right\} < \tau_{[p,q],M}(A_0, z_0) < \infty.$$

Then, every nontrivial solution f of (1.1) that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies

$$\rho_{[p+1,q]}(f, z_0) = \rho_{[p,q]}(A_0, z_0).$$

Theorem 1.13. *([20]) Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying*

$$\max\left\{\rho_{[p,q]}(A_j, z_0) : j \neq s\right\} < \rho_{[p,q]}\left(A_s, z_0\right) < \infty.$$

Then, every nontrivial solution f of (1.1) that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies

$$\rho_{[p+1,q]}(f, z_0) \leq \rho_{[p,q]}\left(A_s, z_0\right) \leq \rho_{[p,q]}(f, z_0).$$

Here is the full generalization of the work of Long and Zeng given in [20] by using the concept of the φ-order. The following theorem seems like to be a classical version that describes the impact of A_0.

Theorem 1.14. *Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$, all together satisfying $\max\left\{\widetilde{\rho}^{0}_{\varphi}\left(A_j, z_0\right) : j \neq 0\right\} < \widetilde{\rho}^{0}_{\varphi}\left(A_0, z_0\right) < \infty$. Then, every nontrivial solution f of (1.1) that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies $\widetilde{\rho}^{1}_{\varphi}(f, z_0) = \widetilde{\rho}^{0}_{\varphi}(A_0, z_0)$.*

The following theorem discusses the case when A_0 still a dominant coefficient but not the only one.

Theorem 1.15. *Let $A_0(z), A_1(z), ..., A_{k-1}(z)$be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ all together satisfying the following conditions*

$$\max\left\{\widetilde{\rho}^{0}_{\varphi}(A_j, z_0) : j \neq 0\right\} \leq \widetilde{\rho}^{0}_{\varphi}(A_0, z_0) < \infty,$$

$$\max\left\{\widetilde{\tau}^{0}_{\varphi}(A_j, z_0) : \widetilde{\rho}^{0}_{\varphi}(A_j, z_0) = \widetilde{\rho}^{0}_{\varphi}(A_0, z_0) > 0\right\} < \widetilde{\tau}^{0}_{\varphi}(A_0, z_0) < \infty.$$

Then, every nontrivial solution f of (1.1) that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies

$$\widetilde{\rho}^{1}_{\varphi}(f, z_0) = \widetilde{\rho}^{0}_{\varphi}(A_0, z_0).$$

For the last theorem, we suppose that the dominant coefficient runs over the set $\{0, 1, 2, ..., k-1\}$.

Theorem 1.16. *Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$. Suppose there exists an integer $s,\ 0 \leq s \leq k-1$ such that the following condition holds*

$$\max\left\{\widetilde{\rho}_{\varphi}^{0}(A_j, z_0) : j \neq s\right\} < \widetilde{\rho}_{\varphi}^{0}\left(A_s, z_0\right) < \infty.$$

Then, every transcendental solution f of (1.1) *that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies*

$$\widetilde{\rho}_{\varphi}^{1}(f, z_0) \leq \widetilde{\rho}_{\varphi}^{0}\left(A_s, z_0\right) \leq \widetilde{\rho}_{\varphi}^{0}(f, z_0).$$

Remark 1.17. The condition that f is analytic in $\overline{\mathbb{C}} - \{z_0\}$ is necessary. The following example shows that there exists a solution f of (1.1) such that $f(z)$ is not analytic in $\overline{\mathbb{C}} - \{z_0\}$ provided that all coefficients $A_j(z)$ $(j = 0, ..., k-1)$ of (1.1) are analytic in $\overline{\mathbb{C}} - \{z_0\}$. For instance, we consider the equation

$$f'' + \exp_2\left\{\frac{1}{z_0 - z}\right\} f' + \frac{2}{z_0 - z}\left(\exp_2\left\{\frac{1}{z_0 - z}\right\} - \frac{1}{z_0 - z}\right) f = 0. \tag{1.2}$$

The function $f(z) = (z_0 - z)^2$ solves (1.2), and $f(z)$ is not analytic in $\overline{\mathbb{C}} - \{z_0\}$. So, in our results, we suppose always that $f(z)$ is analytic in $\overline{\mathbb{C}} - \{z_0\}$.

2. Preliminary results

Now, we are going to focus on the main preliminaries needed for establishing the proofs of our results. We firstly clarify some notations. Denote, the logarithmic measure of a set $E \subset (0, 1)$ by

$$m_l(E) = \int_E \frac{dt}{t}.$$

We also denote by $\nu_g(r)$ the central index of an entire function g in $\mathbb{C}$, for more properties, see ([11], p. 33-35). Finally, denote the central index of an analytic function f in $\overline{\mathbb{C}} - \{z_0\}$ by $\nu_{z_0}(r, f)$, reader may check ([9], p. 996).

Lemma 2.1 ([9], Theorem 8)**.** *Let f be nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$. Then, there exists a set $E_0 \subset (0, 1)$ that has finite logarithmic measure, such that for all $j = 0, 1, \ldots, k$, we have*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{\nu_{z_0}(r, f)}{z_0 - z_r}\right)^j (1 + o(1)),$$

as $r \longrightarrow 0$, $r \notin E_0$, where z_r is a point in the circle $|z - z_0| = r$ that satisfies $|f(z_r)| = \max\{|f(z)| : |z - z_0| = r\}$.

