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# ON MANN-TYPE IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF $\alpha$-HEMICONTRACTIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT. The purpose of this article is to study an implicit iteration process for a finite family of $\alpha$-hemicontractive mappings in Hilbert spaces. Our results extend and generalize the recent results of Husain et al. [12] and Diwan et al. [8] from the classes of hemicontractive and $\alpha$-demicontractive mappings respectively, to the more general class of $\alpha$-hemicontractive mappings.

## 1. INTRODUCTION

Let $H$ be a Hilbert space and $K$ is a nonempty closed convex subset of $H$ and $F(T) = \{x \in H : Tx = x\}$ denotes the set of fixed points of $T$. In the sequel, we give the following definitions which will be useful in this study.

**Definition 1.1.** A mapping $T : K \to K$ is said to be:

(i) strictly pseudocontractive if there exists a constant $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad (1.1)$$

for all $x, y \in K$;

(ii) pseudocontractive if $F(T) \neq \emptyset$ and

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad (1.2)$$

for all $x, y \in K$. The class pseudocontractive mappings properly includes the class nonexpansive and strictly pseudocontractive mappings;

(iii) demicontractive if $F(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \|x - Tx\|^2, \quad (1.3)$$

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for all $p \in F(T)$ and $x \in K$. It is well known that the class demicontractive mappings properly includes the class of quasi-nonexpansive mappings and every strictly pseudocontractive mapping with $F(T) \neq \emptyset$ is demicontractive, but the converse is not true as proved in [13];

(iv) $\alpha$–demicontractive if $F(T) \neq \emptyset$ and there exists a constant $k \in (0, 1)$ such that

$$\|Tx - \alpha p\|^2 \leq \|x - \alpha p\|^2 + k\|x - Tx\|^2, \tag{1.4}$$

for some $\alpha \geq 1$, for all $p \in F(T)$ and $x \in K$. Obviously, every demicontractive mapping is $\alpha$–demicontractive with $\alpha = 1$. However, the converse is not true. Maruster and Maruster [17] demonstrated an example of a real function which is $\alpha$–demicontractive mapping, but not demicontractive mapping;

(v) hemicontractive if $F(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \|x - Tx\|^2, \tag{1.5}$$

for all $p2 \in F(T)$ and $x \in K$. Equivalently, (1.5) can also be written as:

$$\langle x - Tx, x - p\rangle \geq 0, \tag{1.6}$$

for all $x \in K$ and $p \in F(T)$. It is proved in [26] and [18] with examples that the class of pseudocontractive mappings with a nonempty fixed point set is a proper subclass of the class of hemicontractive mappings and the class demicontractive mappings is also a proper subclass of the class of hemicontractive mappings, but the converses fail,

(v) $\alpha$-hemicontractive $F(T) \neq \emptyset$ and

$$\|Tx - \alpha p\|^2 \leq \|x - \alpha p\|^2 + \|x - Tx\|^2, \tag{1.7}$$

for some $\alpha \geq 1$, for all $p \in F(T)$ and $x \in K$. Equivalently, in an inner product space, (1.7) can written as:

$$\langle x - Tx, x - \alpha p\rangle \geq 0, \tag{1.8}$$

for some $\alpha \geq 1$ for all $x \in K$ and $p \in F(T)$. Clearly, every hemicontractive mapping is an $\alpha$-hemicontractive mapping with $\alpha = 1$ and every $\alpha$–demicontractive mapping is an $\alpha$–hemicontractive mapping, but the converse cases are not true as shown by Osilike and Onah [20].

Now, the relation between the mappings defined above is demonstrated as follows:

| | | |
|---|---|---|
| strictly-pseudocontrative $\Longrightarrow$ | demicontrative $\Longrightarrow$ | $\alpha$-demicontrative |
| $\Downarrow$ | $\Downarrow$ | $\Downarrow$ |
| pseudocontractive $\Longrightarrow$ | hemicontractive $\Longrightarrow$ | $\alpha$-hemicontrative. |

Hence, the class of $\alpha$–hemicontractive mappings is the most general of the classes of mappings mentioned above. We give the following examples of mappings which are $\alpha$–hemicontractive, but neither hemicontractive nor $\alpha$–hemicontractive mappings.

