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# SPECTRAL THEOREMS IN THE LAGUERRE HYPERGROUP SETTING

HATEM MEJJAOLI[1,*] AND FIRDOUS A. SHAH[2]

ABSTRACT. We introduce the two-wavelet multiplier operator in the Laguerre hypergroup setting. Knowing the fact that the study of this operator are both theoretically interesting and practically useful, we investigated several subjects of spectral analysis for the new operator. Firstly, we present a comprehensive analysis of the generalized two-wavelet multiplier operator. Next, we introduce and we study the generalized Landau-Pollak-Slepian operator. As applications, some problems of the approximation theory and the uncertainty principles are studied. Finally, we give many results on the boundedness and compactness of the Laguerre two-wavelet multipliers on $L^p_\alpha(\mathbb{K})$, $1 \leq p \leq \infty$.

## 1. INTRODUCTION

Let $\mathbb{H}_d$, be the $2d+1$-dimensional Heisenberg group with the multiplication law

$$(z,t)(z',t') = (z+z', t+t' - Im(zz')).$$

Then $T = \frac{\partial}{\partial t}$ and

$$Z_j = \frac{\partial}{\partial z_j} - i\bar{z}_j \frac{\partial}{\partial t}, \quad \overline{Z_j} = \frac{\partial}{\partial \bar{z}_j} + i\bar{z}_j \frac{\partial}{\partial t}, \quad j = 1, ..., d$$

forms a basis of the left invariant vector fields of $h_d^c$, the complexification of the Lie algebra $h_d$ of $\mathbb{H}_d$, where

$$\frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}, \quad \frac{\partial}{\partial \bar{z}_j} = \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}.$$

Set

$$X_j = \frac{\partial}{\partial x_j} - iy_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + ix_j \frac{\partial}{\partial t}, \quad j = 1, ..., d.$$

Thus $X_1, X_2, ..., X_d, Y_1, ..., Y_d, T$ is a basis of $h_d$. A function $f$ on $\mathbb{H}_d$ is said to be radial if it is invariant under the action of the unitary group $U(d)$. Let

$$L^p_{rad}(\mathbb{H}_d) := \Big\{ f \in L^p(\mathbb{H}_d) : \ f(uz,t) = f(z,t) \text{ for all } u \in U(d) \Big\}.$$

The theory of harmonic analysis on $L^p_{rad}(\mathbb{H}_d)$ were exploited by many authors (see [30, 37, 48]). When one considers the problems of radial functions on the Heisenberg group $\mathbb{H}_d$, the underlying manifold can be

[1]TAIBAH UNIVERSITY, COLLEGE OF SCIENCES, DEPARTMENT OF MATHEMATICS, PO BOX 30002 AL MADINAH AL MUNAWARAH, SAUDI ARABIA

[2]DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KASHMIR, SOUTH CAMPUS, ANANTNAG-192101, JAMMU AND KASHMIR, INDIA

*E-mail addresses*: hatem.mejjaoli@yahoo.fr, fashah@uok.edu.in.

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*Corresponding author.

regarded as the Laguerre hypergroup $\mathbb{K} := [0, \infty) \times \mathbb{R}$. Stempak [49] introduced a generalized translation operator on $\mathbb{K}$, and established the theory of harmonic analysis on $L^2(\mathbb{K}, d\nu_\alpha)$, where the weighted Lebesgue measure $\nu_\alpha$ on $\mathbb{K}$ is given by

$$d\nu_\alpha(x,t) := \frac{x^{2\alpha+1}dxdt}{\pi\Gamma(\alpha+1)}, \quad \alpha \geq 0.$$

Furthermore, Nessibi and Trimèche [39] studied the theory of wavelet analysis. Using this type of wavelet, they gave another inversion formula of the Radon transform on $\mathbb{K}$.

In this paper we are interested in the Laguerre hypergroup $\mathbb{K}$ induced with the Haar measure $d\nu_\alpha(x,t)$. We recall that $(\mathbb{K}, *_\alpha)$ is a commutative hypergroup [39], on which the involution and the Haar measure are respectively given by the homeomorphism $(x,t) \to (x,t)^- = (x,-t)$ and the Radon positive measure $d\nu_\alpha(x,t)$. The unit element of $(\mathbb{K}, *_\alpha)$ is given by $e = (0,0)$.

The dual of a hypergroup is the space of all bounded continuous and multiplicative functions $\chi$ such that $\bar{\chi} = \chi$. The dual of the Laguerre hypergroup $\hat{\mathbb{K}}$ can be topologically identified with the so-called Heisenberg fan [15], i.e., the subset embedded in $\mathbb{R}^2$ given by

$$\cup_{j\in\mathbb{N}} \left\{(\lambda,\mu) \in \mathbb{R}^2 : \mu = |\lambda|(2j+\alpha+1),\ \lambda \neq 0\right\} \cup \left\{(0,\mu) \in \mathbb{R}^2 : \mu \geq 0\right\}.$$

Moreover, the subset $\left\{(0,\mu) \in \mathbb{R}^2 : \mu \geq 0\right\}$ has zero Plancherel measure, therefore it will be usually disregarded. Following [39], in this paper, we identify the dual of the Laguerre hypergroup by $\hat{\mathbb{K}} := \mathbb{R} \times \mathbb{N}$.

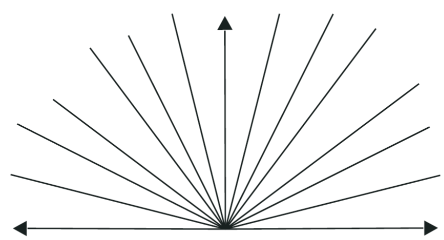

FIGURE 1. Heisenberg fan.

Very recently, the Fourier analysis on $\mathbb{K}$ have been extensively studied with respect to several problems already studied for the Fourier transform; for instance, Hardy's inequality [3], functional spaces [2, 24], Littlewood-Paley g-functions [23], the generalized Wigner transform [10,35], the generalized wavelet transform [36,39], localization operators [35,36], uncertainty principles [40], Titchmarsh's theorems [38] and so on.

Using the properties of the generalized Fourier transform associated with the Laguerre hypergroup, our main aim in this paper is to expose and study some spectral theorems in the time-frequency analysis setting. More precisely, we will study the wavelet multipliers in the spirit of the Wong's point of view. On the other hand we will establish some uncertainty type principles and approximation theorems. The theory of wavelet multipliers has been initiated by He and Wong in [21], developed in the paper [14] by Du and Wong, and detailed in the book [51] by Wong.

The present paper is aimed at exploring contemporary trends in Laguerre hypergroup time-frequency analysis with applications to approximation and spectral theories. Of particular interest shall be the formulation of wavelet multipliers beyond the generalized Fourier domain. Besides, it is also of significant interest to explore the wavelet multipliers in the realm of higher-dimensional signal analysis. Keeping in view the fact that the theory of wavelet multipliers is quite adequate for an efficient time-frequency analysis of signals and has also found numerous applications in several other aspects of science and engineering, including wave propagation, signal processing and quantum optics [41], it is quite lucrative to investigate upon the generalized wavelet multipliers associated with the Laguerre hypergroup. With the advent of time-frequency analysis, the theory of uncertainty principles has gained a considerable attention and it's been extended to a wide class of integral transforms ranging from the classical Fourier to the many recent quadratic-phase Fourier transforms [20,41,42]. The pioneering Donoho-Stark type uncertainty principle [13] asserts that a non-trivial function cannot be precisely concentrated in both the time and frequency domains at the same time. In this paper we are concerned with the Donoho-Stark form of the UP of which we present an improvement in the form of a new general bound for the constant which is involved in the estimate, and a new type of estimation of the same constant in dependence on the signal.

Recently, the field of time-frequency analysis has attired many researchers. For examples, we note that Ben Hamadi and all have studied the generalized Fourier multipliers in [4] and the uncertainty principles associated with some integral transforms [5,6], Ghobber and all in [17–19] have studied the wavelet multipliers in the Bessel setting and the theory of localization for some integral transforms, Lamouchi and all have also studied some problems of time-frequency analysis in [27,28], for the spherical mean operator and for the short time Fourier transform, Mejjaoli in [31–34] has studied the wavelet multipliers in the Dunkl and the deformed Fourier settings, Sraeib in [44,45] has studied the uncertainty principles in the quantum theory and the applications of the deformed Wigner transform to the Localization operators theory, Tantary and all in [43,46] have studied the localization operators and uncertainty principles for the Ridgelet transformation in the Clifford setting.

The main contributions of this article are as follows:

- To prove results on the $L^p$-boundedness and the $L^p$-compactness of the two-wavelet multipliers associated with the generalized Fourier in the Laguerre hypergroup setting.
- To construct and study an example of generalized two-wavelet multipliers. Indeed, we have prove that the generalized two-wavelet multiplier is unitary equivalent to a scalar multiple of the generalized Landau-Pollak-Slepian Operator.
- To give some applications on the generalized two-wavelet multipliers.

The remainder of this paper is arranged as follows. The §2 contains some basic facts needed in the sequel about the Laguerre hypergroup and Schatten-von Neumann classes. In §3 we introduce and we study the two-wavelet multipliers in the setting of the Laguerre hypergroup. More precisely, the Schatten-von Neumann properties of these two-wavelet multipliers are established, and for trace class Laguerre two-wavelet multipliers, the traces and the trace class norm inequalities are presented. In §4, firstly we introduce the generalized Landau-Pollak-Slepian operator. Next, we give the link between this operator and the Laguerre two-wavelet multipliers. As applications, we prove the Donoho-Stark uncertainty principle for the Fourier transformation in the Laguerre hypergroup setting, next we study some spectral problems associated for the generalized Landau-Pollak-Slepian operator. In the last section, under suitable conditions on the symbols and two admissible wavelets, we study the $L^p$ boundedness and compactness of the Laguerre two-wavelet multipliers.

## 2. Preliminaries

In this section we set some notations and we recall some basic results in harmonic analysis related to Laguerre hypergroups and Schatten-von Neumann classes. Main references are [39,51].

### 2.1. **Harmonic analysis on the Laguerre hypergroup.**

We denote by

- $\mathbb{K} := [0,\infty) \times \mathbb{R}$ equipped with the weighted Lebesgue measure $\nu_\alpha$ on $\mathbb{K}$ given by

$$d\nu_\alpha(x,t) := \frac{x^{2\alpha+1}dxdt}{\pi\Gamma(\alpha+1)}, \quad \alpha \geq 0.$$

- For $p \in [1,\infty]$, $p'$ denotes as in all that follows, the conjugate exponent of $p$.
- $L^p_\alpha(\mathbb{K}), 1 \leq p \leq \infty$, the space of measurable functions on $\mathbb{K}$, satisfying

$$\begin{aligned} \|f\|_{L^p_\alpha(\mathbb{K})} &= \left(\int_{\mathbb{K}} |f(x,t)|^p d\nu_\alpha(x,t)\right)^{1/p} < \infty, \quad 1 \leq p < \infty, \\ \|f\|_{L^\infty_\alpha(\mathbb{K})} &= \underset{(x,t)\in\mathbb{K}}{ess\ sup}\ |f(x,t)| < \infty, \quad p = \infty. \end{aligned}$$

- $C_*(\mathbb{K})$ the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable.
- $C_{*,c}(\mathbb{K})$ the subspace of $C_*(\mathbb{K})$ formed by functions with compact support.
- $\mathcal{L}^{(\alpha)}_m$ the Laguerre function defined on $[0,\infty)$ by

$$\mathcal{L}^{(\alpha)}_m(x) = e^{-\frac{x}{2}} L^{(\alpha)}_m(x)/L^{(\alpha)}_m(0),$$

  $L^{(\alpha)}_m$ being the Laguerre polynomial of degree $m$ and order $\alpha$.
- $\hat{\mathbb{K}} := \mathbb{R} \times \mathbb{N}$ equipped with the weighted Lebesgue measure $\gamma_\alpha$ on $\hat{\mathbb{K}}$ given by

$$\int_{\hat{\mathbb{K}}} g(\lambda,m)d\gamma_\alpha(\lambda,m) = \sum_{m=0}^{\infty} L^{(\alpha)}_m(0) \int_{\mathbb{R}} g(\lambda,m)|\lambda|^{\alpha+1}d\lambda.$$

- $L^p_\alpha(\hat{\mathbb{K}}), p \in [1,\infty]$, the space of measurable functions $g : \hat{\mathbb{K}} \to \mathbb{C}$, such that $\|g\|_{L^p_\alpha(\hat{\mathbb{K}})} < \infty$, where

$$\begin{aligned} \|g\|_{L^p_\alpha(\hat{\mathbb{K}})} &= \left(\int_{\hat{\mathbb{K}}} |g(\lambda,m)|^p d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, \\ \|g\|_{L^\infty_\alpha(\hat{\mathbb{K}})} &= \underset{(\lambda,m)\in\hat{\mathbb{K}}}{ess\ sup}\ |g(\lambda,m)| < \infty, \quad p = \infty. \end{aligned}$$

It is well known [39] that for all $(\lambda,m) \in \hat{\mathbb{K}}$, the system

$$\begin{cases} D_1 u(x,t) &= i\lambda u(x,t), \\ D_2 u(x,t) &= -4|\lambda|(m + \frac{\alpha+1}{2})u(x,t) \\ u(0,0) &= 1, \quad \frac{\partial u}{\partial r}(0,t) = 0, \ \forall\, t \in \mathbb{R}, \end{cases}$$

admits a unique solution $\varphi_{\lambda,m}$, given by

$$\varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathcal{L}^{(\alpha)}_m(|\lambda|x^2),$$

where $D_1$ and $D_2$ be the singular partial differential operators, given by

$$\begin{cases} D_1 &= \frac{\partial}{\partial t} \\ D_2 &= \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2}, \quad (x,t) \in (0,\infty)\times\mathbb{R}, \end{cases} \tag{2.1}$$

where $\alpha$ is a nonnegative number. For $\alpha = d-1$, $d$ being a positive integer, the operator $D_2$ is the radial part of the sublaplacian on the Heisenberg group $\mathbb{H}_d$.