Lemma 2.2 ([20], Lemma 2.5)**.** *Let $g : (0, 1) \to \mathbb{R}$, $h : (0, 1) \to \mathbb{R}$ be monotone decreasing functions such that $g(r) \geq h(r)$ possibly outside an exceptional set $E_1 \subset (0, 1)$ that has finite logarithmic measure. Then, for any given $\beta > 1$, there exists a constant $0 < r_0 < 1$ such that for all $r \in (0, r_0)$, we have $g(r^{\beta}) \geq h(r)$.*

Lemma 2.3 ([14])**.** *Let $\varphi \in \Phi$ and f be an entire function. Then, we have*

$$\widetilde{\rho}_{\varphi}^{1}(f) = \limsup_{r \to +\infty} \frac{\varphi(\nu_f(r))}{\log r}.$$

Lemma 2.4. *Let f be a nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$. For a function $\varphi \in \phi$ one has*

$$\widetilde{\rho}_{\varphi}^{1}(f, z_0) = \limsup_{r \to 0} \frac{\varphi(\nu_{z_0}(r, f))}{\log \frac{1}{r}}.$$

Proof. Set $g(w) = f\left(z_0 - \frac{1}{w}\right)$. As the function g is entire ([9], Remark 7), it turns out that

$$\nu_{z_0}(r, f) = \nu_g(R),\ R = \frac{1}{r}.$$

By Lemma 2.3, we have

$$\widetilde{\rho}_{\varphi}^{1}(g) = \limsup_{R \to +\infty} \frac{\varphi(\nu_g(R)))}{\log R}.$$

That gives

$$\widetilde{\rho}_{\varphi}^{1}(f, z_0) = \widetilde{\rho}_{\varphi}^{1}(g) = \limsup_{r\to 0} \frac{\varphi(\nu_{z_0}(r, f))}{\log \frac{1}{r}}.$$

□

Lemma 2.5 ([7]). *Let f be a nonconstant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$, let $\gamma > 1$, $\varepsilon > 0$ be given real constants and $k \in \mathbb{N}$. Then there exist a set $E_2 \subset (0, r_1]$, $r_1 \in (0, 1)$ having finite logarithmic measure and a constant $\lambda > 0$ that depends on γ and k such that for all $|z - z_0| = r \in (0, r_1] \setminus E_2$, we have*

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq \lambda \left[\frac{1}{r^2} T_{z_0}(\frac{r}{\gamma}, f) \log T_{z_0}(r, f) \right]^k.$$

Lemma 2.6. *Let f be a nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\widetilde{\rho}_{\varphi}^{0}(f, z_0) = \rho$. Then, there exists a set $E_3 \subset (0, 1)$ with $m_l(E_3) = +\infty$ such that for all $|z - z_0| = r \in E_3$, we have*

$$\lim_{r\to 0} \frac{\varphi(M_{z_0}(r, f))}{\log \frac{1}{r}} = \rho.$$

Proof. By Definition 1.5, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to 0 satisfying $r_{n+1} < \frac{n}{n+1} r_n$ and

$$\lim_{n\to +\infty} \frac{\varphi(M_{z_0}(r_n, f))}{\log \frac{1}{r_n}} = \rho.$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for every $r \in \left[\frac{n}{n+1} r_n, r_n\right]$, we get

$$\frac{\varphi(M_{z_0}(r_n, f))}{\log \frac{1}{\frac{n}{n+1} r_n}} \leq \frac{\varphi(M_{z_0}(r, f))}{\log \frac{1}{r}} \leq \frac{\varphi(M_{z_0}(\frac{n}{n+1} r_n, f))}{\log \frac{1}{r_n}}.$$

Therefore, since

$$\lim_{n\to +\infty} \frac{\varphi(M_{z_0}(r_n, f))}{\log \frac{1}{\frac{n}{n+1} r_n}} = \lim_{n\to +\infty} \frac{\varphi(M_{z_0}(\frac{n}{n+1} r_n, f))}{\log \frac{1}{r_n}} = \rho,$$

then yielding

$$\lim_{r\to 0} \frac{\varphi(M_{z_0}(r, f))}{\log \frac{1}{r}} = \rho$$

for all $r \in \left[\frac{n}{n+1} r_n, r_n\right]$. By setting $E_3 = \bigcup_{n=n_0}^{+\infty} \left[\frac{n}{n+1} r_n, r_n\right]$, the conclusion follows since E_3 fulfills

$$m_l(E_3) = \sum_{n=n_0}^{+\infty} \int_{\frac{n}{n+1} r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_0}^{+\infty} \log(1 + \frac{1}{n}) = +\infty.$$

□

By analogous logic, we establish the same lemma with the following limit below.

Lemma 2.7. *Let f be a nonconstant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\rho_{\varphi}^{0}(f, z_0) = \rho$. Then, there exists a set $E_4 \subset (0, 1)$ with $m_l(E_4) = +\infty$ such that for all $|z - z_0| = r \in E_4$, we have*

$$\lim_{r\to 0} \frac{\varphi(e^{T_{z_0}(r, f)})}{\log \frac{1}{r}} = \rho.$$

Lemma 2.8. *Let f be a nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\widetilde{\rho}_{\varphi}^{0}(f, z_0) = \rho \in (0, \infty)$ and $\widetilde{\tau}_{\varphi}^{0}(f, z_0) = \tau \in (0, \infty)$. Then, for any given $\beta \in (0, \tau)$, there exists a set $E_5 \subset (0, 1)$ of infinite logarithmic measure such that for $|z - z_0| = r \in E_5$, we have*

$$\varphi(M_{z_0}(r, f)) > \log\left(\frac{\beta}{r^{\rho}}\right).$$

Proof. By Definition 1.7, there exists a sequence $\{r_m\}_{m=1}^{\infty}$ tending to 0 satisfying $r_{m+1} < \frac{m}{m+1} r_m$ and

$$\lim_{m\to+\infty} \frac{\exp\{\varphi(M_{z_0}(r_m, f))\}}{\frac{1}{r_m^{\rho}}} = \tau.$$