Firstly, we give an example of a nonlinear mapping, $T$ which is 2–hemicontractive (i.e., $T$ is $\alpha$–hemicontractive for $\alpha = 2$), but $T$ is not hemicontractive (1–hemicontractive) and hence not pseudocontractive.

**Example 1.2.** [25] Let $\mathbb{R}$ denote the reals with the usual norm and let $K : [1, 4] \subset \mathbb{R}$. Define $T : K \to K$ by

$$Tx = \begin{cases} x^2, & \text{if } x \in [1, 2], \\ 1, & \text{if } x \in (2, 4]. \end{cases}$$

Secondly, we give an example of a nonlinear mappings, $T$ which is $\alpha$–hemicontractive, but $T$ is not $\alpha$—demicontractive as follows:

**Example 1.3.** [25] Let $R$ denote the reals with the usual norm and let $K = (-\infty, 1)$. Define $T : K \to K$ by

$$Tx = \begin{cases} \frac{x}{1-x}, & \text{if } x \in (-\infty, 0], \\ \frac{x}{x-1}, & \text{if } x \in (0, 1). \end{cases}$$

Obviously, the class of $\alpha$-hemicontractive mappings is larger than the classes of hemicontractive and $\alpha$-demicontractive mappings.

On the other hand, iterative methods for nonexpansive mappings have been investigated by many researchers (see for example, [2–4,9–11,19,33] and references there in). The iterative methods for approximating pseudocontractive mappings are far less developed than those of nonexpansive mappings. However, on the other hand pseudocontractions have more powerful applications than nonexpansive mappings in solving nonlinear in- verse problems. In recent years, many authors have studied iterative approximation of fixed of strongly pseudocontractive mappings. Most of them used Mann's iteration process [16]. But in the case of pseudocontractive mapping, it is well known that Mann's iteration fails to converge to fixed point of Lipschitz pseudocontractive mappings in a compact convex subset of a Hilbert space. Chidume and Mutangadura [6] provided an example of a Lipschitz pseudocontractive mapping with a unique fixed point for which the Mann iteration process does not converge in a compact subset of a Hilbert spaces. In 1974, Ishikawa [14] introduced an iteration process which converges to a fixed point of Lipschitz pseudocontractive mapping in a compact convex subset of a Hilbert space. Qihou [26], extended result of Ishikawa to slightly more general class of Lipschitz hemicontractive mappings.

These classes of mappings have been studied by many authors (see for example [7, 12, 13, 15, 17, 18, 20–27, 31] and the references there in). In 2007, Rafiq [27], proposed a Mann-type iteration process for hemicontractive mapping $T$ defined by:

$$\begin{cases} x_0 \in K, \\ x_n = \alpha_n x_{n+1} + (1-\alpha)Tx_n, \end{cases} \quad n \geq 0, \tag{1.9}$$

where $n$ is a real sequence such that $\alpha_n \in [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. They proved the following the following result.

**Theorem 1.4.** *[27] Let K be a compact convex subset of a real Hilbert space H and $T : K \to K$ be a hemicontractive mapping. Let $\{\alpha_n\}$ be a real sequence in [0,1] satisfying $\alpha_n \in [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. For arbitrary $x_0 \in K$, the sequence $\{x_n\}$ is defined by* (1.9)*. Then $\{x_n\}$ converges strongly to a fixed point of $T$.*

But, Song [30] observed that there is a gap in the iteration (1.9) for for hemicontractive mapping $T$ and proved the following theorem:

**Theorem 1.5.** *[29] Let $K$ be a compact convex subset of a real Hilbert space $H$ and $T : K \to K$ is a continuous pseudocontractive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\} \in (0, 1)$ is a real sequence satisfying the condition $\lim_{n\to\infty} \alpha_n = 0$. Let $x_0 \in K$ and $\{x_n\}$ be defined by* (1.9)*. Then $\{x_n\}$ strongly converges to a point of $T$.*