The harmonic analysis on the Laguerre hypergroup $\mathbb{K}$ is generated by the singular operator

$$\mathcal{L}_\alpha := \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2}$$

and the norm

$$N(x,t) = (x^4+t^2)^{\frac{1}{4}}, \quad (x,t) \in \mathbb{K},$$

while its dual $\hat{\mathbb{K}}$ is generated by the differential difference operator

$$\Lambda = \Lambda_1^2 - (2\Lambda_2 + 2\frac{\partial}{\partial\lambda})^2,$$

and the function

$$\mathcal{N}(\lambda,m)=|\lambda|(m+\frac{\alpha+1}{2}),\quad (\lambda,m)\in\hat{\mathbb{K}},$$

The operators $\Lambda_1,\Lambda_2$ are given for a suitable function $g$ on $\hat{\mathbb{K}}$, by

$$\Lambda_1 g(\lambda,m)=\frac{1}{|\lambda|}\Big(m\Delta_+\Delta_- g(\lambda,m)+(\alpha+1)\Delta_+ g(\lambda,m)\Big)$$

$$\Lambda_2 g(\lambda,m)=-\frac{1}{2\lambda}\Big((\alpha+m+1)\Delta_+ g(\lambda,m)+m\Delta_- g(\lambda,m)\Big),$$

where the difference operators $\Delta_+,\Delta_-$ are given for a suitable function $g$ on $\hat{\mathbb{K}}$, by

$$\triangle_+ g(\lambda,m)=g(\lambda,m+1)-g(\lambda,m),$$

$$\triangle_- g(\lambda,m)=\begin{cases} g(\lambda,m)-g(\lambda,m-1), & \text{if} \quad m\geq 1\\ g(\lambda,0), & \text{if} \quad m=0.\end{cases}$$

These operators satisfy some basic properties which can be found in [2,39], namely one has

$$\mathcal{L}_\alpha\varphi_{\lambda,m}(x,t)=-\mathcal{N}(\lambda,m)\varphi_{\lambda,m}(x,t),\quad \Lambda\varphi_{\lambda,m}(x,t)=N^4(x,t)\varphi_{\lambda,m}(x,t).$$

**Definition 2.1.** *Let $f\in C_{*,c}(\mathbb{K})$. For all $(x,t)$ and $(y,s)$ in $\mathbb{K}$, we put*

(2.2)

$$\tau^{(\alpha)}_{(x,t)}f(y,s)=\begin{cases} \frac{1}{2\pi}\displaystyle\int_0^{2\pi} f(\sqrt{x^2+y^2+2xy\cos\theta},s+t+xy\sin\theta)d\theta, & \text{if } \alpha=0\\[2ex] \frac{\alpha}{\pi}\displaystyle\int_0^{2\pi}\int_0^1 f(\sqrt{x^2+y^2+2xyr\cos\theta},s+t+xyr\sin\theta)r(1-r^2)^{\alpha-1}drd\theta, & \text{if } \alpha>0.\end{cases}$$

The operators $\tau^{(\alpha)}_{(x,t)}$, $(x,t)\in\mathbb{K}$, are called generalized translation operators on $\mathbb{K}$.

**Proposition 2.1.** *For all $(\lambda,m)\in\hat{\mathbb{K}}$, the function $\varphi_{\lambda,m}$ satisfies the product formula*

$$\forall (x,t),(y,s)\in\mathbb{K},\quad \varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,s)=\tau^{(\alpha)}_{(x,t)}\varphi_{\lambda,m}(y,s). \tag{2.3}$$

**Corollary 2.1.** *For all $(\lambda,m)\in\hat{\mathbb{K}}$, the function $\varphi_{\lambda,m}$ is infinitely differentiable on $\mathbb{R}^2$, even with respect to the first variable and satisfies*

$$\sup_{(x,t)\in\mathbb{K}}|\varphi_{\lambda,m}(x,t)|=1. \tag{2.4}$$

We denote by

- $\mathcal{S}_*(\mathbb{K})$ the space of functions $f:\mathbb{R}^2\to\mathbb{C}$, even with respect to the first variable, $C^\infty$ on $\mathbb{R}^2$ and rapidly decreasing together with their derivatives, i.e., for all $k,p,q\in\mathbb{N}$ we have

$$N_{k,p,q}(f)=\sup_{(x,t)\in\mathbb{K}}\Big\{(1+x^2+t^2)^k|\frac{\partial^{p+q}}{\partial x^p\partial t^q}f(x,t)|\Big\}<\infty.$$

  Equipped with the topology defined by the semi-norms $N_{k,p,q}$, $\mathcal{S}_*(\mathbb{K})$ is a Fréchet space.
- $\mathcal{S}(\hat{\mathbb{K}})$ the space of functions $g:\hat{\mathbb{K}}\to\mathbb{C}$, such that

  (i) For all $m,p,q,r,s\in\mathbb{N}$, the function

$$\lambda\mapsto\lambda^p\left(|\lambda|(m+\frac{\alpha+1}{2})\right)^q\Lambda_1^r\left(\Lambda_2+\frac{\partial}{\partial\lambda}\right)^s g(\lambda,m)$$

  is bounded and continuous on $\mathbb{R}$, $C^\infty$ on $\mathbb{R}^*$ such that the left and the right derivatives at zero exist.

  (ii) For all $k,p,q\in\mathbb{N}$ we have

$$\nu_{k,p,q}(g)=\sup_{(\lambda,m)\in\mathbb{R}^*\times\mathbb{N}}\left\{\left(1+\lambda^2(1+m^2)\right)^k\left|\Lambda_1^p\left(\Lambda_2+\frac{\partial}{\partial\lambda}\right)^q g(\lambda,m)\right|\right\}<\infty.$$

Equipped with the topology defined by the semi-norms $\nu_{k,p,q}$, $\mathcal{S}(\hat{\mathbb{K}})$ is a Fréchet space.

**Definition 2.2.** *The generalized Fourier transform $\mathcal{F}_\alpha$ is defined on $L^1_\alpha(\mathbb{K})$ by*

$$\mathcal{F}_\alpha(f)(\lambda,m)=\int_{\mathbb{K}}\varphi_{-\lambda,m}(x,t)f(x,t)d\nu_\alpha(x,t),\quad \text{for all } (\lambda,m)\in\hat{\mathbb{K}}. \tag{2.5}$$

**Proposition 2.2.** *Let $f$ be in $L^1_\alpha(\mathbb{K})$. Then*

*(i) For all $m\in\mathbb{N}$, the function $\lambda\mapsto\mathcal{F}_\alpha(f)(\lambda,m)$ is continuous on $\mathbb{R}$.*

*(ii) The function $\mathcal{F}_\alpha(f)$ is bounded on $\hat{\mathbb{K}}$ and satisfies*

$$\|\mathcal{F}_\alpha(f)\|_{L^\infty_\alpha(\hat{\mathbb{K}})}\le\|f\|_{L^1_\alpha(\mathbb{K})}. \tag{2.6}$$

**Theorem 2.1.** *The generalized Fourier transform $\mathcal{F}_\alpha$ is a topological isomorphism from $\mathcal{S}_*(\mathbb{K})$ onto $\mathcal{S}(\hat{\mathbb{K}})$.*

**Theorem 2.2.** *(Plancherel's Theorem for $\mathcal{F}_\alpha$)*

i*) Plancherel's formula for $\mathcal{F}_\alpha$. For all $f$ in $\mathcal{S}_*(\mathbb{K})$ we have*

$$\int_{\hat{\mathbb{K}}}|\mathcal{F}_\alpha(f)(\lambda,m)|^2d\gamma(\lambda,m)=\int_{\mathbb{K}}|f(x,t)|^2d\nu_\alpha(x,t). \tag{2.7}$$

ii*) The generalized Fourier transform $\mathcal{F}_\alpha$ extends to an isometric isomorphism from $L^2_\alpha(\mathbb{K})$ onto $L^2_\alpha(\hat{\mathbb{K}})$.*

**Corollary 2.2.** *For all $f$ and $g$ in $L^2_\alpha(\mathbb{K})$ we have the following Parseval's formula for the generalized Fourier transform $\mathcal{F}_\alpha$*

$$\int_{\mathbb{K}}f(x,t)\overline{g(x,t)}d\nu_\alpha(x,t)=\int_{\hat{\mathbb{K}}}\mathcal{F}_\alpha(f)(\lambda,m)\overline{\mathcal{F}_\alpha(g)(\lambda,m)}d\gamma_\alpha(\lambda,m). \tag{2.8}$$

2.2. **Schatten-von Neumann classes.**

**Notations.** We denote by

- $l^p(\mathbb{N})$, $1\le p\le\infty$, the set of all infinite sequences of real (or complex) numbers $u:=(u_j)_{j\in\mathbb{N}}$, such that

$$\begin{aligned}||u||_p &:= \Big(\sum_{j=1}^\infty|u_j|^p\Big)^{\frac{1}{p}}<\infty,\quad \text{if}\quad 1\le p<\infty,\\ ||u||_\infty &:= \sup_{j\in\mathbb{N}}|u_j|<\infty.\end{aligned}$$

  For $p=2$, we provide this space $l^2(\mathbb{N})$ with the scalar product

$$\langle u,v\rangle_2:=\sum_{j=1}^\infty u_j\overline{v_j}.$$

- $B(L^p_\alpha(\mathbb{K}))$, $1\le p\le\infty$, the space of bounded operators from $L^p_\alpha(\mathbb{K})$ into itself.

**Definition 2.3.** *(i) The singular values $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator $A$ in $B(L^2_\alpha(\mathbb{K}))$ are the eigenvalues of the positive self-adjoint operator $|A|=\sqrt{A^*A}$.*

*(ii) For $1\le p<\infty$, the Schatten class $S_p$ is the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space $S_p$ is equipped with the norm*

$$||A||_{S_p}:=\Big(\sum_{n=1}^\infty(s_n(A))^p\Big)^{\frac{1}{p}}. \tag{2.9}$$

**Remark 2.1.** *We note that the space $S_2$ is the space of Hilbert-Schmidt operators, and $S_1$ is the space of trace class operators.*

**Definition 2.4.** *The trace of an operator $A$ in $S_1$ is defined by*

$$tr(A) = \sum_{n=1}^{\infty} \langle Av_n, v_n \rangle_{L^2_\alpha(\mathbb{K})} \tag{2.10}$$

*where $(v_n)_n$ is any orthonormal basis of $L^2_\alpha(\mathbb{K})$.*

**Remark 2.2.** *If $A$ is positive, then*

$$tr(A) = ||A||_{S_1}. \tag{2.11}$$

*Moreover, a compact operator $A$ on the Hilbert space $L^2_\alpha(\mathbb{K})$ is Hilbert-Schmidt, if the positive operator $A^*A$ is in the space of trace class $S_1$. Then*

$$||A||^2_{HS} := ||A||^2_{S_2} = ||A^*A||_{S_1} = tr(A^*A) = \sum_{n=1}^{\infty} ||Av_n||^2_{L^2_\alpha(\mathbb{K})} \tag{2.12}$$

*for any orthonormal basis $(v_n)_n$ of $L^2_\alpha(\mathbb{K})$.*

**Definition 2.5.** *We define $S_\infty := B(L^2_\alpha(\mathbb{K}))$, equipped with the norm,*

$$||A||_{S_\infty} := \sup_{v \in L^2_\alpha(\mathbb{K}) : ||v||_{L^2_\alpha(\mathbb{K})} = 1} ||Av||_{L^2_\alpha(\mathbb{K})}. \tag{2.13}$$

**Remark 2.3.** *It is obvious that $S_p \subset S_q$, $1 \le p \le q \le \infty$.*

## 3. Laguerre Two-Wavelet Multipliers

3.1. **Introduction.** Let $\sigma \in L^\infty_\alpha(\hat{\mathbb{K}})$, we define the linear operator $M_\sigma : L^2_\alpha(\mathbb{K}) \to L^2_\alpha(\mathbb{K})$ by

$$M_\sigma(f) = \mathcal{F}^{-1}_\alpha(\sigma \mathcal{F}_\alpha(f)). \tag{3.1}$$

This operator is called the generalized multiplier. Moreover, from Plancherel's formula (2.7), it is clair that $M_\sigma$ is bounded with

$$||M_\sigma||_{S_\infty} \le ||\sigma||_{L^\infty_\alpha(\hat{\mathbb{K}})}.$$

**Definition 3.1.** *Let $u, v$ be measurable functions on $\mathbb{K}$ and $\sigma$ measurable function on $\hat{\mathbb{K}}$, we define the Laguerre two-wavelet multiplier operator noted by $\mathcal{P}_{u,v}(\sigma)$, on $L^p_\alpha(\mathbb{K})$, $1 \le p \le \infty$, by*

$$\mathcal{P}_{u,v}(\sigma)(f)(x,t) = \int_{\hat{\mathbb{K}}} \sigma(\lambda,m)\mathcal{F}_\alpha(uf)(\lambda,m)\overline{\varphi_{\lambda,m}(x,t)v(x,t)}d\gamma_\alpha(\lambda,m), \ (x,t) \in \mathbb{K}. \tag{3.2}$$

In accordance with the different choices of the symbols $\sigma$ and the different continuities required, we need to impose different conditions on $u$ and $v$. And then we obtain an operator on $L^p_\alpha(\mathbb{K})$.

It is often more convenient to interpret the definition of $\mathcal{P}_{u,v}(\sigma)$ in a weak sense, that is, for $f$ in $L^p_\alpha(\mathbb{K})$, $p \in [1, \infty]$, and $g$ in $L^{p'}_\alpha(\mathbb{K})$,

$$\langle \mathcal{P}_{u,v}(\sigma)(f), g \rangle_{L^2_\alpha(\mathbb{K})} = \int_{\hat{\mathbb{K}}} \sigma(\lambda,m)\mathcal{F}_\alpha(uf)(\lambda,m)\overline{\mathcal{F}_\alpha(vg)(\lambda,m)}d\gamma_\alpha(\lambda,m). \tag{3.3}$$

**Proposition 3.1.** *Let $p \in [1, \infty)$. The adjoint of the linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_\alpha(\mathbb{K}) \to L^p_\alpha(\mathbb{K})$$

*is $\mathcal{P}_{v,u}(\overline{\sigma}) : L^{p'}_\alpha(\mathbb{K}) \to L^{p'}_\alpha(\mathbb{K})$.*

*Proof.* For all $f$ in $L^p_\alpha(\mathbb{K})$ and $g$ in $L^{p'}_\alpha(\mathbb{K})$ it follows immediately from (3.3)

$$\begin{aligned}\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{L^2_\alpha(\mathbb{K})} &= \int_{\hat{\mathbb{K}}} \sigma(\lambda,m)\mathcal{F}_\alpha(uf)(\lambda,m)\overline{\mathcal{F}_\alpha(vg)(\lambda,m)}d\gamma_\alpha(\lambda,m)\\ &= \overline{\int_{\hat{\mathbb{K}}} \overline{\sigma(\lambda,m)\mathcal{F}_\alpha(uf)(\lambda,m)}\mathcal{F}_\alpha(vg)(\lambda,m)d\gamma_\alpha(\lambda,m)}\\ &= \overline{\langle \mathcal{P}_{v,u}(\overline{\sigma})(g), f\rangle_{L^2_\alpha(\mathbb{K})}} = \langle f, \mathcal{P}_{v,u}(\overline{\sigma})(g)\rangle_{L^2_\alpha(\mathbb{K})}.\end{aligned}$$

Thus we get

$$\mathcal{P}^*_{u,v}(\sigma) = \mathcal{P}_{v,u}(\overline{\sigma}). \tag{3.4}$$