So, there exists a positive integer m_0 such that for all $m \geq m_0$ and for any given $0 < \varepsilon < \tau - \beta$, we have

$$\varphi(M_{z_0}(r_m, f)) \geq \log\left(\frac{\tau - \varepsilon}{r_m^{\rho}}\right). \tag{2.1}$$

Since

$$\lim_{m\to+\infty} \left(\frac{m}{m+1}\right)^{\rho} = 1,$$

then for any given $\beta < \tau - \varepsilon$, there exists a positive integer m_1 such that for all $m \geq m_1$, we have

$$\left(\frac{m}{m+1}\right)^{\rho} > \frac{\beta}{\tau - \varepsilon}. \tag{2.2}$$

Take $m \geq m_2 = \max\{m_1, m_0\}$. By (2.1) and (2.2), for any $r \in \left[\frac{m}{m+1} r_m, r_m\right]$

$$\varphi(M_{z_0}(r, f)) \geq \varphi(M_{z_0}(r_m, f)) \geq \log\left(\frac{\tau - \varepsilon}{r_m^{\rho}}\right)$$

$$\geq \log\left(\frac{\tau - \varepsilon}{r^{\rho}}\left(\frac{m}{m+1}\right)^{\rho}\right) > \log\left(\frac{\beta}{r^{\rho}}\right).$$

Set $E_5 = \bigcup_{m=m_2}^{+\infty} \left[\frac{m}{m+1} r_m, r_m\right]$. Then there holds

$$m_l(E_5) = \sum_{m=m_2}^{+\infty} \int_{\frac{m}{m+1} r_m}^{r_m} \frac{dt}{t} = \sum_{m=m_2}^{+\infty} \log(1 + \frac{1}{m}) = +\infty.$$

□

Lemma 2.9. *Let $A_j(z)$ $(j = 0, 1, ..., k-1)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying all together, the inequality*

$$\widetilde{\rho}_{\varphi}^{0}(A_j, z_0) \leq \rho < \infty,\ j = 0, 1, ..., k-1.$$

Then every nontrivial solution f of (1.1) that is analytic in $\overline{\mathbb{C}} - \{z_0\}$ satisfies $\widetilde{\rho}_{\varphi}^{1}(f, z_0) \leq \rho$.

Proof. Suppose that f $(\not\equiv 0)$ is a solution of equation (1.1) which is analytic in $\overline{\mathbb{C}} - \{z_0\}$. The equation (1.1) implies

$$\left|\frac{f^{(k)}}{f}\right| \leq |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right| + |A_0(z)|. \tag{2.3}$$

By Definition 1.5 and since one has the bound $\widetilde{\rho}_{\varphi}^{0}(A_j, z_0) \leq \rho$ $(j = 0, 1, ..., k-1)$, then for any given $\varepsilon > 0$, there exists $r_2 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_2)$, we get

$$|A_j(z)| \leq \varphi^{-1}\left(\left(\rho + \frac{\varepsilon}{2}\right) \log \frac{1}{r}\right),\ \ (j = 0, 1, ..., k-1). \tag{2.4}$$

By Lemma 2.1, there exists a set $E_0 \subset (0, 1)$ that has finite logarithmic measure, such that for all $j \in \{1, ..., k\}$ and $r \notin E_0$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| = |1 + o(1)| \left(\frac{\nu_{z_0}(r, f)}{r}\right)^{j},\ r \to 0, \tag{2.5}$$

for z in the circle $|z-z_0| = r$ and $|f(z)| = \max_{|z-z_0|=r} |f(z)|$. Together, combining the three estimations (2.3), (2.4) and (2.5), we get for all $|z-z_0| = r \in (0, r_2) \setminus E_0$ and $|f(z)| = M_{z_0}(r, f)$

$$\nu_{z_0}(r, f) \le kr\varphi^{-1}\left(\left(\rho + \frac{\varepsilon}{2}\right)\log\frac{1}{r}\right)|1 + o(1)| \le \varphi^{-1}\left((\rho+\varepsilon)\log\frac{1}{r}\right). \tag{2.6}$$

Finally by Lemma 2.2, Lemma 2.4 and (2.6), the desired conclusion follows. □

Then the following lemma helps to complete the proof of the third theorem.

Lemma 2.10 ([20]). *Let f be a nonconstant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$. Then f enjoys the following two properties: (i) $T_{z_0}(r, \frac{1}{f}) = T_{z_0}(r, f) + O(1)$,*
(ii) $T_{z_0}(r, f') < O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right)$, $r \in (0, r_3] \setminus E_6$, where $E_6 \subset (0, r_3]$ with $m_l(E_6) < \infty$.

Lemma 2.11. *Let $\varphi \in \Phi$ and f, g be two analytic function in $\overline{\mathbb{C}} - \{z_0\}$. Then*
i)

$$\widetilde{\rho}^j_\varphi(f+g, z_0) \le \max\left\{\widetilde{\rho}^j_\varphi(f, z_0), \widetilde{\rho}^j_\varphi(g, z_0)\right\} \text{ for } j = 0, 1,$$
$$\widetilde{\rho}^j_\varphi(fg, z_0) \le \max\left\{\widetilde{\rho}^j_\varphi(f, z_0), \widetilde{\rho}^j_\varphi(g, z_0)\right\} \text{ for } j = 0, 1.$$

ii) If $\widetilde{\rho}^j_\varphi(g, z_0) < \widetilde{\rho}^j_\varphi(f, z_0)$, $(j = 0, 1)$, then $\widetilde{\rho}^j_\varphi(f+g, z_0) = \widetilde{\rho}^j_\varphi(fg, z_0) = \widetilde{\rho}^j_\varphi(f, z_0)$ for $j = 0, 1$.
iii)

$$\widetilde{\rho}^j_\varphi(f', z_0) = \widetilde{\rho}^j_\varphi(f, z_0) \quad \text{for } j = 0, 1.$$

Proof. By Lemma 2.2 in [7], we know that if f is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$ and $g(w) = f(z_0 - \frac{1}{w})$, then $g(w)$ is an entire function in $\mathbb{C}$ and we have $T(R, g) = T_{z_0}(r, f)$, where $R = \frac{1}{r}$. So, by using Theorem 2.1 and Theorem 2.7 in [15], we get the conclusions of Lemma 2.11. □

The next lemma finishes the preliminaries.

