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NOVEL MULTIVARIATE INTEGRAL OPERATORS INCORPORATING TRIGONOMETRIC TRANSFORMATIONS

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ABSTRACT. The creation of new nonlinear multivariate integral operators is motivated by the need for mathematical tools that can handle the complex interdependencies that naturally arise in contemporary applications. From an abstract scientific point of view, it is also necessary to develop new operator theories beyond existing ones to offer original research perspectives. This article contributes to these complementary aspects. We present two nonlinear multivariate integral operators that have the particularity of incorporating trigonometric transformations of the main function. Thanks to their trigonometric nature, they completely stand out from existing operators, offering a new and complete framework. Therefore, we take advantage of advanced mathematical techniques for trigonometric functions to address the challenges they pose. In particular, we show that they have manageable integrals and series expansions, that they are solutions of specific differential and functional equations, and that they are involved in general inequalities of various types (Hölder-type, convex-type, etc.). In the application part, we use some of these properties to propose a wide collection of trigonometric inequalities that are both original and precise. Figures are produced to illustrate them for a direct visual check.

1. MOTIVATIONS

Integral operators are key mathematical tools. They are used to provide a robust framework for analyzing and transforming functions, playing a crucial role in disciplines such as calculus, functional analysis, signal processing, mathematical physics, engineering, and image processing. Their actual research is of importance for introducing new perspectives and addressing specific issues or phenomena beyond the capabilities of existing operators. In the field of univariate analysis, the most renowned linear integral operator is the Fourier operator. We may also mention the Laplace, Sumudu, Elzaki, Natural, Formable, and Jafari operators (see [36], [5], [15], [6], [33], and [22], respectively). Despite a certain variety in the definition, notable linear integral operators using trigonometric functions are relatively rare. The most famous examples are the Fourier-sine and Fourier-cosine operators, both derived from imaginary and real parts of the Fourier operator, respectively. Hence, we may define them as

$$F_S(f)(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \sin(\gamma x) dx$$

and

$$F_C(f)(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \cos(\gamma x) dx,$$

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where $\gamma \in \mathbb{R}$, respectively. There is also the Hartley operator. It is known for its ability to transform real-valued functions into real-valued functions via integration and trigonometric functions. Technically, the Hartley operator of a function f is defined by

$$H(f)(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \operatorname{cas}(\gamma x) dx,$$

where $\gamma \in \mathbb{R}$, and $\operatorname{cas}(t)$ denotes the cosine-and-sine function indicated as

$$\operatorname{cas}(t) = \cos(t) + \sin(t), \tag{1.1}$$

which can also be expressed without a sum as

$$\operatorname{cas}(t) = \sqrt{2} \sin\left(t + \frac{\pi}{4}\right), \quad \operatorname{cas}(t) = \sqrt{2} \cos\left(t - \frac{\pi}{4}\right).$$

Thus, the following relationship holds: $H(f)(\gamma) = F_S(f)(\gamma) + F_C(f)(\gamma)$. One interest of the Hartley operator is its involution property: it is its own inverse, i.e., $H[H(f)](t) = f(t)$. More details on the Hartley operator can be found in [28] and [32]. The trigonometric nature of the Fourier-sine, Fourier-cosine and Hartley operators will be inspirational for the findings in this article.

On a connex topic, important nonlinear integral operators include the nonlinear Fourier, Kamimura, Urysohn and Hammerstein operators (see [14], [23], [4], and [30], respectively). Also, to effectively capture complex dependencies between multiple variables in various applications, the development of innovative multivariate integral operators has attracted attention. Among the family of linear operators of this type, except the classic multivariate Fourier operator, we can refer to the multivariate Hartley, triple Laplace-Aboodh-Sumudu, double Laplace, double Sumudu, triple Laplace, triple Elzaki, double fuzzy Natural and fractional order multiple integral operators (see [35], [1], [25], [25], [26], [16], [18], and [24], respectively). Among the rare integral operators mixing multidimension and nonlinearity, there are the special convolution-type, special Kantorovich, $\mathfrak{C}$, generalized $\mathfrak{C}$ and integral ratio operators (see [2], [7], [8], [9], and [10], respectively).