□

**Proposition 3.2.** *Let $\sigma \in L^1_\alpha(\hat{\mathbb{K}}) \cup L^\infty_\alpha(\hat{\mathbb{K}})$ and let $u, v \in L^2_\alpha(\mathbb{K}) \cap L^\infty_\alpha(\mathbb{K})$. Then*

$$\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{L^2_\alpha(\mathbb{K})} = \langle \bar{v}M_\sigma(uf), g\rangle_{L^2_\alpha(\mathbb{K})}. \tag{3.5}$$

*Proof.* For all $f, g$ in $L^2_\alpha(\mathbb{K})$ it follows immediately from (3.3), (3.1) and Parseval's formula (2.8)

$$\begin{aligned}\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{L^2_\alpha(\mathbb{K})} &= \int_{\hat{\mathbb{K}}} \sigma(\lambda,m)\mathcal{F}_\alpha(uf)(\lambda,m)\overline{\mathcal{F}_\alpha(vg)(\lambda,m)}d\gamma_\alpha(\lambda,m)\\ &= \int_{\hat{\mathbb{K}}} \mathcal{F}_\alpha(M_\sigma(uf))(\lambda,m)\overline{\mathcal{F}_\alpha(vg)(\lambda,m)}d\gamma_\alpha(\lambda,m)\\ &= \int_{\mathbb{K}} M_\sigma(uf)(x,t)\overline{(vg)(x,t)}d\nu_\alpha(x,t) = \langle \bar{v}M_\sigma(uf), g\rangle_{L^2_\alpha(\mathbb{K})}.\end{aligned}$$

Thus the proof is complete. □

In this section, $u$ and $v$ will be any functions in $L^2_\alpha(\mathbb{K}) \cap L^\infty_\alpha(\mathbb{K})$ such that

$$\|u\|_{L^2_\alpha(\mathbb{K})} = \|v\|_{L^2_\alpha(\mathbb{K})} = 1.$$

3.2. **Boundedness for $\mathcal{P}_{u,v}(\sigma)$ on $S_\infty$.** The main result of this subsection is to prove that the linear operators

$$\mathcal{P}_{u,v}(\sigma) : L^2_\alpha(\mathbb{K}) \to L^2_\alpha(\mathbb{K})$$

are bounded for all symbol $\sigma \in L^p_\alpha(\hat{\mathbb{K}})$, $1 \le p \le \infty$. We first consider this problem for $\sigma$ in $L^\infty_\alpha(\hat{\mathbb{K}})$ and next in $L^1_\alpha(\hat{\mathbb{K}})$ and then we conclude by using interpolation theory.

**Proposition 3.3.** *Let $\sigma$ be in $L^\infty_\alpha(\hat{\mathbb{K}})$, then the Laguerre two-wavelet multiplier operator $\mathcal{P}_{u,v}(\sigma)$ is in $S_\infty$ and we have*

$$||\mathcal{P}_{u,v}(\sigma)||_{S_\infty} \leqslant \|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}\|\sigma\|_{L^\infty_\alpha(\hat{\mathbb{K}})}.$$

*Proof.* For all functions $f$ and $g$ in $L^2_\alpha(\mathbb{K})$, we have from Cauchy-Schwarz's inequality

$$\begin{aligned}|\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{L^2_\alpha(\mathbb{K})}| &\leqslant \int_{\hat{\mathbb{K}}} |\sigma(\lambda,m)||\mathcal{F}_\alpha(uf)(\lambda,m)||\overline{\mathcal{F}_\alpha(vg)(\lambda,m)}|d\gamma_\alpha(\lambda,m)\\ &\leqslant \|\sigma\|_{L^\infty_\alpha(\hat{\mathbb{K}})}\|\mathcal{F}_\alpha(uf)\|_{L^2_\alpha(\hat{\mathbb{K}})}\|\mathcal{F}_\alpha(vg)\|_{L^2_\alpha(\hat{\mathbb{K}})}.\end{aligned}$$

Using Plancherel's formula (2.7) we get

$$|\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{L^2_\alpha(\mathbb{K})}| \leqslant \|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}\|\sigma\|_{L^\infty_\alpha(\hat{\mathbb{K}})}\|f\|_{L^2_\alpha(\mathbb{K})}\|g\|_{L^2_\alpha(\mathbb{K})}.$$

Thus,

$$||\mathcal{P}_{u,v}(\sigma)||_{S_\infty} \leqslant \|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}\|\sigma\|_{L^\infty_\alpha(\hat{\mathbb{K}})}.$$

□

**Proposition 3.4.** *Let $\sigma$ be in $L^1_\alpha(\hat{\mathbb{K}})$, then the Laguerre two-wavelet multiplier $\mathcal{P}_{u,v}(\sigma)$ is in $S_\infty$ and we have*

$$||\mathcal{P}_{u,v}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}. \tag{3.6}$$

*Proof.* For every functions $f$ and $g$ in $L^2_\alpha(\mathbb{K})$, from (3.3) we have,

$$\begin{aligned}|\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{L^2_\alpha(\mathbb{K})}| &\leqslant \int_{\hat{\mathbb{K}}} |\sigma(\lambda,m)|\,|\mathcal{F}_\alpha(uf)(\lambda,m)\overline{\mathcal{F}_\alpha(vg)(\lambda,m)}|d\gamma_\alpha(\lambda,m)\\ &\leqslant \|\mathcal{F}_\alpha(uf)\|_{L^\infty_\alpha(\hat{\mathbb{K}})}\|\mathcal{F}_\alpha(vg)\|_{L^\infty_\alpha(\hat{\mathbb{K}})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.\end{aligned}$$

Using relation (2.4) and the Cauchy-Schwarz inequality, we get

$$\|\mathcal{F}_\alpha(uf)\|_{L^\infty_\alpha(\hat{\mathbb{K}})} \leq \|u\|_{L^2_\alpha(\mathbb{K})}\|f\|_{L^2_\alpha(\mathbb{K})}, \quad \|\mathcal{F}_\alpha(vg)\|_{L^\infty_\alpha(\hat{\mathbb{K}})} \leq \|v\|_{L^2_\alpha(\mathbb{K})}\|g\|_{L^2_\alpha(\mathbb{K})}.$$

Hence we deduce that

$$|\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{L^2_\alpha(\mathbb{K})}| \leqslant \|f\|_{L^2_\alpha(\mathbb{K})}\|g\|_{L^2_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.$$

Thus,

$$||\mathcal{P}_{u,v}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.$$

□

We can now associate the Laguerre two-wavelet multiplier

$$\mathcal{P}_{u,v}(\sigma) : L^2_\alpha(\mathbb{K}) \to L^2_\alpha(\mathbb{K})$$

to every symbol $\sigma$ in $L^p_\alpha(\hat{\mathbb{K}})$, $1 \leq p \leq \infty$ and prove that $\mathcal{P}_{u,v}(\sigma)$ is in $S_\infty$. The precise result is the following theorem.

**Theorem 3.1.** *Let $\sigma$ be in $L^p_\alpha(\hat{\mathbb{K}})$, $1 \leq p \leq \infty$. Then there exists a unique bounded linear operator $\mathcal{P}_{u,v}(\sigma)$ : $L^2_\alpha(\mathbb{K}) \to L^2_\alpha(\mathbb{K})$, such that*

$$||\mathcal{P}_{u,v}(\sigma)||_{S_\infty} \leqslant (\|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})})^{\frac{p-1}{p}}\|\sigma\|_{L^p_\alpha(\hat{\mathbb{K}})}.$$

*Proof.* Let $f$ be in $L^2_\alpha(\mathbb{K})$. We consider the following operator

$$\mathcal{T} : L^1_\alpha(\hat{\mathbb{K}}) \bigcap L^\infty_\alpha(\hat{\mathbb{K}}) \quad \to \quad L^2_\alpha(\mathbb{K}),$$

given by

$$\mathcal{T}(\sigma) := \mathcal{P}_{u,v}(\sigma)(f).$$

Then by Proposition 3.3 and Proposition 3.4

$$||\mathcal{T}(\sigma)||_{L^2_\alpha(\mathbb{K})} \leqslant ||f||_{L^2_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})} \tag{3.7}$$

and

$$||\mathcal{T}(\sigma)||_{L^2_\alpha(\mathbb{K})} \leq \|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}||f||_{L^2_\alpha(\mathbb{K})}\|\sigma\|_{L^\infty_\alpha(\hat{\mathbb{K}})}. \tag{3.8}$$

Therefore, by (3.7), (3.8) and the Riesz-Thorin interpolation theorem (see [ [47], Theorem 2] and [ [51], Theorem 2.11]), $\mathcal{T}$ may be uniquely extended to a linear operator on $L^p_\alpha(\hat{\mathbb{K}})$, $1 \leq p \leq \infty$ and we have

$$||\mathcal{P}_{u,v}(\sigma)(f)||_{L^2_\alpha(\mathbb{K})} = ||\mathcal{T}(\sigma)||_{L^2_\alpha(\mathbb{K})} \leq (\|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})})^{\frac{p-1}{p}}||f||_{L^2_\alpha(\mathbb{K})}\|\sigma\|_{L^p_\alpha(\hat{\mathbb{K}})}. \tag{3.9}$$

Since (3.9) is true for arbitrary functions $f$ in $L^2_\alpha(\mathbb{K})$, then we obtain the desired result. □

3.3. **Traces of the Laguerre two-wavelet multipliers.** The main result of this subsection is to prove that, the Laguerre two-wavelet multiplier

$$\mathcal{P}_{u,v}(\sigma) : L^2_\alpha(\mathbb{K}) \to L^2_\alpha(\mathbb{K})$$

is in the Schatten class $S_p$.

**Proposition 3.5.** *Let $\sigma$ be in $L^1_\alpha(\hat{\mathbb{K}})$, then the Laguerre two-wavelet multiplier $\mathcal{P}_{u,v}(\sigma)$ is in $S_2$ and we have*

$$\|\mathcal{P}_{u,v}(\sigma)\|_{S_2} \leqslant \|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.$$

*Proof.* Let $\{\phi_j, j = 1, 2...\}$ be an orthonormal basis for $L^2_\alpha(\mathbb{K})$. Then by (3.3), Fubini's theorem, Parseval's identity and (3.4), we have

$$\begin{aligned}
\sum_{j=1}^{\infty} ||\mathcal{P}_{u,v}(\sigma)(\phi_j)||^2_{L^2_\alpha(\mathbb{K})} &= \sum_{j=1}^{\infty} \langle \mathcal{P}_{u,v}(\sigma)(\phi_j), \mathcal{P}_{u,v}(\sigma)(\phi_j)\rangle_{L^2_\alpha(\mathbb{K})} \\
&= \sum_{j=1}^{\infty} \int_{\hat{\mathbb{K}}} \sigma(\lambda, m)\langle \phi_j, \overline{u\varphi_{\lambda,m}}\rangle_{L^2_\alpha(\mathbb{K})} \overline{\langle \mathcal{P}_{u,v}(\sigma)(\phi_j), \overline{v\varphi_{\lambda,m}}\rangle}_{L^2_\alpha(\mathbb{K})} d\gamma_\alpha(\lambda, m) \\
&= \int_{\hat{\mathbb{K}}} \sigma(\lambda, m) \sum_{j=1}^{\infty} \langle \mathcal{P}^*_{u,v}(\sigma)(\overline{v\varphi_{\lambda,m}}), \phi_j\rangle_{L^2_\alpha(\mathbb{K})} \langle \phi_j, \overline{u\varphi_{\lambda,m}}\rangle_{L^2_\alpha(\mathbb{K})} d\gamma_\alpha(\lambda, m) \\
&= \int_{\hat{\mathbb{K}}} \sigma(\lambda, m)\langle \mathcal{P}^*_{u,v}(\sigma)(\overline{v\varphi_{\lambda,m}}), \overline{u\varphi_{\lambda,m}}\rangle_{L^2_\alpha(\mathbb{K})} d\gamma_\alpha(\lambda, m).
\end{aligned}$$

Thus from (3.6), Proposition 3.1 and (2.4), we get

$$\sum_{j=1}^{\infty} ||\mathcal{P}_{u,v}(\sigma)(\phi_j)||^2_{L^2_\alpha(\mathbb{K})} \le \int_{\hat{\mathbb{K}}} |\sigma(\lambda, m)| \, ||\mathcal{P}^*_{u,v}(\sigma)||_{S_\infty} d\gamma_\alpha(\lambda, m) \le \|\sigma\|^2_{L^1_\alpha(\hat{\mathbb{K}})} < \infty. \tag{3.10}$$

So, by (3.10) and the Proposition 2.8 in the book [51], by Wong,

$$\mathcal{P}_{u,v}(\sigma) : L^2_\alpha(\mathbb{K}) \to L^2_\alpha(\mathbb{K})$$

is in the Hilbert-Schmidt class $S_2$ and hence compact. □

**Proposition 3.6.** *Let $\sigma$ be a symbol in $L^p_\alpha(\hat{\mathbb{K}})$, $1 \leqslant p < \infty$. Then the Laguerre two-wavelet multiplier $\mathcal{P}_{u,v}(\sigma)$ is compact.*

*Proof.* Let $\sigma$ be in $L^p_\alpha(\hat{\mathbb{K}})$ and let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1_\alpha(\hat{\mathbb{K}}) \bigcap L^\infty_\alpha(\hat{\mathbb{K}})$ such that $\sigma_n \to \sigma$ in $L^p_\alpha(\hat{\mathbb{K}})$ as $n \to \infty$. Then by Theorem 3.1

$$||\mathcal{P}_{u,v}(\sigma_n) - \mathcal{P}_{u,v}(\sigma)||_{S_\infty} \le (\|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})})^{\frac{p-1}{p}} ||\sigma_n - \sigma||_{L^p_\alpha(\hat{\mathbb{K}})}.$$

Hence $\mathcal{P}_{u,v}(\sigma_n) \to \mathcal{P}_{u,v}(\sigma)$ in $S_\infty$ as $n \to \infty$. On the other hand as by Proposition 3.5 $\mathcal{P}_{u,v}(\sigma_n)$ is in $S_2$ hence compact, it follows that $\mathcal{P}_{u,v}(\sigma)$ is compact. □

**Theorem 3.2.** *Let $\sigma$ be in $L^1_\alpha(\hat{\mathbb{K}})$. Then $\mathcal{P}_{u,v}(\sigma) : L^2_\alpha(\mathbb{K}) \to L^2_\alpha(\mathbb{K})$ is in $S_1$ and we have*

$$\frac{2}{\|u\|^2_{L^\infty_\alpha(\mathbb{K})} + \|v\|^2_{L^\infty_\alpha(\mathbb{K})}} \|\widetilde{\sigma}\|_{L^1_\alpha(\hat{\mathbb{K}})} \leqslant \|\mathcal{P}_{u,v}(\sigma)\|_{S_1} \leqslant \|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}, \tag{3.11}$$