Lemma 2.12. *Let f_1, f_2 be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying $\rho^0_\varphi(f_1, z_0) = \rho_1 > 0$, $\rho^0_\varphi(f_2, z_0) = \rho_2 < \infty$ and $\rho_2 < \rho_1$. Then, there exists a set $E_7 \subset (0, 1)$ having infinite logarithmic measure such that for all $|z - z_0| = r \in E_7$ one has*

$$\lim_{r\to 0} \frac{T_{z_0}(r, f_2)}{T_{z_0}(r, f_1)} = 0.$$

Proof. By Definition 1.5, for any given ε with $0 < \varepsilon < \frac{\rho_1 - \rho_2}{2}$, there exists $r_4 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_4)$ the following holds

$$T_{z_0}(r, f_2) \le \log\varphi^{-1}\left((\rho_2 + \varepsilon)\log\frac{1}{r}\right). \tag{2.7}$$

Concerning the Lemma 2.7, we deduce the existence of a set E_4 that has infinite logarithmic measure such that for all $|z - z_0| = r \in E_4$

$$T_{z_0}(r, f_1) \ge \log\varphi^{-1}\left((\rho_1 - \varepsilon)\log\frac{1}{r}\right). \tag{2.8}$$

Combining (2.7) and (2.8), it follows that for all $|z - z_0| = r \in E_4 \cap (0, r_4) = E_7$, for sure E_7 has infinite logarithmic measure

$$0 \le \frac{T_{z_0}(r, f_2)}{T_{z_0}(r, f_1)} \le \frac{\log\varphi^{-1}\left((\rho_2+\varepsilon)\log\frac{1}{r}\right)}{\log\varphi^{-1}\left((\rho_1-\varepsilon)\log\frac{1}{r}\right)}$$

as $\rho_2 + \varepsilon < \rho_1 - \varepsilon$. By setting $(\rho_2 + \varepsilon)\log\frac{1}{r} = x$, $\frac{\rho_1-\varepsilon}{\rho_2+\varepsilon} = 1 + \delta$ $(\delta > 0)$ and making use of Proposition 1.2 (i), we get

$$\lim_{r\to 0} \frac{\log\varphi^{-1}\left((\rho_2+\varepsilon)\log\frac{1}{r}\right)}{\log\varphi^{-1}\left((\rho_1-\varepsilon)\log\frac{1}{r}\right)} = \lim_{r\to 0} \frac{\log\varphi^{-1}\left((\rho_2+\varepsilon)\log\frac{1}{r}\right)}{\log\varphi^{-1}\left(\frac{\rho_1-\varepsilon}{\rho_2+\varepsilon}(\rho_2+\varepsilon)\log\frac{1}{r}\right)}$$

$$= \lim_{x\to+\infty} \frac{\log \varphi^{-1}(x)}{\log \varphi^{-1}((1+\delta)x)} = 0.$$

Therefore yielding

$$\lim_{r\to 0} \frac{T_{z_0}(r,f_2)}{T_{z_0}(r,f_1)} = 0.$$

□

3. Proof of the theorems

Proof of Theorem 1.14.

Proof. Suppose that $f\ (\not\equiv 0)$ is a solution of equation (1.1) which is analytic in $\overline{\mathbb{C}} - \{z_0\}$. Set

$$\alpha = \max\left\{\widetilde{\rho}^{0}_{\varphi}(A_j,z_0) : j\neq 0\right\} < \rho = \widetilde{\rho}^{0}_{\varphi}(A_0,z_0).$$

By Definition 1.5, for any given $\varepsilon \in \left(0, \frac{\rho-\alpha}{2}\right)$, there exists $r_5 \in (0,1)$ such that for all $|z-z_0| = r \in (0,r_5)$, we have

$$|A_j(z)| \le \varphi^{-1}\left((\alpha+\varepsilon)\log\frac{1}{r}\right),\ j=1,2,...,k-1. \tag{3.1}$$

By Lemma 2.6 for all ε given above, we conclude the existence a set $E_3 \subset (0,1)$ with infinite logarithmic measure such that for all $|z-z_0| = r \in E_3$ and $|A_0(z)| = M_{z_0}(r,A_0)$

$$|A_0(z)| \ge \varphi^{-1}\left((\rho-\varepsilon)\log\frac{1}{r}\right). \tag{3.2}$$

By Lemma 2.5, there exist a set $E_2 \subset (0,r_1]$ $(r_1 \in (0,1))$ that has finite logarithmic measure and a constant $\lambda > 0$ that depends on $\gamma > 1$ such that for all $|z-z_0| = r \in (0,r_1] \setminus E_2$ the following occurs

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le \lambda\left(\frac{1}{r^2}T_{z_0}\left(\frac{r}{\gamma},f\right)\log T_{z_0}\left(\frac{r}{\gamma},f\right)\right)^{j},\ j=1,...,k. \tag{3.3}$$

By (1.1), we get

$$|A_0(z)| \le \left|\frac{f^{(k)}}{f}\right| + \cdots + |A_j(z)|\left|\frac{f^{(j)}}{f}\right| + \cdots + |A_1(z)|\left|\frac{f'}{f}\right|. \tag{3.4}$$