To the best of our knowledge, multivariate nonlinear integral operators involving trigonometric functions have received relatively limited attention in the literature. Nevertheless, they could be very promising for the creation of new theories and mathematical models. Indeed, trigonometric functions are well-mastered tools, and their basic properties can enrich those of any derived operator. Thus, as surprising as it may seem, their use is a particularly original approach in a multivariate nonlinear context. In light of this, keeping in mind the constructions of the Fourier-sine, Fourier-cosine and generalized $\mathfrak{C}$ operators, we define new multivariate nonlinear integral operators having the following general form:

$$G(f)(\alpha, \beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} g[\beta f(\mathbf{x})] d\mathbf{x},$$

with $g(t) \in \{\sin(t), \cos(t)\}$, and α and β are two tuning parameters (the mathematical details will be given later). Consequently, a trigonometric function depending on β and a power function driven by α both involve the transformation of the primary function f into the integral. In this article, we emphasize that such multivariate nonlinear trigonometric integral operators have several notable features and can open the horizons of new applications. Among them are solutions to sophisticated functional and differential equations. In addition, we can expand them into simple infinite series with manageable coefficients for further mathematical manipulations. As an important aspect, thanks to knowledge of the basic trigonometric functions, they satisfy general inequalities, including convex, concave, and Hölder-type inequalities. These inequalities can offer an alternative view to existing findings and also innovate, taking advantage of a high level of precision. Furthermore, we can express these multivariate nonlinear trigonometric integral operators in simple forms for a broad range of univariate and multivariate functions f. With the aim of giving examples of applications, we combine some of the above properties to establish new trigonometric inequalities, which remain a contemporary topic (see [13], [3], [11] and [12], among others). To provide a direct visual check on them, figures are created. We thus make a theoretical and practical contribution to the field of analysis by examining multivariate nonlinear trigonometric integral operators that have not received enough attention, despite promising perspectives.

The article is structured as follows: The two proposed operators and their fundamental characteristics are covered in detail in Section 2. In Section 3, some general inequalities are established. The two operators are applied to a comprehensive panel of univariate and multivariate functions in Sections 4 and 5. Section 6 shows how the findings can be exploited to innovate in the field of inequalities. Section 7 provides a conclusion.

2. Definition and fundamental characteristics

2.1. **Definition.** The two proposed multivariate nonlinear trigonometric integral operators are described in detail below.

Definition 2.1. Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, n be a positive integer, and $\mathcal{X} = \mathcal{X}_n = \times_{i=1}^{n}[0,1] = [0,1]^n$. Let us set $\mathbf{x} = \mathbf{x}_n = (x_1, \ldots, x_n)$ and $d\mathbf{x} = d\mathbf{x}_n = \prod_{i=1}^{n} dx_i$. Let $f(\mathbf{x}) = f(x_1, \ldots, x_n)$, $\mathbf{x} \in \mathcal{X}$, be a non-negative function. The two considered multivariate nonlinear trigonometric integral operators of f are presented below.

- We define the sine (S) operator of f as

$$S(f)(\alpha, \beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \sin[\beta f(\mathbf{x})] d\mathbf{x}.$$

- We define the cosine (C) operator of f as

$$C(f)(\alpha, \beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \cos[\beta f(\mathbf{x})] d\mathbf{x}.$$

Since $|\cos(t)| \in [0,1]$ and $|\sin(t)| \in [0,1]$, it is clear that the S and C operators exist if we suppose that $\int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} d\mathbf{x} < \infty$. Further integral hypotheses can be posed using Riemann rules, integral comparisons, etc. On the mathematical notion of existence used in this article, the remark below must be taken into account.

Remark 2.2. *Throughout the article, for ease of reading and to relax the possible list of assumptions, all mathematical quantities used are assumed to exist in the mathematical sense, that is, all sums converge, all integrals converge, etc. If, on a case-by-case basis, an involved quantity does not exist, the corresponding result becomes invalid.*

The non-negative assumption on f is connected to the presence of α. This point is clarified in the remark below.

Remark 2.3. *If α is an integer, the non-negative assumption on f can be removed. For the sake of uniformity throughout the article, we will, however, suppose that f is non-negative to be more flexible on the possible values of α.*

The definition $\mathcal{X}$ is chosen in such a way that $\int_{\mathcal{X}} d\mathbf{x} = 1$. It can, however, be modulated to any bounded domain, but the coming theory needs to be adjusted accordingly. Also, the remark below nuances the nature of the sub-intervals used in the original expression of $\mathcal{X}$.

Remark 2.4. *In the article, $\mathcal{X}$ can be changed to $\mathcal{X} = \times_{i=1}^{n} I_i$, with $I_i \in \{[0,1], (0,1], [0,1), (0,1)\}$ for each $i = 1, \ldots, n$.*

We can remark that $S(f)(\alpha,\beta)$ and $C(f)(\alpha,\beta)$ are the imaginary and real parts, respectively, of the following multivariate complex integral operator:

$$T(f)(\alpha,\beta)=\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\exp[i\beta f(\mathbf{x})]d\mathbf{x},$$

where $i^2=-1$. It can be considered a special functional variant of the multivariate Fourier transform.