*where $\widetilde{\sigma}$ is given by*

$$\widetilde{\sigma}(\lambda, m) = \langle \mathcal{P}_{u,v}(\sigma)\, \varphi_{\lambda,m} u,\, \varphi_{\lambda,m} v\rangle_{L^2_\alpha(\mathbb{K})}, \quad (\lambda, m) \in \mathbb{K}.$$

*Proof.* Since $\sigma$ is in $L^1_\alpha(\hat{\mathbb{K}})$, by Proposition 3.5, $\mathcal{P}_{u,v}(\sigma)$ is in $S_2$. Using [51, Theorem 2.2], there exists an orthonormal basis $\{\phi_j,\ j=1,2...\}$ for the orthogonal complement of the kernel of the operator $\mathcal{P}_{u,v}(\sigma)$, consisting of eigenvectors of $|\mathcal{P}_{u,v}(\sigma)|$ and $\{\psi_j,\ j=1,2...\}$ an orthonormal set in $L^2_\alpha(\mathbb{K})$, such that

$$\mathcal{P}_{u,v}(\sigma)(f)=\sum_{j=1}^{\infty}s_j\langle f,\phi_j\rangle_{L^2_\alpha(\mathbb{K})}\psi_j, \tag{3.12}$$

where $s_j,\ j=1,2...$ are the positive singular values of $\mathcal{P}_{u,v}(\sigma)$ corresponding to $\phi_j$. Then, we get

$$\|\mathcal{P}_{u,v}(\sigma)\|_{S_1}=\sum_{j=1}^{\infty}s_j=\sum_{j=1}^{\infty}\langle\mathcal{P}_{u,v}(\sigma)(\phi_j),\psi_j\rangle_{L^2_\alpha(\mathbb{K})}.$$

Thus, by Fubini's theorem, Parseval's identity, Bessel's inequality, Cauchy-Schwarz's inequality, relation (2.4), and the fact $||u||_{L^2_\alpha(\mathbb{K})}=||v||_{L^2_\alpha(\mathbb{K})}=1$, we get

$$\begin{aligned}
\|\mathcal{P}_{u,v}(\sigma)\|_{S_1} &= \sum_{j=1}^{\infty}\langle\mathcal{P}_{u,v}(\sigma)(\phi_j),\psi_j\rangle_{L^2_\alpha(\mathbb{K})}\\
&= \sum_{j=1}^{\infty}\int_{\hat{\mathbb{K}}}\sigma(\lambda,m)\mathcal{F}_\alpha(u\phi_j)(\lambda,m)\overline{\mathcal{F}_\alpha(v\psi_j)(\lambda,m)}d\gamma_\alpha(\lambda,m)\\
&= \int_{\hat{\mathbb{K}}}\sigma(\lambda,m)\sum_{j=1}^{\infty}\langle\phi_j,\overline{u\varphi_{\lambda,m}}\rangle_{L^2_\alpha(\mathbb{K})}\langle\overline{v\varphi_{\lambda,m}},\psi_j\rangle_{L^2_\alpha(\mathbb{K})}d\gamma_\alpha(\lambda,m)\\
&\leq \int_{\hat{\mathbb{K}}}|\sigma(\lambda,m)|\Big(\sum_{j=1}^{\infty}|\langle\phi_j,\overline{u\varphi_{\lambda,m}}\rangle_{L^2_\alpha(\mathbb{K})}|^2\Big)^{\frac12}\Big(\sum_{j=1}^{\infty}|\langle\overline{v\varphi_{\lambda,m}},\psi_j\rangle_{L^2_\alpha(\mathbb{K})}|^2\Big)^{\frac12}d\gamma_\alpha(\lambda,m)\\
&\leq \int_{\hat{\mathbb{K}}}|\sigma(\lambda,m)|\,||\overline{u\varphi_{\lambda,m}}||_{L^2_\alpha(\mathbb{K})}||\overline{v\varphi_{\lambda,m}}||_{L^2_\alpha(\mathbb{K})}d\gamma_\alpha(\lambda,m)\\
&\leqslant \|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.
\end{aligned}$$

Thus

$$\|\mathcal{P}_{u,v}(\sigma)\|_{S_1}\leqslant\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.$$

We now prove that $\mathcal{P}_{u,v}(\sigma)$ satisfies the first member of (3.11). It is easy to see that $\widetilde{\sigma}$ belongs to $L^1_\alpha(\hat{\mathbb{K}})$, and using formula (3.12), we get

$$\begin{aligned}
|\widetilde{\sigma}(\lambda,m)| &= \Big|\langle\mathcal{P}_{u,v}(\sigma)(\ \varphi_{\lambda,m}u),\ \varphi_{\lambda,m}v\rangle_{L^2_\alpha(\mathbb{K})}\Big|\\
&= \Big|\sum_{j=1}^{\infty}s_j\langle\ \varphi_{\lambda,m}u,\phi_j\rangle_{L^2_\alpha(\mathbb{K})}\langle\psi_j,\ \varphi_{\lambda,m}v\rangle_{L^2_\alpha(\mathbb{K})}\Big|\\
&\leqslant \frac12\sum_{j=1}^{\infty}s_j\Big(\Big|\langle\ \varphi_{\lambda,m}u,\phi_j\rangle_{L^2_\alpha(\mathbb{K})}\Big|^2+\Big|\langle\ \varphi_{\lambda,m}v,\psi_j\rangle_{L^2_\alpha(\mathbb{K})}\Big|^2\Big).
\end{aligned}$$

By Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\hat{\mathbb{K}}}|\widetilde{\sigma}(\lambda,m)|d\gamma_\alpha(\lambda,m) &\leq \frac12\sum_{j=1}^{\infty}s_j\Big(\int_{\hat{\mathbb{K}}}|\langle\ \varphi_{\lambda,m}u,\phi_j\rangle_{L^2_\alpha(\mathbb{K})}|^2d\gamma_\alpha(\lambda,m)\\
&\quad+\int_{\hat{\mathbb{K}}}|\langle\ \varphi_{\lambda,m}v,\psi_j\rangle_{L^2_\alpha(\mathbb{K})}|^2d\gamma_\alpha(\lambda,m)\Big).
\end{aligned}$$

Then using Plancherel's formula given by relation (2.7), we get

$$\int_{\hat{\mathbb{K}}} |\widetilde{\sigma}(\lambda,m)| d\gamma_\alpha(\lambda,m) \le \frac{\|u\|^2_{L^\infty_\alpha(\mathbb{K})} + \|v\|^2_{L^\infty_\alpha(\mathbb{K})}}{2} \sum_{j=1}^{\infty} s_j = \frac{\|u\|^2_{L^\infty_\alpha(\mathbb{K})} + \|v\|^2_{L^\infty_\alpha(\mathbb{K})}}{2} \|\mathcal{P}_{u,v}(\sigma)\|_{S_1}.$$

The proof is complete. □

**Corollary 3.1.** *For $\sigma$ in $L^1_\alpha(\hat{\mathbb{K}})$, we have the following trace formula*

$$tr(\mathcal{P}_{u,v}(\sigma)) = \int_{\hat{\mathbb{K}}} \sigma(\lambda,m) \langle \overline{v\varphi_{\lambda,m}}, \overline{u\varphi_{\lambda,m}} \rangle_{L^2_\alpha(\mathbb{K})} d\gamma_\alpha(\lambda,m). \tag{3.13}$$

*Proof.* Let $\{\phi_j, j = 1, 2...\}$ be an orthonormal basis for $L^2_\alpha(\mathbb{K})$. From Theorem 3.2, the Laguerre two-wavelet multiplier $\mathcal{P}_{u,v}(\sigma)$ belongs to $S_1$, then by the definition of the trace given by the relation (2.10), Fubini's theorem and Parseval's identity, we have

$$\begin{aligned} tr(\mathcal{P}_{u,v}(\sigma)) &= \sum_{j=1}^{\infty} \langle \mathcal{P}_{u,v}(\sigma)(\phi_j), \phi_j \rangle_{L^2_\alpha(\mathbb{K})} \\ &= \sum_{j=1}^{\infty} \int_{\hat{\mathbb{K}}} \sigma(\lambda,m) \langle \phi_j, \overline{u\varphi_{\lambda,m}} \rangle_{L^2_\alpha(\mathbb{K})} \overline{\langle \phi_j, \overline{v\varphi_{\lambda,m}} \rangle}_{L^2_\alpha(\mathbb{K})} d\gamma_\alpha(\lambda,m) \\ &= \int_{\hat{\mathbb{K}}} \sigma(\lambda,m) \sum_{j=1}^{\infty} \langle \phi_j, \overline{u\varphi_{\lambda,m}} \rangle_{L^2_\alpha(\mathbb{K})} \langle \overline{v\varphi_{\lambda,m}}, \phi_j \rangle_{L^2_\alpha(\mathbb{K})} d\gamma_\alpha(\lambda,m) \\ &= \int_{\hat{\mathbb{K}}} \sigma(\lambda,m) \langle \overline{v\varphi_{\lambda,m}}, \overline{u\varphi_{\lambda,m}} \rangle_{L^2_\alpha(\mathbb{K})} d\gamma_\alpha(\lambda,m), \end{aligned}$$

and the proof is complete. □

In the following we give the main result of this subsection.

**Corollary 3.2.** *Let $\sigma$ be in $L^p_\alpha(\hat{\mathbb{K}})$, $1 \leqslant p \leqslant \infty$. Then, the Laguerre two-wavelet multiplier $\mathcal{P}_{u,v}(\sigma) : L^2_\alpha(\mathbb{K}) \longrightarrow L^2_\alpha(\mathbb{K})$ is in $S_p$ and we have*

$$\|\mathcal{P}_{u,v}(\sigma)\|_{S_p} \leqslant (\|u\|_{L^\infty_\alpha(\mathbb{K})} \|v\|_{L^\infty_\alpha(\mathbb{K})})^{\frac{p-1}{p}} \|\sigma\|_{L^p_\alpha(\hat{\mathbb{K}})}.$$

*Proof.* The result follows from Proposition 3.3, Theorem 3.2 and by interpolation (See [51, Theorem 2.10 and Theorem 2.11]). □

**Remark 3.1.** *If $u = v$ and if $\sigma$ is a real valued and nonnegative function in $L^1_\alpha(\hat{\mathbb{K}})$ then*

$$\mathcal{P}_{u,v}(\sigma) : L^2_\alpha(\mathbb{K}) \to L^2_\alpha(\mathbb{K})$$

*is a positive operator. So, by (2.11) and Corollary 3.1*

$$||\mathcal{P}_{u,v}(\sigma)||_{S_1} = \int_{\hat{\mathbb{K}}} \sigma(\lambda,m) ||\varphi_{\lambda,m} u||^2_{L^2_\alpha(\mathbb{K})} d\gamma_\alpha(\lambda,m). \tag{3.14}$$

Now we state a result concerning the trace of products of Laguerre two-wavelet multipliers.

**Corollary 3.3.** *Let $\sigma_1$ and $\sigma_2$ be any real-valued and non-negative functions in $L^1_\alpha(\hat{\mathbb{K}})$. We assume that $u = v$ and $u$ is a function in $L^2_\alpha(\mathbb{K})$ such that $||u||_{L^2_\alpha(\mathbb{K})} = 1$. Then, the Laguerre two-wavelet multipliers $\mathcal{P}_{u,v}(\sigma_1)$, $\mathcal{P}_{u,v}(\sigma_2)$ are positive trace class operators and*

$$\begin{aligned} \Big|\Big|\Big(\mathcal{P}_{u,v}(\sigma_1)\,\mathcal{P}_{u,v}(\sigma_2)\Big)^n\Big|\Big|_{S_1} &= tr\Big(\mathcal{P}_{u,v}(\sigma_1)\,\mathcal{P}_{u,v}(\sigma_2)\Big)^n \\ &\le \Big(tr\big(\mathcal{P}_{u,v}(\sigma_1)\big)\Big)^n \Big(tr\big(\mathcal{P}_{u,v}(\sigma_2)\big)\Big)^n \\ &= \Big|\Big|\mathcal{P}_{u,v}(\sigma_1)\Big|\Big|^n_{S_1} \Big|\Big|\mathcal{P}_{u,v}(\sigma_2)\Big|\Big|^n_{S_1}, \end{aligned}$$

*for all natural numbers $n$.*

*Proof.* By Theorem 1 in the paper [29] by Liu we know that if $A$ and $B$ are in the trace class $S_1$ and are positive operators, then

$$\forall\, n\in\mathbb{N},\quad tr(AB)^n \leq \Big(tr(A)\Big)^n\Big(tr(B)\Big)^n.$$

So, if we take $A=\mathcal{P}_{u,v}(\sigma_1)$, $B=\mathcal{P}_{u,v}(\sigma_2)$ and we invoke the previous remark, the proof is complete. □

## 4. The Generalized Landau-Pollak-Slepian Operator

4.1. **Trace formula.** Let $R$ and $R_1$ and $R_2$ be positive numbers. We define the linear operators

$$Q_R: L^2_\alpha(\hat{\mathbb{K}})\longrightarrow L^2_\alpha(\hat{\mathbb{K}}),\ P_{R_1}: L^2_\alpha(\hat{\mathbb{K}})\longrightarrow L^2_\alpha(\hat{\mathbb{K}}),\ P_{R_2}: L^2_\alpha(\hat{\mathbb{K}})\longrightarrow L^2_\alpha(\hat{\mathbb{K}}),$$

by

$$Q_Rh=\chi_{B_{\hat{\mathbb{K}}}(0,R)}h,\ P_{R_1}h=\mathcal{F}_\alpha(\chi_{B_{\mathbb{K}}(0,R_1)}(\mathcal{F}_\alpha)^{-1}(h)),\ P_{R_2}h=\mathcal{F}_\alpha(\chi_{B_{\mathbb{K}}(0,R_2)}(\mathcal{F}_\alpha)^{-1}(h)),$$

where

$$B_{\mathbb{K}}(0,R_i):=\Big\{(x,t)\in\mathbb{K}: N(x,t)<R_i\Big\},\ i=1,2$$

and

$$B_{\hat{\mathbb{K}}}(0,R):=\Big\{(\mu,n)\in\hat{\mathbb{K}}:\mathcal{N}(\mu,n)<R\Big\}.$$

We adapt the proof of Proposition 20.1 in the book [51] by Wong, we prove the following.