As the last step, let $E_8 = (0,r_1] \cap (0,r_5) \cap E_3 \setminus E_2$, obviously E_8 has infinite logarithmic measure. Consequently, the combination between (3.1), (3.2), (3.3) and (3.4) gives for any given $\varepsilon \in \left(0, \frac{\rho-\alpha}{2}\right)$ and for all $|z-z_0| = r \in E_8$,

$$\varphi^{-1}\left((\rho-\varepsilon)\log\frac{1}{r}\right) \le \lambda k\left(\frac{1}{r}T_{z_0}\left(\frac{r}{\gamma},f\right)\right)^{2k}\varphi^{-1}\left((\alpha+\varepsilon)\log\frac{1}{r}\right). \tag{3.5}$$

Now, we prove that $\rho^{1}_{\varphi}(f,z_0) \ge \rho$. By contradiction, we suppose that

$$\rho_1 = \rho^{1}_{\varphi}(f,z_0) < \rho.$$

Then, for any given $\varepsilon \in \left(0, \min\left\{\frac{\rho-\alpha}{2}, \frac{\rho-\rho_1}{2}\right\}\right)$, there exists $r_6 \in (0,1)$ such that for all $|z-z_0| = r \in (0,r_6)$, we have

$$T_{z_0}(r,f) \le \varphi^{-1}\left((\rho_1+\varepsilon)\log\frac{1}{r}\right). \tag{3.6}$$

By (3.5) and (3.6), for any given $\varepsilon \in \left(0, \min\left\{\frac{\rho-\alpha}{2}, \frac{\rho-\rho_1}{2}\right\}\right)$ and $|z-z_0| = r \in (0,r_6) \cap E_8$, we obtain

$$\varphi^{-1}\left((\rho-\varepsilon)\log\frac{1}{r}\right) \le \frac{\lambda k}{r^{2k}}\left(\varphi^{-1}\left((\rho_1+\varepsilon)\log\frac{\gamma}{r}\right)\right)^{2k}\varphi^{-1}\left((\alpha+\varepsilon)\log\frac{1}{r}\right).$$

Since $\rho - \varepsilon > \max\{\rho_1 + \varepsilon, \alpha + \varepsilon\}$, we get

$$\varphi^{-1}\left((\rho-\varepsilon)\log\frac{1}{r}\right) \leq \frac{\lambda k}{r^{2k}}\left(\varphi^{-1}\left(\max\{\rho_1+\varepsilon,\alpha+\varepsilon\}\log\frac{\gamma}{r}\right)\right)^{2k+1}. \tag{3.7}$$

Applying the logarithm on the both sides of (3.7), we find

$$\frac{2k\log r}{(2k+1)\log\varphi^{-1}\left(\max\{\rho_1+\varepsilon,\alpha+\varepsilon\}\log\frac{\gamma}{r}\right)} + \frac{\log\varphi^{-1}\left((\rho-\varepsilon)\log\frac{1}{r}\right)}{(2k+1)\log\varphi^{-1}\left(\max\{\rho_1+\varepsilon,\alpha+\varepsilon\}\log\frac{\gamma}{r}\right)}$$

$$\leq \frac{\log\lambda k}{(2k+1)\log\varphi^{-1}\left(\max\{\rho_1+\varepsilon,\alpha+\varepsilon\}\log\frac{\gamma}{r}\right)} + 1. \tag{3.8}$$

Set $\varphi^{-1}\left(\max\{\rho_1+\varepsilon,\alpha+\varepsilon\}\log\frac{\gamma}{r}\right) = x$. Then $\log r = -\frac{\varphi(x)}{\max\{\rho_1+\varepsilon,\alpha+\varepsilon\}} + \log\gamma$ and by using Karamata's theorem ([22]), $\varphi(e^t) = t^{o(1)}$, $t \longrightarrow +\infty$, we immediately get

$$\lim_{r\to 0}\frac{\log r}{\log\varphi^{-1}\left(\max\{\rho_1+\varepsilon,\alpha+\varepsilon\}\log\frac{\gamma}{r}\right)}$$

$$= \lim_{x\to+\infty}\left(-\frac{\varphi(x)}{\max\{\rho_1+\varepsilon,\alpha+\varepsilon\}\log x} + \frac{\log\gamma}{\log x}\right)$$

$$= \lim_{x\to+\infty}\left(-\frac{(\log x)^{o(1)}}{\max\{\rho_1+\varepsilon,\alpha+\varepsilon\}\log x}\right) = 0. \tag{3.9}$$

As we did earlier in Lemma 2.12, we have

$$\lim_{r\to 0}\frac{\log\varphi^{-1}\left((\rho-\varepsilon)\log\frac{1}{r}\right)}{\log\varphi^{-1}\left(\max\{\rho_1+\varepsilon,\alpha+\varepsilon\}\log\frac{\gamma}{r}\right)} = +\infty. \tag{3.10}$$

The right hand side in (3.8) is finite, while the left hand side is infinite, thus a contradiction holds, i.e.,

$$\rho_\varphi^1(f, z_0) \geq \rho.$$

Thus, by Remark 1.6 and as $\varphi \in \phi$, we obtain

$$\tilde{\rho}_\varphi^1(f, z_0) \geq \rho.$$

Finally, by using Lemma 2.9, we get

$$\tilde{\rho}_\varphi^1(f, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0).$$

□

Proof of Theorem 1.15.