In 2013, Thakur [28] proved the following result:

**Theorem 1.6.** *Let $K$ be a nonempty convex subset of an arbitrary Banach space $X$ and $T_i; i = 1, 2$ be two uniformly continuously and $\phi$-hemicontractive mappings. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n^i\}_{n=1}^{i}$, $i = 1, 2$ be real sequences in [0,1] such that $\sum_{n=1}^{2} \beta_n^i + \alpha_n = 1$ and satisfying the conditions (i) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, and (ii) $\lim_{n\to\infty}(1-\alpha_n) = 0$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is the sequence generated from an arbitrary $x_0 \in K$ by*

$$x_n = \alpha x_{n-1} + \beta_n^i T_i x_n, \; n \geq 1. \tag{1.10}$$

*Then the following conditions are equivalent:*

(a) *$\{x_n\}_{n=1}^{\infty}$ converges strongly to the common fixed q of $T_i$, $ii = 1, 2$,*

(a) $\lim_{n\to\infty} T_i x_n = q$, $i = 1, 2$,
(a) $\{T_i x_n\}_{n=1}^{\infty}$, $i = 1, 2$ *are bounded.*

In 2013, Thakur [31] proved the following result:

**Theorem 1.7.** *[31]Let $K$ be a compact convex subset of a Hilbert space $H$. Let $N \geq 1$ be an integer. For each $n \geq 1$, assume that $\{\lambda_i^{(n)}\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_i^{(n)} = 1$ and $\infty_{n\geq 1}\lambda_i^{(n)} > 0$ for all $1 \leq i \leq N$. For each $1 \leq i \leq N$, let $T_i : K \to K$ be a hemicontractive mappings and the family $\{T_i\}_{i=1}^N$ satisfies the condition $\mathbb{B}$. For arbitrary chosen $x_0 \in K$, let a sequence generated by the algorithm*

$$x_n = \alpha x_{n-1} + \sum_{i=1}^{N} \lambda_i^{(n)} T_i x_n, \ n \geq 1, \tag{1.11}$$

*where the sequence $\{\alpha\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. Then $\{x_n\}$ converges strongly to common fixed point of the family $\{T_i\}_{i=1}^N$.*

In 2013, Hussain et al. [12] introduced the following Mann-type implicit iteration process associated with a family of continuous hemicontractive mappings to prove a strong convergence results in Hilbert spaces.

$$\begin{cases} x_0 \in K, \\ x_n = \alpha_n x_{n+1} + \sum_{i=1}^k \beta_n^i T x_n, \end{cases} \quad n \geq 0, \tag{1.12}$$

where $\alpha_n, \beta_n^i \in [0,1]$, $i = 1, 2, \cdots, k$ such that $\alpha_n + \sum_{i=1}^k \beta_n^i = 1$ and some appropriate conditions hold.

**Theorem 1.8.** *[12] Let $K$ be a compact convex subset of a real Hilbert space $H$ and $T_i : K \to K$, $i = 1, 2, \cdots, k$, be a family of continuous hemicontractive mappings . Let $\alpha_n, \beta_n^i \in [0,1]$ be such that $\alpha_n + \sum_{i=1}^m \beta_n^i = 1$ and satisfying $\{\alpha_n\}, \beta_n^i \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$, $i = 1, 2, \cdots, k$. Then, for arbitrary $x_0 \in K$, the sequence $\{x_n\}$ defined by* (1.12) *converges strongly to a common fixed point of $\cap_i^k F(T_i) \neq \emptyset$.*

Recently, Diwan et al. [8] established strong convergence result for family of continuous $\alpha$–demicontractive mappings using the Mann–type implicit iteration process [17] introduced by Hussain et al. [12], their results extended the corresponding results of Hussain et al. [12].