**Proposition 4.1.** *The operators $Q_R: L^2_\alpha(\hat{\mathbb{K}})\longrightarrow L^2_\alpha(\hat{\mathbb{K}})$, $P_{R_i}: L^2_\alpha(\hat{\mathbb{K}})\longrightarrow L^2_\alpha(\hat{\mathbb{K}})$, $i=1,2$, are self-adjoint projections.*

The bounded linear operator $P_{R_2}Q_RP_{R_1}: L^2_\alpha(\hat{\mathbb{K}})\longrightarrow L^2_\alpha(\hat{\mathbb{K}})$ it is called the generalized Landau-Pollak-Slepian operator. We can show that the generalized Landau-Pollak-Slepian operator is in fact a Laguerre two-wavelet multiplier.

**Theorem 4.1.** *Let $u$ and $v$ be the functions on $\mathbb{K}$ defined by*

$$u=\frac{1}{\sqrt{\nu_\alpha(B_{\mathbb{K}}(0,R_1))}}\chi_{B_{\mathbb{K}}(0,R_1)},\quad v=\frac{1}{\sqrt{\nu_\alpha(B_{\mathbb{K}}(0,R_2))}}\chi_{B_{\mathbb{K}}(0,R_2)},$$

*where*

$$\forall\, s>0,\quad \nu_\alpha(B_{\mathbb{K}}(0,s)):=\int_{B_{\mathbb{K}}(0,s)}d\nu_\alpha(x,t).$$

*Then the generalized Landau-Pollak-Slepian operator $P_{R_2}Q_RP_{R_1}: L^2_\alpha(\hat{\mathbb{K}})\longrightarrow L^2_\alpha(\hat{\mathbb{K}})$ is unitary equivalent to a scalar multiple of the Laguerre two-wavelet multiplier*

$$\mathcal{P}_{u,v}(\chi_{B_{\hat{\mathbb{K}}}(0,R)}): L^2_\alpha(\mathbb{K})\longrightarrow L^2_\alpha(\mathbb{K}).$$

*In fact*

$$P_{R_2}Q_RP_{R_1}=C_\alpha(R_1,R_2)\mathcal{F}_\alpha(\mathcal{P}_{u,v}(\chi_{B_{\hat{\mathbb{K}}}(0,R)}))(\mathcal{F}_\alpha)^{-1}, \tag{4.1}$$

*where*

$$C_\alpha(R_1,R_2):=\sqrt{\nu_\alpha(B_{\mathbb{K}}(0,R_1))\nu_\alpha(B_{\mathbb{K}}(0,R_2))}.$$

*Proof.* It is easy to see that $u$ and $v$ belong to $L^2_\alpha(\mathbb{K})\cap L^\infty_\alpha(\mathbb{K})$ and

$$||u||_{L^2_\alpha(\mathbb{K})}=||v||_{L^2_\alpha(\mathbb{K})}=1.$$

On the other hand we have

$$\langle\mathcal{P}_{u,v}(\chi_{B_{\hat{\mathbb{K}}}(0,R)})(f),g\rangle_{L^2_\alpha(\mathbb{K})}=\int_{\mathbb{K}}M_{\chi_{B_{\hat{\mathbb{K}}}(0,R)}}(uf)(x,t)\overline{(vg)(x,t)}d\nu_\alpha(x,t).$$

By simple calculations we find

$$\begin{aligned}\langle \mathcal{P}_{u,v}(\chi_{B_{\hat{\mathbb{K}}}(0,R)})(f),g\rangle_{L^2_\alpha(\mathbb{K})} &= \frac{1}{C_\alpha(R_1,R_2)}\int_{B_{\hat{\mathbb{K}}}(0,R)} P_{R_1}(\mathcal{F}_\alpha(f))(\lambda,m)\overline{P_{R_2}(\mathcal{F}_\alpha(g))(\lambda,m)}d\gamma_\alpha(\lambda,m)\\ &= \frac{1}{C_\alpha(R_1,R_2)}\int_{\hat{\mathbb{K}}} Q_R P_{R_1}(\mathcal{F}_\alpha(f))(\lambda,m)\overline{P_{R_2}(\mathcal{F}_\alpha(g))(\lambda,m)}d\gamma_\alpha(\lambda,m)\\ &= \frac{1}{C_\alpha(R_1,R_2)}\langle Q_R P_{R_1}(\mathcal{F}_\alpha(f)),P_{R_2}(\mathcal{F}_\alpha(g))\rangle_{L^2_\alpha(\hat{\mathbb{K}})}\\ &= \frac{1}{C_\alpha(R_1,R_2)}\langle P_{R_2}Q_R P_{R_1}(\mathcal{F}_\alpha(f)),\mathcal{F}_\alpha(g)\rangle_{L^2_\alpha(\hat{\mathbb{K}})}\\ &= \frac{1}{C_\alpha(R_1,R_2)}\langle \mathcal{F}_\alpha^{-1}P_{R_2}Q_R P_{R_1}(\mathcal{F}_\alpha(f)),g\rangle_{L^2_\alpha(\mathbb{K})}\end{aligned}$$

for all $f,g$ in $\mathcal{S}_*(\mathbb{K})$ and hence the proof is complete. □

The next result gives a formula for the trace of the generalized Landau-Pollak-Slepian operator

$$P_{R_2}Q_R P_{R_1}: L^2_\alpha(\hat{\mathbb{K}}) \longrightarrow L^2_\alpha(\hat{\mathbb{K}}).$$

**Corollary 4.1.** *We have*

$$tr(P_{R_2}Q_R P_{R_1}) = \int_{B_{\hat{\mathbb{K}}}(0,R)}\int_{B_{\mathbb{K}}(0,\min(R_1,R_2))} |\varphi_{\lambda,m}(x,t)|^2 d\nu_\alpha(x,t)d\gamma_\alpha(\lambda,m).$$

*Proof.* The result is an immediate consequence of Theorem 4.1 and Corollary 3.1. □

**Remark 4.1.** *(i) Let $\Sigma_1,\Sigma_2\subset\mathbb{K}$, $S\subset\widehat{\mathbb{K}}$ be a measurable subsets with $0<\nu_\alpha(\Sigma_i),\gamma_\alpha(S)<\infty$, $i=1,2$. Using similar ideas used in Theorem 4.1, we prove that*

(4.2) $$P_{\Sigma_2}Q_S P_{\Sigma_1} = C_\alpha(\Sigma_1,\Sigma_2)\mathcal{F}_\alpha(\mathcal{P}_{u,v}(\chi_S))\mathcal{F}_\alpha^{-1},$$

*where*

$$Q_S h = \chi_S h,\ P_{\Sigma_i}h = \mathcal{F}_\alpha(\chi_{\Sigma_i}(\mathcal{F}_\alpha^{-1})(h)),\ i=1,2,$$
$$u = \frac{1}{\sqrt{\nu_\alpha(\Sigma_1)}}\chi_{\Sigma_1},\quad v = \frac{1}{\sqrt{\nu_\alpha(\Sigma_2)}}\chi_{\Sigma_2}$$

*and*

$$C_\alpha(\Sigma_1,\Sigma_2) := \sqrt{\nu_\alpha(\Sigma_1)\nu_\alpha(\Sigma_2)}.$$

*(ii) Let $\Sigma\subset\mathbb{K}$, $S\subset\mathbb{K}$ be a pair of measurable subsets with $0<\nu_\alpha(\Sigma),\gamma_\alpha(S)<\infty$. Using similar ideas used in Corollary 4.1, we obtain*

$$tr(P_\Sigma Q_S P_\Sigma) = \int_S\int_\Sigma |\phi_{x,t}(\lambda,m)|^2 d\nu_\alpha(x,t)d\gamma_\alpha(\lambda,m).$$

### 4.2. **Donoho-Stark type uncertainty principle.**

One would like to find nonzero functions $f\in L^2_\alpha(\mathbb{K})$, which are timelimited on a subset $S\subset\widehat{\mathbb{K}}$ (i.e. $\mathrm{supp}f\subset S$) and bandlimited on a subset $\Sigma\subset\mathbb{K}$ (i.e. $\mathrm{supp}\mathcal{F}_\alpha(f)\subset\Sigma$), for sets $S$ and $\Sigma$ with finite measure. Unfortunately, such functions do not exist, because if $f$ is time and bandlimited on subsets of finite measure, then $f=0$. As a result, it is natural to replace the exact support by the essential support, and to focus on functions that are essentially time and bandlimited to a bounded region like $\Sigma\times S$ in the time-frequency plane. To do this, we introduce the following operators

$$E_\Sigma f = \chi_\Sigma f,\ F_S f = \mathcal{F}_\alpha^{-1}(\chi_S\mathcal{F}_\alpha(f)),\quad f\in L^2_\alpha(\mathbb{K}).$$

Concerning the meaning of “concentration” and “not too small” sets we adapt of a well-known notion from Fourier analysis (cf. [1,13]).

**Definition 4.1.** *Let $0\le\varepsilon<1$ and let $S\subset\widehat{\mathbb{K}}$, $\Sigma\subset\mathbb{K}$ be a pair of measurable subsets. Then*

(1) *a nonzero function $f\in L^2_\alpha(\mathbb{K})$ is $\varepsilon$-concentrated on $\Sigma$ if $\|E_{\Sigma^c}f\|_{L^2_\alpha(\mathbb{K})}\le\varepsilon\|f\|_{L^2_\alpha(\mathbb{K})}$,*
(2) *a nonzero function $f\in L^2_\alpha(\mathbb{K})$ is $\varepsilon$-bandlimited on $\mathbb{K}$ if $\|F_{S^c}f\|_{L^2_\alpha(\mathbb{K})}\le\varepsilon\|f\|_{L^2_\alpha(\mathbb{K})}$,*

(3) *a nonzero function* $f \in L^2_\alpha(\mathbb{K})$ *is* $\varepsilon$*-localized with respect to an operator*

$$\mathcal{L} : L^2_\alpha(\mathbb{K}) \to L^2_\alpha(\mathbb{K})$$

*if*

$$\|\mathcal{L}f - f\|_{L^2_\alpha(\mathbb{K})} \le \varepsilon \|f\|_{L^2_\alpha(\mathbb{K})}.$$

Here $A^c = \mathbb{K}\backslash A$ is the complement of $A$ in $\mathbb{K}$. Notice also that, the $\varepsilon$-concentration measure was introduced in [13, 25, 26], and the idea of $\varepsilon$-localization has been recently introduced in [1], which arises from the concept of pseudospectra of linear operators.

If $\varepsilon = 0$ in the $\varepsilon$-concentration measures, then $\Sigma$ and $S$ are respectively the exact support of $f$ and $\mathcal{F}_\alpha(f)$, moreover when $\varepsilon \in (0,1)$, $\Sigma$ and $S$ may be considered as the *essential* support of $f$ and $\mathcal{F}_\alpha(f)$ respectively. From Landau's point of view [25], $m$ is said to be an $\varepsilon$-approximated eigenvalue of the operator $L$, if there exists a unit $L^2$-norm function $f \in L^2_\alpha(\mathbb{K})$, such that

$$\|Lf - mf\|_{L^2_\alpha(\mathbb{K})} \le \varepsilon.$$

So that, if a function $f \in L^2_\alpha(\mathbb{K})$ is $\varepsilon$-localized with respect to an operator $L$, then $f$ is called an $\varepsilon$-approximated eigenfunction of $L$ with pseudoeigenvalue $1$. In particular, when $\varepsilon = 0$, then every function $f \in L^2_\alpha(\mathbb{K})$ which is $\varepsilon$-localized with respect to the operator $L$ is an eigenfunction of such operator corresponding to the eigenvalue $1$.

Now let $S \subset \widehat{\mathbb{K}}$, $\Sigma \subset \mathbb{K}$ be a pair of measurable subsets. We put

$$\mathcal{L}_S(f) := \mathcal{P}_{u,v}(\chi_S)(f),\ \mathcal{L}_\Sigma(f) := \chi_\Sigma f, \quad f \in L^2_\alpha(\mathbb{K}),$$

where we assume that $u$ and $v$ satisfy $\|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})} = 1$.

The main of this subsection is to prove the following Donoho-Stark type uncertainty principle.

**Theorem 4.2.** *Let* $\varepsilon_S, \varepsilon_\Sigma \in (0,1)$ *such that* $\varepsilon_S + \varepsilon_\Sigma < 1$. *If* $f \in L^2_\alpha(\mathbb{K})$ *is* $\varepsilon_S$*-localized with respect to* $\mathcal{L}_S$ *and* $\varepsilon_\Sigma$*-localized with respect to* $\mathcal{L}_\Sigma$ *then,*

$$\gamma_\alpha(S)\nu_\alpha(\Sigma) \ge 1 - \varepsilon_S - \varepsilon_\Sigma. \tag{4.3}$$

*Proof.* From Proposition 3.3, we have

$$\begin{aligned} \|f - \mathcal{L}_\Sigma\mathcal{L}_S f\|_{L^2_\alpha(\mathbb{K})} &\le \|f - \mathcal{L}_\Sigma f\|_{L^2_\alpha(\mathbb{K})} + \|\mathcal{L}_\Sigma f - \mathcal{L}_\Sigma\mathcal{L}_S f\|_{L^2_\alpha(\mathbb{K})} \\ &\le \|\mathcal{L}_\Sigma f - f\|_{L^2_\alpha(\mathbb{K})} + \|\mathcal{L}_\Sigma\|_{S_\infty}\|\mathcal{L}_S f - f\|_{L^2_\alpha(\mathbb{K})} \\ &\le (\varepsilon_\Sigma + \varepsilon_S)\|f\|_{L^2_\alpha(\mathbb{K})}. \end{aligned}$$

Therefore

$$\begin{aligned} \|\mathcal{L}_\Sigma\mathcal{L}_S f\|_{L^2_\alpha(\mathbb{K})} &\ge \|f\|_{L^2_\alpha(\mathbb{K})} - \|f - \mathcal{L}_\Sigma\mathcal{L}_S f\|_{L^2_\alpha(\mathbb{K})} \\ &\ge (1 - \varepsilon_S - \varepsilon_\Sigma)\|f\|_{L^2_\alpha(\mathbb{K})}. \end{aligned}$$

Thus from Proposition 3.4 it follows that

$$\begin{aligned} 1 - \varepsilon_S - \varepsilon_\Sigma &\le \|\mathcal{L}_\Sigma\mathcal{L}_S\|_{S_\infty} \\ &\le \|\mathcal{L}_S\|_{S_\infty}\|\mathcal{L}_\Sigma\|_{S_\infty} \\ &\le \gamma_\alpha(S)\nu_\alpha(\Sigma). \end{aligned}$$

This proves the desired result. □

**Corollary 4.2.** *If* $f \in L^2_\alpha(\mathbb{K})$ *is an eigenfunction of* $\mathcal{L}_S$ *and* $\mathcal{L}_\Sigma$ *corresponding to the same eigenvalue* $1$, *then*

$$\gamma_\alpha(S)\nu_\alpha(\Sigma) \ge 1. \tag{4.4}$$

*Proof.* Notice that, when $\varepsilon_S = \varepsilon_\Sigma = 0$ we have in this case $\Sigma = \mathrm{supp} f$, $S = \mathrm{supp}\mathcal{F}_\alpha(f)$ and we proceed as above theorem we obtain the result. □

**Remark 4.2.** *(1) As a first result, we can remark that the essential supports $S$ and $\Sigma$ cannot be too small.*

*(2) The result involves the couple $(\mathcal{L}_\Sigma f, \mathcal{L}_S f)$ and the rectangle $\Sigma \times S$ analogously to the Donoho-Stark UP which involves the couple $(f, \mathcal{F}(f))$ and the same rectangle.*

*(3) The estimate*

$$\gamma_\alpha(S)\nu_\alpha(\Sigma) \geq 1 - \varepsilon_S - \varepsilon_\Sigma$$

*is stronger then the classical Donoho-Stark estimate*

$$\gamma_\alpha(S)\nu_\alpha(\Sigma) \geq (1 - \varepsilon_S - \varepsilon_\Sigma)^2.$$

### 4.3. **Approximation inequalities.**

Let $S \subset \widehat{\mathbb{K}}$, $\Sigma \subset \mathbb{K}$ be a pair of measurable subsets with $0 < \nu_\alpha(\Sigma), \gamma_\alpha(S) < \infty$. Let $\varepsilon_S, \varepsilon_\Sigma \in [0, 1)$. We denote by $L^2_\alpha(\varepsilon_\Sigma, \varepsilon_S, \mathbb{K})$ the subspace of $L^2_\alpha(\mathbb{K})$ consisting of functions that are $\varepsilon_\Sigma$-concentrated on $\Sigma$ and $\varepsilon_S$-bandlimited on $S$ (clearly $L^2_\alpha(0, 0, \mathbb{K}) = \emptyset$).