Proof. Suppose that $f\ (\not\equiv 0)$ is a solution of equation (1.1) which is analytic in $\overline{\mathbb{C}} - \{z_0\}$. By an analogous progress, set $\tilde{\rho}_\varphi^0(A_0, z_0) = \rho$, $\tilde{\tau}_\varphi^0(A_0, z_0) = \tau$. If $\max\{\tilde{\rho}_\varphi^0(A_j, z_0) : j = 1, ..., k-1\} < \tilde{\rho}_\varphi^0(A_0, z_0) = \rho$, then by Theorem 1.14, we obtain $\tilde{\rho}_\varphi^1(f, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0)$. Suppose that $\max\{\tilde{\rho}_\varphi^0(A_j, z_0) : j = 1, 2, ..., k-1\} = \tilde{\rho}_\varphi^0(A_0, z_0) = \rho\ (0 < \rho < +\infty)$ and $\max\{\tilde{\tau}_\varphi^0(A_j, z_0) : \tilde{\rho}_\varphi^0(A_j, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0) > 0\} < \tilde{\tau}_\varphi^0(A_0, z_0) = \tau$ $(0 < \tau < +\infty)$. Then, there exists a set $I \subseteq \{1, 2, ..., k-1\}$ such that $\tilde{\rho}_\varphi^0(A_j, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0) = \rho\ (j \in I)$ and $\tilde{\tau}_\varphi^0(A_j, z_0) < \tilde{\tau}_\varphi^0(A_0, z_0)\ (j \in I)$. Thus, we choose β_1, β_2 satisfying

$$\max\{\tilde{\tau}_\varphi^0(A_j, z_0) : (j \in I)\} < \beta_1 < \beta_2 < \tilde{\tau}_\varphi^0(A_0, z_0) = \tau.$$

From Definition 1.7, there exists $r_7 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_7)$

$$|A_j(z)| \leq \varphi^{-1}\left(\log\frac{\beta_1}{r^\rho}\right)\quad (j \in I) \tag{3.11}$$

and

$$|A_j(z)| \le \varphi^{-1}\left(\log \frac{1}{r^{\rho_1}}\right) \le \varphi^{-1}\left(\log \frac{\beta_1}{r^{\rho}}\right) \ (j \in \{1, ..., k-1\} \setminus I), \tag{3.12}$$

where $0 < \rho_1 < \rho$. We now turns to Lemma 2.5, it claims the existence of a set $E_2 \subset (0, r_1]$ $(r_1 \in (0,1))$ having finite logarithmic measure and a constant $\lambda > 0$ that depends on some given $\gamma > 1$ such that for all $|z - z_0| = r \in (0, r_1] \setminus E_2$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le \lambda \left(\frac{1}{r^2} T_{z_0}\left(\frac{r}{\gamma}, f\right) \log T_{z_0}\left(\frac{r}{\gamma}, f\right)\right)^j, \ j = 1, ..., k. \tag{3.13}$$

By Lemma 2.8, there exists a set $E_5 \subset (0,1)$ of infinite logarithmic measure such that for all $|z - z_0| = r \in E_5$

$$\varphi(M_{z_0}(r, A_0)) > \log \frac{\beta_2}{r^{\rho}}$$

equivalently

$$M_{z_0}(r, A_0) > \varphi^{-1}\left(\log \frac{\beta_2}{r^{\rho}}\right). \tag{3.14}$$

Set $E_9 = E_5 \cap (0, r_7) \cap (0, r_1] \setminus E_2$, for sure E_9 has infinite logarithmic measure. Combining (3.11), (3.12), (3.13) and (3.14) with (3.4) we get for all $|z - z_0| = r \in E_9$,

$$\varphi^{-1}\left(\log \frac{\beta_2}{r^{\rho}}\right) \le \lambda k \left(\frac{1}{r} T_{z_0}(\frac{r}{\gamma}, f)\right)^{2k} \varphi^{-1}\left(\log \frac{\beta_1}{r^{\rho}}\right). \tag{3.15}$$

The last inequality implies $\rho^1_{\varphi}(f, z_0) \ge \rho$. To see why, we assume that $\rho^1_{\varphi}(f, z_0) < \rho$. Then, there exists $r_8 \in (0,1)$ such that for all $|z - z_0| = r \in (0, r_8)$, we have

$$T_{z_0}(r, f) \le \varphi^{-1}\left(\rho_2 \log \frac{1}{r}\right) \tag{3.16}$$

for some $\rho_2 < \rho$. Consequently, by (3.15) and (3.16), for all $|z - z_0| = r \in E_9 \cap (0, r_8)$, we get

$$\begin{aligned} \varphi^{-1}\left(\log \frac{\beta_2}{r^{\rho}}\right) &\le \frac{\lambda k}{r^{2k}} \left(\varphi^{-1}\left(\log \frac{\gamma^{\rho_2}}{r^{\rho_2}}\right)\right)^{2k} \varphi^{-1}\left(\log \frac{\beta_1}{r^{\rho}}\right) \\ &\le \frac{\lambda k}{r^{2k}} \left(\varphi^{-1}\left(\log \frac{\beta_1}{r^{\rho}}\right)\right)^{2k+1}. \end{aligned}$$

Applying the logarithm on both sides, we find

$$\begin{aligned} &\frac{2k \log r}{(2k+1) \log \varphi^{-1}\left(\log \frac{\beta_1}{r^{\rho}}\right)} \\ &+ \frac{\log \varphi^{-1}\left(\log \frac{\beta_2}{r^{\rho}}\right)}{(2k+1) \log \varphi^{-1}\left(\log \frac{\beta_1}{r^{\rho}}\right)} \le \frac{\log(\lambda k)}{(2k+1) \log \varphi^{-1}\left(\log \frac{\beta_1}{r^{\rho}}\right)} + 1. \end{aligned} \tag{3.17}$$

As we did before

$$\lim_{r \to 0} \frac{\log r}{\log \varphi^{-1}\left(\log \frac{\beta_1}{r^{\rho}}\right)} = 0$$

and

$$\lim_{r \to 0} \frac{\log \varphi^{-1}\left(\log \frac{\beta_2}{r^{\rho}}\right)}{(2k+1) \log \varphi^{-1}\left(\log \frac{\beta_1}{r^{\rho}}\right)} = +\infty$$

because $\beta_2 > \beta_1$. Since the right hand side of the inequality (3.17) is bounded by 1, thus taking limits yielding $+\infty \le 1$ which is a contradiction. Hence, by Remark 1.6 and as $\varphi \in \phi$, we get $\rho^1_{\varphi}(f, z_0) = \widetilde{\rho}^1_{\varphi}(f, z_0) \ge \rho$. Finally, by applying Lemma 2.9, the desired conclusion of Theorem 1.15 will be proved. □

Proof of Theorem 1.16.