**Theorem 1.9.** *[8] Let $K$ be a compact convex subset of a real Hilbert space $H$ and $T_i : K \to K$, $i = 1, 2, \cdots, k$, be a family of continuous $\alpha$-demicontractive mappings. Let $\alpha_n, \beta_n^i \in [0,1]$ be such that $\alpha_n + \sum_{i=1}^k \beta_n^i = 1$ and satisfying $\{\alpha_n\}, \beta_n^i \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$, $i = 1, 2, \cdots, k$. Then, for arbitrary $x_0 \in K$, the sequence $\{x_n\}$ defined by* (1.12) *converges strongly to a common fixed point of $\cap_i^k F(T_i) \neq \emptyset$.*

At this juncture, it becomes natural to investigate whether strong convergence theorem can still be obtained when the class of mappings is extended to $\alpha$-hemicontractive mappings which is more general than the aforementioned class of mappings. Motivated by the above results, the purpose of this paper is to prove strong convergence of (1.12) for the finite family of continuous $\alpha$–hemicontractive mappings. As demonstrated by Osilike and Onah [25] the class of $\alpha$–hemicontractive mappings properly contains the classes of hemicontractive and $\alpha$–demicontractive mappings. Hence our results extend and generalize the results of Hussain et al. [12], Diwan et al. [8] and other corresponding results in the literature.

## 2. Preliminaries

In the sequel, we will need the following lemmas.

**Lemma 2.1.** *[30] Suppose that $\{\rho_n\}$ and $\{\sigma\}$ are two sequences of real nonnegative numbers such that, for some real number $N_0 \geq 1$,*

$$\rho_{n+1} \leq \rho_n + \sigma_n, \tag{2.1}$$

*for all $n \geq N_0$. Then we have the following:*

(1) *If $\sum \sigma_n < \infty$, then $\lim \rho_n$ exists.*

(2) *If $\sum \sigma_n < \infty$ and $\{\rho_n\}$ has a subsequence converging to zero, then $\lim \rho_n = 0$.*

**Lemma 2.2.** *[32] For all $x, y \in H$ and $\lambda \in [0,1]$, the following well known identity holds:*

$$\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\| - \lambda(1-\lambda)\|x-y\|^2. \tag{2.2}$$

**Lemma 2.3.** *[12] Let $H$ be a Hilbert space. Then for all $x, x_i \in H, i = 1, 2, \cdots, k$,*

$$\|\delta x + \sum_{i=1}^{k} \beta^i x_i\|^2 = \delta\|x\|^2 + \sum_{i=1}^{k} \beta^i \|x_i\|^2 - \sum_{i=1}^{k} \beta^i \delta \|x_i - x\|^2 - \sum_{i,j=1, i\neq j}^{k} \beta^i \beta^j \|x_i - x\|^2,$$

*where $\delta, \beta^i \in [0,1]$, $i = 1, 2, \cdots, k$, and $\delta + \sum_{i=1}^{k} \beta^i = 1$.*

## 3. Main Result

**Theorem 3.1.** *Let $K$ be a compact convex subset of a real Hilbert space $H$ and $T_i : K \to K$, $i = 1, 2, \cdots, k$, be a family of continuous $\alpha$-hemicontractive mappings. Let $\delta_n, \beta_n^i \in [0,1]$ be such that $\delta_n + \sum_{i=1}^{k} \beta_n^i = 1$ and satisfying $\{\delta_n\}, \beta_n^i \subset [\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1), i = 1, 2, \cdots, k$. Then, for arbitrary $x_0 \in K$, the sequence $\{x_n\}$ defined by* (1.12) *converges strongly to a common fixed point of $\cap_i^k F(T_i) \neq \emptyset$.*

*Proof.* Let $\alpha p \in \cap_i^k F(T_i) \neq \emptyset$. Since $T_1, T_2, \cdots, T_k$ are $\alpha$-hemicontractive, therefore for some $\alpha \geq 1$, we obtain

$$\|T_i x_n - \alpha p\|^2 \leq \|x_n - \alpha p\|^2 + \|x_n - T_i x_n\|^2. \tag{3.1}$$