We define the phase space restriction operator by

$$L_{\Sigma,S} := E_\Sigma F_S E_\Sigma.$$

It is clear that the operator $L_{\Sigma,S} : L^2_\alpha(\mathbb{K}) \to L^2_\alpha(\mathbb{K})$, special case of the generalized Landau-Pollak-Slepian operator, is compact, self-adjoint and then can be diagonalized as

$$L_{\Sigma,S} f = \sum_{n=1}^{\infty} \lambda_n \langle f, \varphi_n \rangle_{L^2_\alpha(\mathbb{K})} \varphi_n, \tag{4.5}$$

where $\{\lambda_n = \lambda_n(\Sigma, S)\}_{n=1}^{\infty}$ are the positive eigenvalues arranged in a non-increasing manner

$$\lambda_n \leq \cdots \leq \lambda_1 < 1, \tag{4.6}$$

and $\{\varphi_n = \varphi_n(\Sigma, S)\}_{n=1}^{\infty}$ is the corresponding orthonormal set of eigenfunctions. In particular

$$\|L_{\Sigma,S}\|_{S_\infty} = \lambda_1, \tag{4.7}$$

where $\lambda_1$ is the first eigenvalue corresponding to the first eigenfunction $\varphi_1$ of the compact operator $L_{\Sigma,S}$. This eigenfunction realizes the maximum of concentration on the set $S \times \Sigma$. On the other hand, since $\varphi_n$ is an eigenfunction of $L_{\Sigma,S}$ with eigenvalue $\lambda_n$, then

$$\|L_{\Sigma,S}\,\varphi_n - \varphi_n\|_{L^2_\alpha(\mathbb{K})} = \langle \varphi_n - L_{\Sigma,S}\,\varphi_n, \varphi_n \rangle_{L^2_\alpha(\mathbb{K})} = 1 - \lambda_n, \tag{4.8}$$

and

$$\begin{aligned} \|L_{\Sigma,S}\left(L_{\Sigma,S}\,\varphi_n\right) - L_{\Sigma,S}\,\varphi_n\|_{L^2_\alpha(\mathbb{K})} &= \lambda_n^{-1} \langle L_{\Sigma,S}\,\varphi_n - L_{\Sigma,S}\left(L_{\Sigma,S}\,\varphi_n\right), L_{\Sigma,S}\,\varphi_n \rangle_{L^2_\alpha(\mathbb{K})} \\ &= \lambda_n(1 - \lambda_n) = (1 - \lambda_n)\|L_{\Sigma,S}\,\varphi_n\|_{L^2_\alpha(\mathbb{K})}. \end{aligned} \tag{4.9}$$

Thus, for all $n$, the functions $\varphi_n$ and $L_{\Sigma,S}\,\varphi_n$ are $(1 - \lambda_n)$-localized with respect to $L_{\Sigma,S}$. More generally, we have the following comparisons of the measures of localization.

**Proposition 4.2.** *Let $\varepsilon, \varepsilon_\Sigma, \varepsilon_S \in (0, 1)$.*

1. *If $f \in L^2_\alpha(\varepsilon_\Sigma, \varepsilon_S, \mathbb{K})$, then $f$ is $(\varepsilon_S + \varepsilon_\Sigma)$-localized with respect to $F_S E_\Sigma$ and $(\varepsilon_S + 2\varepsilon_\Sigma)$-localized with respect to $L_{\Sigma,S}$.*
2. *If $f \in L^2_\alpha(\mathbb{K})$ is $\varepsilon$-localized with respect to $L_{\Sigma,S}$, then*

$$\langle f - L_{\Sigma,S} f, f \rangle_{L^2_\alpha(\mathbb{K})} \leq (\varepsilon^2 + \varepsilon)\|f\|^2_{L^2_\alpha(\mathbb{K})}. \tag{4.10}$$

(3) *If $f \in L^2_\alpha(\mathbb{K})$ satisfies*

$$\langle f - L_{\Sigma,S} f, f\rangle_{L^2_\alpha(\mathbb{K})} \le \varepsilon \|f\|^2_{L^2_\alpha(\mathbb{K})}, \tag{4.11}$$

*then $f$ is $\sqrt{\varepsilon}$-localized with respect to $L_{\Sigma,S}$.*

(4) *If $f \in L^2_\alpha(\varepsilon_\Sigma, \varepsilon_S, \mathbb{K})$, then*

$$\langle f - L_{\Sigma,S} f, f\rangle_{L^2_\alpha(\mathbb{K})} < (\varepsilon_S + 2\varepsilon_\Sigma) \|f\|^2_{L^2_\alpha(\mathbb{K})}. \tag{4.12}$$

*Proof.* (1) Recall that $\|F_S\|_{S_\infty} = \|E_\Sigma\|_{S_\infty} = 1$. First we have

$$\begin{aligned}
\|F_S E_\Sigma f - f\|_{L^2_\alpha(\mathbb{K})} &\le \|E_\Sigma f - f\|_{L^2_\alpha(\mathbb{K})} + \|F_S E_\Sigma f - E_\Sigma f\|_{L^2_\alpha(\mathbb{K})} \\
&\le \|E_{\Sigma^c} f\|_{L^2_\alpha(\mathbb{K})} + \|E_\Sigma\|_{S_\infty} \|F_{S^c} f\|_{L^2_\alpha(\mathbb{K})} \\
&\le (\varepsilon_S + \varepsilon_\Sigma) \|f\|_{L^2_\alpha(\mathbb{K})}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|L_{\Sigma,S} f - f\|_{L^2_\alpha(\mathbb{K})} &\le \|E_\Sigma F_S E_\Sigma f - E_\Sigma f\|_{L^2_\alpha(\mathbb{K})} + \|E_\Sigma f - f\|_{L^2_\alpha(\mathbb{K})} \\
&\le \|E_\Sigma\|_{S_\infty} \|F_S E_\Sigma f - f\|_{L^2_\alpha(\mathbb{K})} + \|E_\Sigma f - f\|_{L^2_\alpha(\mathbb{K})} \\
&\le (\varepsilon_S + 2\varepsilon_\Sigma) \|f\|_{L^2_\alpha(\mathbb{K})}.
\end{aligned}$$

Thus the first result is proved.

(2) Now since

$$\begin{aligned}
2\langle f - L_{\Sigma,S} f, f\rangle_{L^2_\alpha(\mathbb{K})} &= \|L_{\Sigma,S} f - f\|^2_{L^2_\alpha(\mathbb{K})} + \|f\|^2_{L^2_\alpha(\mathbb{K})} - \|L_{\Sigma,S} f\|^2_{L^2_\alpha(\mathbb{K})} \\
&\le \|L_{\Sigma,S} f - f\|^2_{L^2_\alpha(\mathbb{K})} + \Big(\|L_{\Sigma,S} f - f\|_{L^2_\alpha(\mathbb{K})} + \|L_{\Sigma,S} f\|_{L^2_\alpha(\mathbb{K})}\Big)^2 - \|L_{\Sigma,S} f\|^2_{L^2_\alpha(\mathbb{K})} \\
&= 2\|L_{\Sigma,S} f - f\|^2_{L^2_\alpha(\mathbb{K})} + 2\|L_{\Sigma,S} f - f\|_{L^2_\alpha(\mathbb{K})} \|L_{\Sigma,S} f\|_{L^2_\alpha(\mathbb{K})},
\end{aligned}$$

and as $\|L_{\Sigma,S}\|_{S_\infty} \le 1$, then

$$\langle f - L_{\Sigma,S} f, f\rangle_{L^2_\alpha(\mathbb{K})} \le \|L_{\Sigma,S} f - f\|^2_{L^2_\alpha(\mathbb{K})} + \|L_{\Sigma,S} f - f\|_{L^2_\alpha(\mathbb{K})} \|f\|_{L^2_\alpha(\mathbb{K})} \le (\varepsilon^2 + \varepsilon) \|f\|^2_{L^2_\alpha(\mathbb{K})}, \tag{4.13}$$

and the second result follows.

(3) On the other hand, since

$$\Big\langle (L_{\Sigma,S})^2 f, f \Big\rangle_{L^2_\alpha(\mathbb{K})} \le \langle L_{\Sigma,S} f, f\rangle_{L^2_\alpha(\mathbb{K})}, \tag{4.14}$$

and as $L_{\Sigma,S}$ is self-adjoint, then

$$\|L_{\Sigma,S} f - f\|^2_{L^2_\alpha(\mathbb{K})} = \Big\langle (I - L_{\Sigma,S})^2 f, f \Big\rangle_{L^2_\alpha(\mathbb{K})} \le \langle (I - L_{\Sigma,S}) f, f\rangle_{L^2_\alpha(\mathbb{K})} \le \varepsilon \|f\|^2_{L^2_\alpha(\mathbb{K})}. \tag{4.15}$$

(4) Finally, since

$$\langle f - L_{\Sigma,S} f, f\rangle_{L^2_\alpha(\mathbb{K})} = \langle E_{\Sigma^c} f, f\rangle_{L^2_\alpha(\mathbb{K})} + \langle E_\Sigma f, F_{S^c} f\rangle_{L^2_\alpha(\mathbb{K})} + \langle F_S E_\Sigma f, E_{\Sigma^c} f\rangle_{L^2_\alpha(\mathbb{K})},$$

then we obtain the last result. □

The estimate (4.11) is equivalent to

$$\langle L_{\Sigma,S} f, f\rangle_{L^2_\alpha(\mathbb{K})} \ge (1 - \varepsilon) \|f\|^2_{L^2_\alpha(\mathbb{K})}, \tag{4.16}$$

and we denote by $L^2_\alpha(\varepsilon, S, \Sigma, \mathbb{K})$ the subspace of $L^2_\alpha(\mathbb{K})$ consisting of functions $f \in L^2_\alpha(\mathbb{K})$ satisfying (4.16). Hence from (4.8) and (4.9) we have,

$$\forall\, n \ge 1, \qquad \varphi_n,\ L_{\Sigma,S}\, \varphi_n \in L^2_\alpha(1 - \lambda_n, S, \Sigma, \mathbb{K}). \tag{4.17}$$

Moreover from Proposition 4.2, if $f \in L^2_\alpha(\varepsilon_\Sigma, \varepsilon_S, \mathbb{K})$, then $f \in L^2_\alpha(\varepsilon_S + 2\varepsilon_\Sigma, S, \Sigma, \mathbb{K})$, and if $f$ is $\varepsilon$-localized with respect to $L_{\Sigma,S}$, then $f \in L^2_\alpha(2\varepsilon, S, \Sigma, \mathbb{K})$. Therefore we are interested to study the following optimization problem

$$\text{Maximize} \qquad \langle L_{\Sigma,S} f, f\rangle_{L^2_\alpha(\mathbb{K})}, \qquad \|f\|_{L^2_\alpha(\mathbb{K})} = 1, \tag{4.18}$$

which aims to look for orthonormal functions in $L^2_\alpha(\mathbb{K})$, which are approximately time and band-limited to a bounded region like $\Sigma \times S$. It follows that the number of eigenfunctions of $L_{\Sigma,S}$ whose eigenvalues are very close to one, are an optimal solutions to the problem (4.18), since if $\varphi_n$ is an eigenfunction of $L_{\Sigma,S}$ with eigenvalue $\lambda_n \geq (1-\varepsilon)$, we have from the spectral representation,

$$\langle L_{\Sigma,S}\, \varphi_n, \varphi_n\rangle_{L^2_\alpha(\mathbb{K})} = \lambda_n \geq (1-\varepsilon). \tag{4.19}$$

We denote by $N(\varepsilon, \Sigma, S)$ the number of eigenvalues $\lambda_n$ of $L_{\Sigma,S}$ which are close to one, in the sense that

$$\lambda_1 \geq \cdots \geq \lambda_{N(\varepsilon,\Sigma,S)} \geq 1-\varepsilon > \lambda_{1+N(\varepsilon,\Sigma,S)} \geq \cdots, \tag{4.20}$$

and we denote by $V_{N(\varepsilon,\Sigma,S)} = \operatorname{span}\{\varphi_n\}_{n=1}^{N(\varepsilon,\Sigma,S)}$ the span of the first eigenfunctions of $L_{\Sigma,S}$ corresponding to the largest eigenvalues $\{\lambda_n\}_{n=1}^{N(\varepsilon,\Sigma,S)}$.

Therefore, by (4.19) and (4.17), each eigenfunction $\varphi_n$ and its resulting function $L_{\Sigma,S}\,\varphi_n$ are in $L^2_\alpha(\varepsilon, S, \Sigma, \mathbb{K})$, if and only if $1 \leq n \leq N(\varepsilon, \Sigma, S)$. Now, if $f \in V_{N(\varepsilon,\Sigma,S)}$, then

$$\langle L_{\Sigma,S} f, f\rangle_{L^2_\alpha(\mathbb{K})} = \sum_{n=1}^{N(\varepsilon,\Sigma,S)} \lambda_n \left|\langle f, \varphi_n\rangle_{L^2_\alpha(\mathbb{K})}\right|^2 \geq \lambda_{N(\varepsilon,\Sigma,S)} \sum_{n=1}^{N(\varepsilon,\Sigma,S)} \left|\langle f, \varphi_n\rangle_{L^2_\alpha(\mathbb{K})}\right|^2 \geq (1-\varepsilon)\|f\|^2_{L^2_\alpha(\mathbb{K})}.$$

Thus $V_{N(\varepsilon,\Sigma,S)}$ determines the subspace of $L^2_\alpha(\mathbb{K})$ with maximum dimension that is in $L^2_\alpha(\varepsilon, S, \Sigma, \mathbb{K})$. Motivated by the recent paper [50] in the Gabor setting, we obtain the following theorem that characterizes functions that are in $L^2_\alpha(\varepsilon, S, \Sigma, \mathbb{K})$.