The idea of using the nonlinear composition of f comes from the construction of the $\mathfrak{C}$ operator studied in [8]. This approach is, however, original in the context of trigonometric transformation.

In this article, we will emphasize the fact that the S and C operators take advantage of their trigonometric nature to benefit from manageable properties. The most basic of them are described in the next subsection.

2.2. **Basic properties.** A list of basic properties satisfied by the S and C operators is presented in the proposition below.

Proposition 2.5. *Let $\alpha\in\mathbb{R}$, $\beta\in\mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x}\in\mathcal{X}$, be a non-negative function.*

(1) *For $f=a$ with $a\geq 0$, we have*

$$S(f)(\alpha,\beta)=a^{\alpha}\sin(\beta a),\quad C(f)(\alpha,\beta)=a^{\alpha}\cos(\beta a).$$

(2) *We have*

$$S(f)(\alpha,-\beta)=-S(f)(\alpha,\beta),\quad C(f)(\alpha,-\beta)=C(f)(\alpha,\beta).$$

(3) *We have*

$$S(f)(\alpha,0)=0,\quad C(f)(\alpha,0)=\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}d\mathbf{x}.$$

(4) *By assuming that there exists an integer m such that $\beta f(\mathbf{x})\in[2m\pi,2m\pi+\pi]$ for any $\mathbf{x}\in\mathcal{X}$ so that $\sin[\beta f(\mathbf{x})]\geq 0$, and setting $g(\mathbf{x})=f(\mathbf{x})\left\{\sin[\beta f(\mathbf{x})]\right\}^{1/\alpha}$, $\mathbf{x}\in\mathcal{X}$, we have*

$$C(g)(\alpha,0)=S(f)(\alpha,\beta).$$

(5) *We have*

$$S(f)(0,\beta)=\int_{\mathcal{X}}\sin[\beta f(\mathbf{x})]d\mathbf{x},\quad C(f)(0,\beta)=\int_{\mathcal{X}}\cos[\beta f(\mathbf{x})]d\mathbf{x}.$$

(6) *By assuming that $[f(\mathbf{x})]^{\alpha}\cos[\beta f(\mathbf{x})]\in[0,1]$ and setting $h(\mathbf{x})=(1/\beta)\arcsin\left\{[f(\mathbf{x})]^{\alpha}\cos[\beta f(\mathbf{x})]\right\}$, $\mathbf{x}\in\mathcal{X}$, we have*

$$S(h)(0,\beta)=C(f)(\alpha,\beta).$$

(7) *By assuming that $[f(\mathbf{x})]^{\alpha}\sin[\beta f(\mathbf{x})]\in[0,1]$ and setting $\ell(\mathbf{x})=(1/\beta)\arccos\left\{[f(\mathbf{x})]^{\alpha}\sin[\beta f(\mathbf{x})]\right\}$, $\mathbf{x}\in\mathcal{X}$, we have*

$$C(\ell)(0,\beta)=S(f)(\alpha,\beta).$$

(8) *By setting $q(\mathbf{x})=f(\mathbf{x}_*)$, where $\mathbf{x}_*=(x_1^*,\ldots,x_n^*)$ with $x_i^*\in\{x_i,1-x_i\}$ for each $i=1,\ldots,n$, we have*

$$S(q)(\alpha,\beta)=S(f)(\alpha,\beta),\quad C(q)(\alpha,\beta)=C(f)(\alpha,\beta).$$

(9) *For any $a\in\mathbb{R}$, we have*

$$S(f+a)(0,\beta)=\cos(\beta a)S(f)(0,\beta)+\sin(\beta a)C(f)(0,\beta).$$

(10) *For any $a\in\mathbb{R}$, we have*

$$C(f+a)(0,\beta)=\cos(\beta a)C(f)(0,\beta)-\sin(\beta a)S(f)(0,\beta).$$

Proof.