From (1.12), (3.1) and Lemma 2.3, we get

$$\begin{aligned}
\|x_n - \alpha p\|^2 &= \left\|\delta_n x_{n+1} + \sum_{i=1}^{k} \beta_n^i T_i x_n - \alpha p\right\|^2 \\
&= \left\|(\delta_n x_{n-1} - \alpha p) + \sum_{i=1}^{k} \beta_n^i (T x_n - \alpha p)\right\|^2 \\
&\leq \delta_n \|x_{n-1} - \alpha p\|^2 + \sum_{i=1}^{k} \beta_n^i \|T_i x_n - \alpha p\|^2 \\
&\quad - \sum_{i=1}^{k} \beta^i \delta_n \|x_{n-1} - T_i x_n\|^2 - \sum_{i,j=1, i\neq j}^{k} \beta^i \beta^j \|T_i x_n - T_j x_n\|^2 \\
&\leq \delta_n \|x_{n-1} - \alpha p\|^2 + \sum_{i=1}^{k} \beta_n^i \|T x_n - \alpha p\|^2 \\
&\quad - \sum_{i=1}^{k} \beta^i \delta_n \|x_{n-1} - T_i x_n\|^2 \\
&\leq \delta_n \|x_{n-1} - \alpha p\|^2 + \sum_{i=1}^{k} \beta_n^i \|x_n - \alpha p\|^2 + \sum_{i=1}^{k} \beta_n^i \|x_n - T_i x_n\|^2 \\
&\quad - \sum_{i=1}^{k} \beta^i \delta_n \|x_{n-1} - T_i x_n\|^2
\end{aligned} \tag{3.2}$$

Observe that

$$\|x_n - T_i x_n\| = \|\delta_n x_{n+1} + \sum_{i=1}^{k} \beta_n^i T_i x_n - T_i x_n\|^2 = \delta_n^2 \|x_{n-1} - T_i x_n\|^2. \tag{3.3}$$

From (3.2) and (3.3), we have

$$\begin{aligned}\|x_n - \alpha p\|^2 &\leq \delta_n \|x_{n-1} - \alpha p\|^2 + \sum_{i=1}^{k} \beta_n^i \|x_n - \alpha p\|^2 - \sum_{i=1}^{k} \delta_n (1-\delta_n)\beta^i \|x_{n-1} - T_i x_n\|^2 \\ &\leq \|x_{n-1} - \alpha p\|^2 - \sum_{i=1}^{k} (1-\delta_n)\beta^i \|x_{n-1} - T_i x_n\|^2. \end{aligned} \tag{3.4}$$

Using the condition $\{\delta_n\}, \beta_n^i \subset [\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$, $i = 1, 2, \cdots, k$, we have

$$\|x_n - \alpha p\|^2 \leq \|x_{n-1} - \alpha p\|^2 - \varepsilon(1-\varepsilon) \sum_{i=1}^{k} \|x_{n-1} - T_i x_n\|^2, \tag{3.5}$$

for all fixed points $\alpha p \in \cap_i^k F(T_i) \neq \emptyset$. It follows from (3.5) that

$$\varepsilon(1-\varepsilon) \sum_{i=1}^{k} \|x_{n-1} - T_i x_n\|^2 \leq \|x_{n-1} - \alpha p\|^2 - \|x_n - \alpha p\|^2,$$

and thus, for all $i = 1, 2, \cdots, k$, we get

$$\varepsilon(1-\varepsilon) \sum_{j=1}^{\infty} \|x_{j-1} - T_j x_j\|^2 \leq \sum_{j=1}^{\infty} (\|x_{j-1} - \alpha p\|^2 - \|x_j - \alpha p\|^2) = \|x_0 - \alpha p\|^2.$$