**Proposition 4.3.** *Let $\varepsilon \in (0,1)$ be a fixed real number. Let $f_{\ker}$ denote the orthogonal projection of $f$ onto the kernel $\operatorname{Ker}(L_{\Sigma,S})$ of $L_{\Sigma,S}$. Then a function $f$ is in $L^2_\alpha(\varepsilon, S, \Sigma, \mathbb{K})$ if and only if,*

$$\sum_{n=1}^{N(\varepsilon,\Sigma,S)} (\lambda_n + \varepsilon - 1)\left|\langle f, \varphi_n\rangle_{L^2_\alpha(\mathbb{K})}\right|^2 \geq (1-\varepsilon)\|f_{\ker}\|^2_{L^2_\alpha(\mathbb{K})} + \sum_{n=1+N(\varepsilon,\Sigma,S)}^{\infty} (1-\varepsilon-\lambda_n)\left|\langle f, \varphi_n\rangle_{L^2_\alpha(\mathbb{K})}\right|^2.$$

*Proof.* For a given function $f \in L^2_\alpha(\mathbb{K})$, write

$$f = \sum_{n=1}^{\infty} \langle f, \varphi_n\rangle_{L^2_\alpha(\mathbb{K})}\varphi_n + f_{\ker}, \tag{4.21}$$

where $f_{\ker} \in \operatorname{Ker}(L_{\Sigma,S})$. Then

$$\langle L_{\Sigma,S} f, f\rangle_{L^2_\alpha(\mathbb{K})} = \sum_{n=1}^{\infty} \lambda_n \left|\langle f, \varphi_n\rangle_{L^2_\alpha(\mathbb{K})}\right|^2. \tag{4.22}$$

So the function $f$ is in $L^2_\alpha(\varepsilon, S, \Sigma, \mathbb{K})$ if and only if

$$\sum_{n=1}^{\infty} \lambda_n \left|\langle f, \varphi_n\rangle_{L^2_\alpha(\mathbb{K})}\right|^2 \geq (1-\varepsilon)\left(\|f_{\ker}\|^2_{L^2_\alpha(\mathbb{K})} + \sum_{n=1}^{\infty}\left|\langle f, \varphi_n\rangle_{L^2_\alpha(\mathbb{K})}\right|^2\right), \tag{4.23}$$

and the conclusion follows. □

While a function $f$ that is in $L^2_\alpha(\varepsilon, S, \Sigma, \mathbb{K})$ does not necessarily lies in some subspace
$V_N = \operatorname{span}\{\varphi_n\}_{n=1}^{N}$, it can be approximated using a finite number of such eigenfunctions.
Let $\varepsilon_0 \in (0,1)$ be a fixed real number and let $\mathcal{P}$ the orthogonal projection onto the subspace $V_{N(\varepsilon_0,\Sigma,S)}$.

**Proposition 4.4.** *Let $f$ be a function in $L^2_\alpha(\varepsilon, S, \Sigma, \mathbb{K})$. Then*

$$\left\| f - \sum_{n=1}^{N(\varepsilon_0,\Sigma,S)} \langle f, \varphi_n\rangle_{L^2_\alpha(\mathbb{K})}\varphi_n \right\|_{L^2_\alpha(\mathbb{K})} \leq \sqrt{\frac{\varepsilon}{\varepsilon_0}}\, \|f\|_{L^2_\alpha(\mathbb{K})}. \tag{4.24}$$

*Proof.* By an easy adaptation of the proof of Proposition 3.3 in [50], we can conclude that

$$\|\mathcal{P}f\|^2_{L^2_\alpha(\mathbb{K})} \geq (1-\varepsilon/\varepsilon_0)\|f\|^2_{L^2_\alpha(\mathbb{K})}. \tag{4.25}$$

It then follows,

$$\|f\|^2_{L^2_\alpha(\mathbb{K})} = \|\mathcal{P}f + (f-\mathcal{P}f)\|^2_{L^2_\alpha(\mathbb{K})} = \|\mathcal{P}f\|^2_{L^2_\alpha(\mathbb{K})} + \|f-\mathcal{P}f\|^2_{L^2_\alpha(\mathbb{K})}.$$

Thus

$$\|f-\mathcal{P}f\|^2_{L^2_\alpha(\mathbb{K})} = \|f\|^2_{L^2_\alpha(\mathbb{K})} - \|\mathcal{P}f\|^2_{L^2_\alpha(\mathbb{K})} \leq \|f\|^2_{L^2_\alpha(\mathbb{K})} - (1-\varepsilon/\varepsilon_0)\|f\|^2_{L^2_\alpha(\mathbb{K})} = \varepsilon/\varepsilon_0\|f\|^2_{L^2_\alpha(\mathbb{K})}.$$

This completes the proof of the theorem. □

Consequently and from Proposition 4.2, we immediately deduce the following approximation results.

**Corollary 4.3.** *Let $\varepsilon, \varepsilon_\Sigma, \varepsilon_S \in (0,1)$.*

(1) *If $f \in L^2_\alpha(\varepsilon_\Sigma, \varepsilon_S, \mathbb{K})$, then*

$$\left\| f - \sum_{n=1}^{N(\varepsilon_0,\Sigma,S)} \langle f, \varphi_n\rangle_{L^2_\alpha(\mathbb{K})}\varphi_n \right\|_{L^2_\alpha(\mathbb{K})} \leq \sqrt{\frac{2\varepsilon_S+\varepsilon_\Sigma}{\varepsilon_0}}\, \|f\|_{L^2_\alpha(\mathbb{K})}. \tag{4.26}$$

(2) *If $f \in L^2_\alpha(\mathbb{K})$ is $\varepsilon$-localized with respect to $L_{\Sigma,S}$, then*

$$\left\| f - \sum_{n=1}^{N(\varepsilon_0,\Sigma,S)} \langle f, \varphi_n\rangle_{L^2_\alpha(\mathbb{K})}\varphi_n \right\|_{L^2_\alpha(\mathbb{K})} \leq \sqrt{\frac{2\varepsilon}{\varepsilon_0}}\, \|f\|_{L^2_\alpha(\mathbb{K})}. \tag{4.27}$$

## 5. $L^p$ Boundedness and Compactness of $\mathcal{P}_{u,v}(\sigma)$

5.1. **Boundedness for symbols in $L^p_\alpha(\hat{\mathbb{K}})$.** For $1 \leq p \leq \infty$, let $\sigma \in L^1_\alpha(\hat{\mathbb{K}})$, $v \in L^p_\alpha(\mathbb{K})$ and $u \in L^{p'}_\alpha(\mathbb{K})$ .

We are going to show that $\mathcal{P}_{u,v}(\sigma)$ is a bounded operator on $L^p_\alpha(\mathbb{K})$. Let us start with the following propositions.

**Proposition 5.1.** *Let $\sigma$ be in $L^1_\alpha(\hat{\mathbb{K}})$, $u \in L^\infty_\alpha(\mathbb{K})$ and $v \in L^1_\alpha(\mathbb{K})$, then the Laguerre two-wavelet multiplier*

$$\mathcal{P}_{u,v}(\sigma) : L^1_\alpha(\mathbb{K}) \longrightarrow L^1_\alpha(\mathbb{K})$$

*is a bounded linear operator and we have*

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^1_\alpha(\mathbb{K}))} \leqslant \|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^1_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.$$

*Proof.* For every function $f$ in $L^1_\alpha(\mathbb{K})$, we have from the relations (3.2), (2.6) and (2.4)

$$\begin{aligned} ||\mathcal{P}_{u,v}(\sigma)(f)||_{L^1_\alpha(\mathbb{K})} &\leqslant \int_{\hat{\mathbb{K}}}\int_{\mathbb{K}} |\sigma(\lambda,m)|\,|\mathcal{F}_\alpha(uf)(\lambda,m)|\,|\varphi_{\lambda,m}(x,t)v(x,t)|d\gamma_\alpha(\lambda,m)d\nu_\alpha(x,t) \\ &\leqslant \|f\|_{L^1_\alpha(\mathbb{K})}||u||_{L^\infty_\alpha(\mathbb{K})}||v||_{L^1_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})} \end{aligned}$$

Thus,

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^1_\alpha(\mathbb{K}))} \leqslant ||u||_{L^\infty_\alpha(\mathbb{K})}||v||_{L^1_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.$$

□

**Proposition 5.2.** *Let $\sigma$ be in $L^1_\alpha(\hat{\mathbb{K}})$, $u \in L^1_\alpha(\mathbb{K})$ and $v \in L^\infty_\alpha(\mathbb{K})$, then the Laguerre two-wavelet multiplier*

$$\mathcal{P}_{u,v}(\sigma) : L^\infty_\alpha(\mathbb{K}) \longrightarrow L^\infty_\alpha(\mathbb{K})$$

*is a bounded linear operator and we have*

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^\infty_\alpha(\mathbb{K}))} \leqslant \|u\|_{L^1_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.$$

*Proof.* Let $f$ be in $L^\infty_\alpha(\mathbb{K})$. As above from the relations (3.2), (2.6) and (2.4)

$$\begin{aligned}\forall\,(x,t) \in \mathbb{K}, \quad |\mathcal{P}_{u,v}(\sigma)(f)(x,t)| &\leqslant \int_{\hat{\mathbb{K}}} |\sigma(\lambda,m)|\,|\mathcal{F}_\alpha(uf)(\lambda,m)|\,|\varphi_{\mu,\lambda}(x,t)v(x,t)|d\gamma_\alpha(\lambda,m)\\ &\leqslant \|f\|_{L^\infty_\alpha(\mathbb{K})}||u||_{L^1_\alpha(\mathbb{K})}||v||_{L^\infty_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}\end{aligned}$$

Thus,

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^\infty_\alpha(\mathbb{K}))} \leqslant ||u||_{L^1_\alpha(\mathbb{K})}||v||_{L^\infty_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.$$

□

**Remark 5.1.** *Proposition 5.2 is also a corollary of Proposition 5.1, since the adjoint of*

$$\mathcal{P}_{v,u}(\overline{\sigma}) : L^1_\alpha(\mathbb{K}) \to L^1_\alpha(\mathbb{K})$$

*is $\mathcal{P}_{u,v}(\sigma) : L^\infty_\alpha(\mathbb{K}) \to L^\infty_\alpha(\mathbb{K})$.*

Using an interpolation of Propositions 5.1 and 5.2, we get the following result.

**Theorem 5.1.** *Let $u$ and $v$ be functions in $L^1_\alpha(\mathbb{K}) \cap L^\infty_\alpha(\mathbb{K})$. Then for all $\sigma$ in $L^1_\alpha(\hat{\mathbb{K}})$, there exists a unique bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_\alpha(\mathbb{K}) \longrightarrow L^p_\alpha(\mathbb{K}), \quad 1 \leq p \leq \infty,$$

*such that*

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_\alpha(\mathbb{K}))} \leqslant \|u\|^{\frac{1}{p'}}_{L^1_\alpha(\mathbb{K})}\|v\|^{\frac{1}{p}}_{L^1_\alpha(\mathbb{K})}\|u\|^{\frac{1}{p}}_{L^\infty_\alpha(\mathbb{K})}\|v\|^{\frac{1}{p'}}_{L^\infty_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.$$

With a Schur technique, we can obtain an $L^p$-boundedness result as in the previous Theorem, but the estimate for the norm $||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_\alpha(\mathbb{K}))}$ is cruder.

**Theorem 5.2.** *Let $\sigma$ be in $L^1_\alpha(\hat{\mathbb{K}})$, $u$ and $v$ in $L^1_\alpha(\mathbb{K}) \bigcap L^\infty_\alpha(\mathbb{K})$. Then there exists a unique bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_\alpha(\mathbb{K}) \longrightarrow L^p_\alpha(\mathbb{K}), \quad 1 \leq p \leq \infty,$$

*such that*

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_\alpha(\mathbb{K}))} \leqslant \max(\|u\|_{L^1_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}, \|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^1_\alpha(\mathbb{K})})\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.$$

*Proof.* Let $\mathcal{N}$ be the function defined on $\mathbb{K} \times \mathbb{K}$ by

$$\mathcal{N}(x,t;s,y) = \int_{\hat{\mathbb{K}}} \sigma(\lambda,m)\overline{\varphi_{\lambda,m}(x,t)v(x,t)}\varphi_{\lambda,m}(s,y)u(s,y)d\gamma_\alpha(\lambda,m). \tag{5.1}$$

We have

$$\mathcal{P}_{u,v}(\sigma)(f)(x,t) = \int_{\mathbb{K}} \mathcal{N}(x,t;s,y)f(s,y)d\nu_\alpha(s,y).$$

By simple calculations, it is easy to see that

$$\int_{\mathbb{K}} |\mathcal{N}(x,t;s,y)|d\nu_\alpha(x,t) \leq \|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^1_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})},\ (s,y) \in \mathbb{K},$$

and

$$\int_{\mathbb{K}} |\mathcal{N}(x,t;s,y)|d\nu_\alpha(s,y) \leq \|u\|_{L^1_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})},\ (x,t) \in \mathbb{K}.$$

Thus by Schur Lemma (cf. [16]), we can conclude that

$$\mathcal{P}_{u,v}(\sigma) : L^p_\alpha(\mathbb{K}) \longrightarrow L^p_\alpha(\mathbb{K})$$

is a bounded linear operator for $1 \leq p \leq \infty$, and we have

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_\alpha(\mathbb{K}))} \leqslant \max(\|u\|_{L^1_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}, \|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^1_\alpha(\mathbb{K})})\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.$$

□

**Remark 5.2.** *The previous Theorem tells us that the unique bounded linear operator on $L^p_\alpha(\mathbb{K})$, $1 \leq p \leq \infty$, obtained by interpolation in Theorem 5.1 is in fact the integral operator on $L^p_\alpha(\mathbb{K})$ with kernel $\mathcal{N}$ given by (5.1).*

We can give another version of the $L^p$-boundedness. Firstly we generalize and we improve Proposition 5.2.