(1) For $f = a$ with $a \geq 0$, since $\int_{\mathcal{X}} d\mathbf{x} = 1$, we have

$$S(f)(\alpha, \beta) = \int_{\mathcal{X}} a^{\alpha} \sin(\beta a) d\mathbf{x} = a^{\alpha} \sin(\beta a) \int_{\mathcal{X}} d\mathbf{x} = a^{\alpha} \sin(\beta a)$$

and

$$C(f)(\alpha, \beta) = \int_{\mathcal{X}} a^{\alpha} \cos(\beta a) d\mathbf{x} = a^{\alpha} \cos(\beta a) \int_{\mathcal{X}} d\mathbf{x} = a^{\alpha} \cos(\beta a).$$

(2) Since the sine function is an odd function, it is obvious that

$$\begin{aligned} S(f)(\alpha, -\beta) &= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \sin[-\beta f(\mathbf{x})] d\mathbf{x} \\ &= -\int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \sin[\beta f(\mathbf{x})] d\mathbf{x} = -S(f)(\alpha, \beta). \end{aligned}$$

Conversely, since the cosine function is an even function, we obtain

$$\begin{aligned} C(f)(\alpha, -\beta) &= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \cos[-\beta f(\mathbf{x})] d\mathbf{x} \\ &= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \cos[\beta f(\mathbf{x})] d\mathbf{x} = C(f)(\alpha, \beta). \end{aligned}$$

(3) Since $\sin(0) = 0$ and $\cos(0) = 1$, it is obvious that

$$S(f)(\alpha, 0) = \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \sin[0 \times f(\mathbf{x})] d\mathbf{x} = 0$$

and

$$C(f)(\alpha, 0) = \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \cos[0 \times f(\mathbf{x})] d\mathbf{x} = \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} d\mathbf{x}.$$

(4) By substituting, we have

$$\begin{aligned} C(g)(\alpha, 0) &= \int_{\mathcal{X}} [g(\mathbf{x})]^{\alpha} d\mathbf{x} = \int_{\mathcal{X}} \left[f(\mathbf{x}) \left\{ \sin[\beta f(\mathbf{x})] \right\}^{1/\alpha} \right]^{\alpha} d\mathbf{x} \\ &= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \sin[\beta f(\mathbf{x})] d\mathbf{x} = S(f)(\alpha, \beta). \end{aligned}$$

(5) Since $[f(\mathbf{x})]^0 = 1$, we directly have

$$S(f)(0, \beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^0 \sin[\beta f(\mathbf{x})] d\mathbf{x} = \int_{\mathcal{X}} \sin[\beta f(\mathbf{x})] d\mathbf{x}$$

and

$$C(f)(0, \beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^0 \cos[\beta f(\mathbf{x})] d\mathbf{x} = \int_{\mathcal{X}} \cos[\beta f(\mathbf{x})] d\mathbf{x}.$$

(6) By substituting and using $\sin[\arcsin(t)] = t$, we have

$$\begin{aligned} S(h)(0, \beta) &= \int_{\mathcal{X}} \sin[\beta h(\mathbf{x})] d\mathbf{x} = \int_{\mathcal{X}} \sin\left[\beta \times \frac{1}{\beta} \arcsin\left\{ [f(\mathbf{x})]^{\alpha} \cos[\beta f(\mathbf{x})] \right\} \right] d\mathbf{x} \\ &= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \cos[\beta f(\mathbf{x})] d\mathbf{x} = C(f)(\alpha, \beta). \end{aligned}$$

(7) By substituting and using $\cos[\arccos(t)] = t$, we have

$$\begin{aligned} C(\ell)(0, \beta) &= \int_{\mathcal{X}} \cos[\beta \ell(\mathbf{x})] d\mathbf{x} = \int_{\mathcal{X}} \cos\left[\beta \times \frac{1}{\beta} \arccos\left\{ [f(\mathbf{x})]^{\alpha} \sin[\beta f(\mathbf{x})] \right\} \right] d\mathbf{x} \\ &= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \sin[\beta f(\mathbf{x})] d\mathbf{x} = S(f)(\alpha, \beta). \end{aligned}$$

(8) Let us consider the multivariate change of variables $\mathbf{x}_* = (x_1^*, \ldots, x_n^*)$ with $x_i^* \in \{x_i, 1-x_i\}$ for each $i = 1, \ldots, n$. The changed domain of integration is $\mathcal{X}_* = \mathcal{X}$, and the associated Jacobian is $(-1)^r$, where r denotes the number of x_i^* such that $x_i^* = 1 - x_i$ for any $i = 1, \ldots, n$. As a result, we have

$$\begin{aligned} S(q)(\alpha,\beta) &= \int_{\mathcal{X}_*} [f(\mathbf{x}_*)]^\alpha \sin[\beta f(\mathbf{x}_*)] d\mathbf{x}_* = \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \sin[\beta f(\mathbf{x})] |(-1)^r| d\mathbf{x} \\ &= \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \sin[\beta f(\mathbf{x})] d\mathbf{x} = S(f)(\alpha,\beta). \end{aligned}$$

Similarly, we prove that $C(q)(\alpha,\beta) = C(f)(\alpha,\beta)$.