Hence, for all $i, 2, \cdots, k$, we obtain

$$\sum_{j=1}^{\infty} \|x_{j-1} - T_j x_j\|^2 < \infty. \tag{3.6}$$

Thus, for each $i = 1, 2, \cdots, k$, from (3.6), it implies that

$$\lim_{n\to\infty} \|x_{n-1} - T_i x_n\|^2 = 0. \tag{3.7}$$

Hence, for each $i = 1, 2, \cdots, k$ and from (3.3), we obtain

$$\lim_{n\to\infty} \|x_n - T_i x_n\|^2 = 0. \tag{3.8}$$

Since K is compact, then there exists a subsequence $\{x_{n_j}\}$ of $x_n$ which converges to a common fixed point of $\cap_i^k F(T_i)$, say $\alpha q$. Since (3.5) holds for all fixed points of $\cap_i^k F(T_i)$, we have

$$\|x_n - \alpha p\|^2 \leq \|x_{n-1} - \alpha p\|^2 - \varepsilon(1-\varepsilon) \sum_{i=1}^{k} \|x_{n-1} - T_i x_n\|^2. \tag{3.9}$$

Now, from (3.7) and Lemma 2.1, it follows that $\|x_n - \alpha p\| \to 0$ as $n \to \infty$. This completes the proof. □

**Theorem 3.2.** *Let $H, K, T_i, i = 1, 2, \cdots, k$, be as in Theorem 3.1 and $\delta_n, \beta_n^i \in [0,1]$ be such that $\delta_n + \sum_{i=1}^{k} \beta_n^i = 1$ and satisfying $\{\delta_n\}, \beta_n^i \subset [\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1), i = 1, 2, \cdots, k$. If $P_K : H \to K$ is the projection operator of $H$ onto $K$, then the sequence $\{x_n\}$ defined iterative by*

$$x_n = P_K(\delta_n x_{n+1} + \sum_{i=1}^{k} \beta_n^i T_i x_n), \tag{3.10}$$

*for each $n \geq 0$ converges strongly a common fixed point in $\cap_i^k F(T_i)$.*

*Proof.* It well known that $P_K$ is nonexpansive (see [3]) and K is a Chebyshev subset of H and so $P_K$ is a single value mapping. Hence, we have the following estimate

$$\begin{aligned}
\|x_n-\alpha p\|^2+\| &\leq \left\|P_K(\delta_n x_{n+1}+\sum_{i=1}^{k}\beta_n^i T_i x_n-\alpha p)\right\|^2\\
&\leq \left\|(\delta_n x_{n-1}-\alpha p)+\sum_{i=1}^{k}\beta_n^i(Tx_n-\alpha p)\right\|^2\\
&\leq \delta_n\|x_{n-1}-\alpha p\|^2+\sum_{i=1}^{k}\beta_n^i\|Tx_n-\alpha p\|^2\\
&\quad -\sum_{i=1}^{k}\beta^i\delta_n\|x_{n-1}-T_i x_n\|^2\\
&\leq \delta_n\|x_{n-1}-\alpha p\|^2+\sum_{i=1}^{k}\beta_n^i\|x_n-\alpha p\|^2-\sum_{i=1}^{k}\delta_n(1-\delta_n)\beta^i\|x_{n-1}-T_i x_n\|^2\\
&\leq \delta_n\|x_{n-1}-\alpha p\|^2+\sum_{i=1}^{k}\beta_n^i\|x_n-\sum_{i=1}^{k}\delta_n(1-\delta_n)\beta^i\|x_{n-1}-T_i x_n\|^2.
\end{aligned}$$

The set $K\cup T(K)$ is compact, it follows that $\{\|x_n-T_ix_n\|\}$ is bounded. Following the same argument as exactly as in the proof of Theorem 3.1, the conclusion is obtained.