**Proposition 5.3.** *Let $\sigma$ be in $L^1_\alpha(\hat{\mathbb{K}})$, $v \in L^p_\alpha(\mathbb{K})$ and $u \in L^{p'}_\alpha(\mathbb{K})$, for $1 < p \leq \infty$, then the Laguerre two-wavelet multiplier*

$$\mathcal{P}_{u,v}(\sigma) : L^p_\alpha(\mathbb{K}) \longrightarrow L^p_\alpha(\mathbb{K})$$

*is a bounded linear operator, and we have*

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_\alpha(\mathbb{K}))} \leqslant \|u\|_{L^{p'}_\alpha(\mathbb{K})}\|v\|_{L^p_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.$$

*Proof.* For any $f \in L^p_\alpha(\mathbb{K})$, consider the linear functional

$$\begin{array}{rccc} \mathcal{I}_f : & L^{p'}_\alpha(\mathbb{K}) & \rightarrow & \mathbb{C} \\ & g & \mapsto & \langle g, \mathcal{P}_{u,v}(\sigma)(f)\rangle_{L^2_\alpha(\mathbb{K})}. \end{array}$$

From the relation (3.3)

$$\begin{aligned} |\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{L^2_\alpha(\mathbb{K})}| &\leqslant \int_{\hat{\mathbb{K}}} |\sigma(\lambda,m)||\mathcal{F}_\alpha(uf)(\lambda,m)|\,|\mathcal{F}_\alpha(vg)(\lambda,m)|d\gamma_\alpha(\lambda,m) \\ &\leqslant \|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}\|\mathcal{F}_\alpha(uf)\|_{L^\infty_\alpha(\hat{\mathbb{K}})}\|\mathcal{F}_\alpha(vg)\|_{L^\infty_\alpha(\hat{\mathbb{K}})}. \end{aligned}$$

Using the relation (2.5), (2.4) and Hölder's inequality, we get

$$|\langle \mathcal{P}_{u,v}(\sigma)(f), g\rangle_{L^2_\alpha(\mathbb{K})}| \leqslant \|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}\|u\|_{L^{p'}_\alpha(\mathbb{K})}\|v\|_{L^p_\alpha(\mathbb{K})}\|f\|_{L^p_\alpha(\mathbb{K})}\|g\|_{L^{p'}_\alpha(\mathbb{K})}.$$

Thus, the operator $\mathcal{I}_f$ is a continuous linear functional on $L^{p'}_\alpha(\mathbb{K})$, and the operator norm

$$||\mathcal{I}_f||_{B(L^{p'}_\alpha(\mathbb{K}))} \leqslant \|u\|_{L^{p'}_\alpha(\mathbb{K})}\|v\|_{L^p_\alpha(\mathbb{K})}\|f\|_{L^p_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.$$

As $\mathcal{I}_f(g) = \langle g, \mathcal{P}_{u,v}(\sigma)(f)\rangle_{L^2_\alpha(\mathbb{K})}$, by the Riesz representation theorem, we have

$$||\mathcal{P}_{u,v}(\sigma)(f)||_{L^{p'}_\alpha(\mathbb{K})} = ||\mathcal{I}_f||_{B(L^{p'}_\alpha(\mathbb{K}))} \leqslant \|u\|_{L^{p'}_\alpha(\mathbb{K})}\|v\|_{L^p_\alpha(\mathbb{K})}\|f\|_{L^p_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})},$$

which establishes the proposition. □

Combining Proposition 5.1 and Proposition 5.3, we have the following theorem.

**Theorem 5.3.** *Let $\sigma$ be in $L^1_\alpha(\hat{\mathbb{K}})$, $v \in L^p_\alpha(\mathbb{K})$ and $u \in L^{p'}_\alpha(\mathbb{K})$, for $1 \leq p \leq \infty$, then the Laguerre two-wavelet multiplier*

$$\mathcal{P}_{u,v}(\sigma) : L^p_\alpha(\mathbb{K}) \longrightarrow L^p_\alpha(\mathbb{K})$$

*is a bounded linear operator, and we have*

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_\alpha(\mathbb{K}))} \leqslant \|u\|_{L^{p'}_\alpha(\mathbb{K})}\|v\|_{L^p_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}.$$

We can now state and prove the main result in this subsection.

**Theorem 5.4.** *Let $\sigma$ be in $L^r_\alpha(\hat{\mathbb{K}})$, $r \in [1,2]$, and $u, v \in L^1_\alpha(\mathbb{K}) \bigcap L^\infty_\alpha(\mathbb{K})$. Then there exists a unique bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_\alpha(\mathbb{K}) \longrightarrow L^p_\alpha(\mathbb{K})$$

*for all $p \in [r, r']$, and we have*

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_\alpha(\mathbb{K}))} \leqslant C_1^t C_2^{1-t} \|\sigma\|_{L^r_\alpha(\hat{\mathbb{K}})}, \tag{5.2}$$

*where*

$$\begin{aligned} C_1 &= \Big(\|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^1_\alpha(\mathbb{K})}\Big)^{\frac{2}{r}-1}\Big(\|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}\Big)^{\frac{1}{r'}}, \\ C_2 &= \Big(\|u\|_{L^1_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}\Big)^{\frac{2}{r}-1}\Big(\|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}\Big)^{\frac{1}{r'}}, \end{aligned}$$

*and*

$$\frac{t}{r} + \frac{1-t}{r'} = \frac{1}{p}.$$

*Proof.* Consider the linear functional

$$\begin{aligned} \mathcal{I} : \Big(L^1_\alpha(\hat{\mathbb{K}}) \cap L^2_\alpha(\hat{\mathbb{K}})\Big) \times \Big(L^1_\alpha(\mathbb{K}) \cap L^2_\alpha(\mathbb{K})\Big) &\rightarrow L^1_\alpha(\mathbb{K}) \cap L^2_\alpha(\mathbb{K}) \\ (\sigma, f) &\mapsto \mathcal{P}_{u,v}(\sigma)(f). \end{aligned}$$

Then by Proposition 5.1 and Theorem 3.1

$$||\mathcal{I}(\sigma,f)||_{L^1_\alpha(\mathbb{K})} \leqslant \|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^1_\alpha(\mathbb{K})}||f||_{L^1_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})} \tag{5.3}$$

and

$$||\mathcal{I}(\sigma,f)||_{L^2_\alpha(\mathbb{K})} \leq \sqrt{\|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}}\,||f||_{L^2_\alpha(\mathbb{K})}\|\sigma\|_{L^2_\alpha(\hat{\mathbb{K}})}. \tag{5.4}$$

Therefore, by (5.3), (5.4) and the the multi-linear interpolation theory, see Section 10.1 in [8] for reference, we get a unique bounded linear operator

$$\mathcal{I} : L^r_\alpha(\hat{\mathbb{K}}) \times L^r_\alpha(\mathbb{K}) \to L^r_\alpha(\mathbb{K})$$

such that

$$||\mathcal{I}(\sigma,f)||_{L^r_\alpha(\mathbb{K})} \leq C_1||f||_{L^r_\alpha(\mathbb{K})}\|\sigma\|_{L^r_\alpha(\hat{\mathbb{K}})}, \tag{5.5}$$

where

$$C_1 = \Big(\|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^1_\alpha(\mathbb{K})}\Big)^{\theta}\Big(\|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}\Big)^{\frac{1-\theta}{2}}$$

and

$$\frac{\theta}{1} + \frac{1-\theta}{2} = \frac{1}{r}.$$

By the definition of $\mathcal{I}$, we have

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^r_\alpha(\mathbb{K}))} \leqslant C_1\|\sigma\|_{L^r_\alpha(\hat{\mathbb{K}})}.$$

As the adjoint of $\mathcal{P}_{u,v}(\sigma)$ is $\mathcal{P}_{v,u}(\overline{\sigma})$, so $\mathcal{P}_{u,v}(\sigma)$ is a bounded linear map on $L^{r'}(\mathbb{K})$ with its operator norm

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^{r'}_\alpha(\mathbb{K}))} = ||\mathcal{P}_{v,u}(\overline{\sigma})||_{B(L^r_\alpha(\mathbb{K}))} \leq C_2\|\sigma\|_{L^r_\alpha(\hat{\mathbb{K}})}, \tag{5.6}$$

where

$$C_2 = \Big(\|u\|_{L^1_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}\Big)^{\theta}\Big(\|u\|_{L^\infty_\alpha(\mathbb{K})}\|v\|_{L^\infty_\alpha(\mathbb{K})}\Big)^{\frac{1-\theta}{2}}.$$

Using an interpolation of (5.5) and (5.6), we have that, for any $p \in [r, r']$,

$$||\mathcal{P}_{u,v}(\sigma)||_{B(L^p_\alpha(\mathbb{K}))} \leqslant C_1^t C_2^{1-t}\|\sigma\|_{L^r_\alpha(\hat{\mathbb{K}})},$$

with

$$\frac{t}{r}+\frac{1-t}{r'}=\frac{1}{p}.$$

□

5.2. **Compactness of $\mathcal{P}_{u,v}(\sigma)$.**

**Proposition 5.4.** *Under the same hypothesis of Theorem 5.1, the Laguerre two-wavelet multiplier $\mathcal{P}_{u,v}(\sigma)$ : $L^1_\alpha(\mathbb{K})\longrightarrow L^1_\alpha(\mathbb{K})$ is compact.*

*Proof.* Let $(f_n)_{n\in\mathbb{N}}\in L^1_\alpha(\mathbb{K})$ such that $f_n\rightharpoonup 0$ weakly in $L^1_\alpha(\mathbb{K})$ as $n\to\infty$. It is enough to prove that

$$\lim_{n\to\infty}||\mathcal{P}_{u,v}(\sigma)(f_n)||_{L^1_\alpha(\mathbb{K})}=0.$$

We have

$$||\mathcal{P}_{u,v}(\sigma)(f_n)||_{L^1_\alpha(\mathbb{K})}\leqslant\int_{\hat{\mathbb{K}}}\int_{\mathbb{K}}|\sigma(\lambda,m)|\,|\langle f_n,\varphi_{\lambda,m}u\rangle_{L^2_\alpha(\mathbb{K})}|\,|\varphi_{\lambda,m}(x,t)v(x,t)|d\gamma_\alpha(\lambda,m)d\nu_\alpha(x,t). \tag{5.7}$$

Now using the fact that $f_n\rightharpoonup 0$ weakly in $L^1_\alpha(\mathbb{K})$, we deduce that

$$\forall\,(x,t)\in\mathbb{K},\ \forall\,(\lambda,m)\in\hat{\mathbb{K}},\quad \lim_{n\to\infty}|\sigma(\lambda,m)|\,|\langle f_n,\varphi_{\lambda,m}u\rangle_{L^2_\alpha(\mathbb{K})}|\,|\varphi_{\lambda,m}(x,t)v(x,t)|=0. \tag{5.8}$$

On the other hand as $f_n\rightharpoonup 0$ weakly in $L^1_\alpha(\mathbb{K})$ as $n\to\infty$, then there exists a positive constant $C$ such that $||f_n||_{L^1_\alpha(\mathbb{K})}\leq C$.

Thus using Hölder's inequality and relation (2.4), we get for all $(x,t)\in\mathbb{K}$, for all $(\lambda,m)\in\hat{\mathbb{K}}$,

$$|\sigma(\lambda,m)|\,|\langle f_n,\varphi_{\lambda,m}u\rangle_{L^2_\alpha(\mathbb{K})}|\,|\varphi_{\lambda,m}(x,t)v(x,t)|\leq C|\sigma(\lambda,m)|\,||u||_{L^\infty_\alpha(\mathbb{K})}\,|v(x,t)|. \tag{5.9}$$

So, from Fubini's theorem and previous relation, we obtain

$$\begin{aligned}&\int_{\hat{\mathbb{K}}}\int_{\mathbb{K}}|\sigma(\lambda,m)|\,|\langle f_n,\varphi_{\lambda,m}u\rangle_{L^2_\alpha(\mathbb{K})}|\,|\varphi_{\lambda,m}(x,t)v(x,t)|d\gamma_\alpha(\lambda,m)d\nu_\alpha(x,t)\\ &\leq C||u||_{L^\infty_\alpha(\mathbb{K})}\int_{\hat{\mathbb{K}}}|\sigma(\lambda,m)|\int_{\mathbb{K}}|v(x,t)|d\nu_\alpha(x,t)d\gamma_\alpha(\lambda,m)\\ &\leq C||u||_{L^\infty_\alpha(\mathbb{K})}||v||_{L^1_\alpha(\mathbb{K})}\|\sigma\|_{L^1_\alpha(\hat{\mathbb{K}})}<\infty.\end{aligned} \tag{5.10}$$

Thus from the relations (5.7), (5.8), (5.9), (5.10) and the Lebesgue dominated convergence theorem we deduce that

$$\lim_{n\to\infty}||\mathcal{P}_{u,v}(\sigma)(f_n)||_{L^1_\alpha(\mathbb{K})}=0$$

and the proof is complete. □

In the following we give three results for compactness of the Laguerre two-wavelet multiplier operators.

**Theorem 5.5.** *Under the hypothesis of Theorem 5.1, the bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma):L^p_\alpha(\mathbb{K})\longrightarrow L^p_\alpha(\mathbb{K})$$

*is compact for $1\leq p\leq\infty$.*

*Proof.* From the previous proposition, we only need to show that the conclusion holds for $p=\infty$. In fact, the operator $\mathcal{P}_{u,v}(\sigma):L^\infty_\alpha(\mathbb{K})\longrightarrow L^\infty_\alpha(\mathbb{K})$ is the adjoint of the operator

$$\mathcal{P}_{v,u}(\overline{\sigma}):L^1_\alpha(\mathbb{K})\longrightarrow L^1_\alpha(\mathbb{K}),$$

which is compact by the previous Proposition. Thus by the duality property,

$$\mathcal{P}_{u,v}(\sigma):L^\infty_\alpha(\mathbb{K})\longrightarrow L^\infty_\alpha(\mathbb{K})$$

is compact. Finally, by an interpolation of the compactness on $L^1_\alpha(\mathbb{K})$ and on $L^\infty_\alpha(\mathbb{K})$ such as the one given on pages 202 and 203 of the book [7] by Bennett and Sharpley, the proof is complete. □

The following result is an analogue of Theorem 5.4 for compact operators.

**Theorem 5.6.** *Under the hypotheses of Theorem 5.4, the bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_\alpha(\mathbb{K}) \longrightarrow L^p_\alpha(\mathbb{K})$$

*is compact for all* $p \in [r, r']$.

*Proof.* The result is an immediate consequence of an interpolation of Corollary 3.2 and Proposition 5.4. See again pages 202 and 203 of the book [7] by Bennett and Sharpley for the interpolation used. □

Using similar ideas as above we can prove the following.

**Theorem 5.7.** *Under the hypothesis of Theorem 5.3, the bounded linear operator*

$$\mathcal{P}_{u,v}(\sigma) : L^p_\alpha(\mathbb{K}) \longrightarrow L^p_\alpha(\mathbb{K})$$

*is compact for* $1 \le p \le \infty$.

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