Proof. Let that the dominant coefficient is unique and runs over the set $\{0, 1, ..., k-1\}$. In other words, there exists $s \in \{0, 1, ..., k-1\}$ such that

$$\max\left\{\widetilde{\rho}^{0}_{\varphi}(A_j, z_0) : j \neq s\right\} < \widetilde{\rho}^{0}_{\varphi}(A_s, z_0).$$

Assume that $f(z) \not\equiv 0$ is a rational solution of (1.1) which is analytic in $\overline{\mathbb{C}} - \{z_0\}$. Then $f^{(s)}(z) \not\equiv 0$ and by (1.1), we have

$$A_s(z) f^{(s)}(z) = -f^{(k)}(z) - \sum_{\substack{j=0 \\ j\neq s}}^{k-1} A_j(z) f^{(j)}(z).$$

By Lemma 2.11, it follows that

$$\widetilde{\rho}^{0}_{\varphi}(A_s, z_0) = \widetilde{\rho}^{0}_{\varphi}(A_s f^{(s)}, z_0) = \widetilde{\rho}^{0}_{\varphi}\left(-f^{(k)} - \sum_{j=0,\, j\neq s}^{k-1} A_j f^{(j)}, z_0\right)$$
$$\leq \max_{j=0,1,\cdots,k-1,\, j\neq s}\left\{\widetilde{\rho}^{0}_{\varphi}(A_j, z_0)\right\},$$

which is a contradiction. Hence, f must be a transcendental.
Suppose that f is a transcendental solution of equation (1.1) which is analytic in $\overline{\mathbb{C}} - \{z_0\}$. The equation (1.1) yields

$$m_{z_0}(r, A_s) \leq \sum_{j=0,\, j\neq s}^{k} m_{z_0}\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + \sum_{j=0,\, j\neq s}^{k-1} m_{z_0}(r, A_j) + \log k. \tag{3.18}$$

By Lemma 2.10, there exists a set $E_6 \subset (0, r_3]$ for fixed $r_3 \in (0, 1)$ which has finite logarithmic measure such that for all $|z - z_0| = r \in (0, r_3] \setminus E_6$, we have

$$T_{z_0}(r, f') < O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right).$$

Consequently

$$T_{z_0}(r, f^{(j)}) < O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right).$$

Then, it follows

$$\sum_{j=0,\, j\neq s}^{k} m_{z_0}\left(r, \frac{f^{(j)}}{f^{(s)}}\right) \leq O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right). \tag{3.19}$$

By Lemma 2.12, there exists a set $E_7 \subset (0, 1)$ with infinite logarithmic measure such that for all $|z - z_0| = r \in E_7$

$$\lim_{r\to 0}\frac{T_{z_0}(r, A_j)}{T_{z_0}(r, A_s)} = 0,\ j \neq s,$$

so for any given $\varepsilon \in \left(0, \frac{1}{2(k-1)}\right)$

$$m_{z_0}(r, A_j) \leq \varepsilon m_{z_0}(r, A_s),\ j \neq s. \tag{3.20}$$

By (3.18), (3.19) and (3.20), we conclude that for all $|z - z_0| = r \in E_7 \cap (0, r_3] \setminus E_6$,

$$m_{z_0}(r, A_s) \leq O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right) + (k-1)\,\varepsilon m_{z_0}(r, A_s) + \log k.$$

So

$$(1 - (k-1)\,\varepsilon)\, m_{z_0}(r, A_s) \leq O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right). \tag{3.21}$$

Since $\varepsilon \in \left(0, \frac{1}{2(k-1)}\right)$, then $1-(k-1)\,\varepsilon > 1-(k-1)\,\frac{1}{2(k-1)} = \frac{1}{2}$. From (3.21), we get

$$\frac{1}{2} m_{z_0}(r, A_s) \leq O\left(T_{z_0}(r,f) + \log\frac{1}{r}\right),$$

that is by f is a transcendental

$$m_{z_0}(r, A_s) \leq O\left(T_{z_0}(r,f) + \log\frac{1}{r}\right) \leq O\left(T_{z_0}(r,f)\right).$$

Using this, we get

$$\limsup_{r\to 0}\frac{\varphi(\exp(m_{z_0}(r,A_s)))}{\log\frac{1}{r}} \leq \limsup_{r\to 0}\frac{\varphi(\exp(cT_{z_0}(r,f))}{\log\frac{1}{r}},$$

where $c>0$ is some constant. By using Proposition 1.2 case (iii), we obtain

$$\limsup_{r\to 0}\frac{\varphi(\exp(m_{z_0}(r,A_s)))}{\log\frac{1}{r}} \leq \limsup_{r\to 0}\frac{(1+o(1))\,\varphi(e^{T_{z_0}(r,f)})}{\log\frac{1}{r}} = \rho^0_\varphi(f, z_0).$$

Therefore,

$$\rho^0_\varphi(A_s, z_0) \leq \rho^0_\varphi(f, z_0).$$

It remains to show that $\rho^1_\varphi(f, z_0) \leq \rho^0_\varphi(A_s, z_0)$. By Lemma 2.9, it follows

$$\rho^1_\varphi(f, z_0) \leq \rho^0_\varphi(A_s, z_0).$$

So we have the double inequality

$$\rho^1_\varphi(f, z_0) \leq \rho^0_\varphi(A_s, z_0) \leq \rho^0_\varphi(f, z_0)$$

and by Remark 1.6 this leads

$$\widetilde{\rho}^1_\varphi(f, z_0) \leq \widetilde{\rho}^0_\varphi(A_s, z_0) \leq \widetilde{\rho}^0_\varphi(f, z_0).$$

□

4. Examples

Here, we provide some examples that illustrate all what we did before.