□

**Theorem 3.3.** *Let K be a convex subset of a real Hilbert space H and $T : K \to K$ $i = 1, 2, \cdots k$, be a family of continuous $\alpha$-hemicontractive mappings of K such that $\cap_i^k F(T_i)$ and there exists one map $T \in \{T_i\}_{i=1}^k$ that is semicompact. $\delta_n, \beta_n^i \in [0,1]$ be such that $\delta_n + \sum_{i=1}^k \beta_n^i = 1$ and satisfying $\{\delta_n\}, \beta_n^i \subset [\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1), i = 1, 2, \cdots, k$. Then, for arbitrary $x_0 \in K$, the sequence $\{x_n\}$ defined by*

$$x_n=\delta_n x_{n+1}+\sum_{i=1}^{k}\beta_n^i T_i x_n, \tag{3.11}$$

*for each $n \geq 0$ converges strongly a common fixed point in $\cap_i^k F(T_i) \neq \emptyset$.*

*Proof.* From Theorem 3.1, for all $\alpha p \in \cap_i^k F(T_i) \neq \emptyset$, we have $\|x_n - \alpha p\|$ exists and $\lim_{n\to\infty} \|x_n - T_ix_n\| = 0$, for each $i = 1, 2, \cdots, k$. Thus $\{x_n\}$ is bounded; then by the hypothesis that there exists one map $T \in \{T_i\}_{i=1}^k$ that is semicompact, we may assume that $T_1$ is semicompact without loss of generality. Therefore

$$\lim_{n\to\infty}\|x_n-T_1x|\|=0, \tag{3.12}$$

and by the definition of being semicompact there exist a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to \alpha q \in K(j \to \infty)$.

Thus,

$$\|\alpha q-T_i\alpha q\|=\lim_{j\to\infty}\|x_{n_j}-T_ix_{n_j}\|=0\ \forall\ i\in\{1,2,\cdots,k\}$$

□

**Theorem 3.4.** *Let $K$ be a compact convex subset of a real Hilbert space $H$ and $T_i : K \to K$, $i = 1, 2, \cdots, k$, be a family of Lipschitz $\alpha$-hemicontractive mappings. Let $\delta_n, \beta_n^i \in [0,1]$ be such that $\delta_n + \sum_{i=1}^k \beta_n^i = 1$ and satisfying $\{\delta_n\}, \beta_n^i \subset [\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1), i = 1, 2, \cdots, k$. Then, for arbitrary $x_0 \in K$, the sequence $\{x_n\}$ defined by* (1.12) *converges strongly to a common fixed point of $\cap_i^k F(T_i) \neq \emptyset$.*

**Theorem 3.5.** *Let $H, K, T_i, i = 1, 2, \cdots, k$, be as in Theorem 3.4 and $\delta_n \beta_n^i \in [0,1]$ be such that $\delta_n + \sum_{i=1}^{k} \beta_n^i = 1$ and satisfying $\{\delta_n\}, \beta_n^i \subset [\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1), i = 1, 2, \cdots, k$. If $P_K : H \to K$ is the projection operator of $H$ onto $K$, then the sequence $\{x_n\}$ defined iterative by*

$$x_n = P_K(\delta_n x_{n+1} + \sum_{i=1}^{k} \beta_n^i T_i x_n), \tag{3.13}$$

*for each $n \geq 0$ converges strongly a common fixed point in $\cap_i^k F(T_i)$.*

*Remark* 3.6. For $k = 2$, we can choose the following control parameters: $\delta_n = \frac{1}{4} - \frac{1}{(n+1)^2}$, $\beta_n^1 = \frac{1}{4}$ and $\delta_n^2 = \frac{1}{2} + \frac{1}{(n+1)^2}$.

*Remark* 3.7. Theorem 3.1 extends and improves the corresponding results of Diwan et al. ( [8], Theorem 2.4) and Hussain eta l. ( [12], Theorem 2.1) from the classes of $\alpha$–demicontractive mapping and hemicontractive mapping respectively, to the more general class of $\alpha$-hemicontractive mappings.

*Remark* 3.8. Theorem 3.2 extends and improves the corresponding results of Diwan et al. ( [8], Theorem 2.5) and Hussain eta l. ( [12], Theorem 2.2) from the classes of $\alpha$–demicontractive mapping and hemicontractive mapping respectively, to the more general class of $\alpha$–hemicontractive mappings.

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