Example 4.1. Consider the equation

$$f'' + \left(\frac{1}{(z-z_0)^2} + \frac{2}{z-z_0}\right) f' - \frac{1}{(z-z_0)^4} e^{\frac{2}{z-z_0}} f = 0. \tag{4.1}$$

It is not hard to see that $f(z) = \exp\left(\exp\frac{1}{z-z_0}\right)$ which is analytic in $\overline{\mathbb{C}} - \{z_0\}$ is a solution for (4.1). Notice that, the function $\varphi(t) = \log\log t = \log_2 t$ is a function of ϕ, that is φ is unbounded, increasing and $\psi(t) = \varphi(e^t) = \log t$ is clearly slowly growing. A hand wavy calculations give

$$\widetilde{\rho}^0_\varphi(A_1, z_0) = 0 < \ \widetilde{\rho}^0_\varphi(A_0, z_0) = 1.$$

Loosely speaking A_0 is a dominant coefficient so by Theorem 1.14, we conclude that

$$\widetilde{\rho}^1_\varphi(f, z_0) = \widetilde{\rho}^0_\varphi(A_0, z_0) = 1.$$

On the other hand, a simple computation gives

$$M_{z_0}(r, f) = e^{e^{\frac{1}{r}}}.$$

Therefore

$$\widetilde{\rho}^1_\varphi(f, z_0) = \limsup_{r\to 0}\frac{\varphi(\log M_{z_0}(r,f))}{\log\frac{1}{r}} = \limsup_{r\to 0}\frac{\log\log\log\left(e^{e^{\frac{1}{r}}}\right)}{\log\frac{1}{r}} = 1.$$

This emphasizes the conclusion of Theorem 1.14.

Example 4.2. Consider the equation

$$f'' + \left(\left(1 + \frac{1}{z^2}\right)e^{\frac{1}{z}} + \frac{2z+1}{z^2}\right)f' + \frac{e^{\frac{2}{z}}}{z^2}f = 0. \tag{4.2}$$

It is not hard to see that $f(z) = \exp\left(\exp\frac{1}{z}\right)$ which is analytic in $\overline{\mathbb{C}}\setminus\{0\}$ is a solution for (4.2). Notice that, the function $\varphi(t) = \log\log t = \log_2 t$ is a function of ϕ, that is φ is unbounded, increasing and $\psi(t) = \varphi(e^t) = \log t$ is clearly slowly growing. A hand wavy calculations give

$$\widetilde{\rho}^0_\varphi(A_1, 0) = \widetilde{\rho}^0_\varphi(A_0, 0) = 1,$$

and

$$\widetilde{\tau}^0_\varphi(A_1, 0) = 1 < \widetilde{\tau}^0_\varphi(A_0, 0) = 2.$$

Lossly speaking A_0 is a dominant coefficient so by Theorem 1.15, we conclude that

$$\widetilde{\rho}^1_\varphi(f, 0) = \widetilde{\rho}^0_\varphi(A_0, 0) = 1.$$

This confirm the conclusion of Theorem 1.15.

Example 4.3. Consider the equation

$$f''' - e^{-\frac{1}{z}}f'' + \left(\frac{2}{z} - \frac{5}{z^2} - \frac{6}{z^3} - \frac{1}{z^4}\right)f' + \left(\frac{2}{z^3} + \frac{1}{z^4}\right)f = 0.$$

This equation accepts the analytic function f in $\overline{\mathbb{C}}\setminus\{0\}$ given by $f(z) = e^{\frac{1}{z}} + 1$. By letting $\varphi = \log_2 \in \phi$ and setting

$$A_0(z) = \frac{2}{z^3} + \frac{1}{z^4},$$

$$A_1(z) = \frac{2}{z} - \frac{5}{z^2} - \frac{6}{z^3} - \frac{1}{z^4},$$

$$A_2(z) = -\exp\left(-\frac{1}{z}\right).$$

We see that $\widetilde{\rho}^0_\varphi(A_0, 0) = \widetilde{\rho}^0_\varphi(A_1, 0) = 0$ and $\widetilde{\rho}^0_\varphi\,(A_2, 0) = 1$. So, the coefficient A_2 is the dominant. Therefore, by Theorem 1.16, one gets

$$\widetilde{\rho}^1_\varphi(f, 0) \le 1 \le \widetilde{\rho}^0_\varphi(f, 0).$$

While simple calculations give

$$\widetilde{\rho}^0_\varphi(f, 0) = \limsup_{r\to 0}\frac{\log\log\left(e^{\frac{1}{r}} + 1\right)}{\log\frac{1}{r}} = 1,$$

$$\widetilde{\rho}^1_\varphi(f, 0) = \limsup_{r\to 0}\frac{\log\log\log\left(e^{\frac{1}{r}} + 1\right)}{\log\frac{1}{r}} = 0.$$

Consequently the conclusion of Theorem 1.16 holds.

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