(9) For any $a \in \mathbb{R}$, by using the following standard trigonometric formula: $\sin(t+v) = \sin(t)\cos(v) + \cos(t)\sin(v)$, we obtain

$$\begin{aligned} S(f+a)(0,\beta) &= \int_{\mathcal{X}} \sin\{\beta[f(\mathbf{x}) + a]\} d\mathbf{x} = \int_{\mathcal{X}} \sin[\beta f(\mathbf{x}) + \beta a] d\mathbf{x} \\ &= \cos(\beta a) \int_{\mathcal{X}} \sin[\beta f(\mathbf{x})] d\mathbf{x} + \sin(\beta a) \int_{\mathcal{X}} \cos[\beta f(\mathbf{x})] d\mathbf{x} \\ &= \cos(\beta a) S(f)(0,\beta) + \sin(\beta a) C(f)(0,\beta). \end{aligned}$$

(10) For any $a \in \mathbb{R}$, by using the following standard trigonometric formula: $\cos(t+v) = \cos(t)\cos(v) - \sin(t)\sin(v)$, we get

$$\begin{aligned} C(f+a)(0,\beta) &= \int_{\mathcal{X}} \cos\{\beta[f(\mathbf{x}) + a]\} d\mathbf{x} = \int_{\mathcal{X}} \cos[\beta f(\mathbf{x}) + \beta a] d\mathbf{x} \\ &= \cos(\beta a) \int_{\mathcal{X}} \cos[\beta f(\mathbf{x})] d\mathbf{x} - \sin(\beta a) \int_{\mathcal{X}} \sin[\beta f(\mathbf{x})] d\mathbf{x} \\ &= \cos(\beta a) C(f)(0,\beta) - \sin(\beta a) S(f)(0,\beta). \end{aligned}$$

The desired properties are established. □

Based on this proposition, the following comments hold:

- $S(f)(\alpha, 0)$ and $C(f)(\alpha, 0)$ are defined with compound trigonometric functions as the main ingredients, i.e., $\cos[\beta f(\mathbf{x})]$ and $\sin[\beta f(\mathbf{x})]$, respectively.
- Some specific functions make a bridge between the S and C operators.
- If we can determine the S or C operators of a function f, then we have the S or C operators of some modified versions of f, including its "x-flipping version", i.e., $q(\mathbf{x}) = f(\mathbf{x}_*)$, where $\mathbf{x}_* = (x_1^*, \ldots, x_n^*)$ with $x_i^* \in \{x_i, 1-x_i\}$ for each $i = 1, \ldots, n$, and its translated version, i.e., $f + a$, with $a \in \mathbb{R}$.

The sign of the S and C operators can be determined directly under some specific assumptions. This is presented in the result below.

Proposition 2.6. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$, be a non-negative function.*

(1) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi + \pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have $S(f)(\alpha,\beta) \geq 0$.*

(2) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi - \pi, 2m\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have $S(f)(\alpha,\beta) \leq 0$.*

(3) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi - \pi), (1/2)(4m\pi + \pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have $C(f)(\alpha,\beta) \geq 0$.*

(4) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi + \pi), (1/2)(4m\pi + 3\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have $C(f)(\alpha,\beta) \leq 0$.*

Proof.

(1) For any integer m and $t \in [2m\pi, 2m\pi+\pi]$, we have $\sin(t) \geq 0$. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi+\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have $\sin[\beta f(\mathbf{x})] \geq 0$. We also have $[f(\mathbf{x})]^\alpha \sin[\beta f(\mathbf{x})] \geq 0$, which implies that

$$S(f)(\alpha, \beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \sin[\beta f(\mathbf{x})] d\mathbf{x} \geq 0.$$

(2) Conversely, for any integer m and $t \in [2m\pi - \pi, 2m\pi]$, we have $\sin(t) \leq 0$. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi - \pi, 2m\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have $\sin[\beta f(\mathbf{x})] \leq 0$. Hence, we have $[f(\mathbf{x})]^\alpha \sin[\beta f(\mathbf{x})] \leq 0$, and the integration over $\mathcal{X}$ keeps the sign; we have $S(f)(\alpha, \beta) \leq 0$.

(3) For any integer m and $t \in [(1/2)(4m\pi - \pi), (1/2)(4m\pi + \pi)]$, we have $\cos(t) \geq 0$. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi - \pi), (1/2)(4m\pi + \pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have $\cos[\beta f(\mathbf{x})] \geq 0$. Hence, we get $[f(\mathbf{x})]^\alpha \cos[\beta f(\mathbf{x})] \geq 0$, which implies that $C(f)(\alpha, \beta) \geq 0$.

(4) Conversely, for any integer m and $t \in [(1/2)(4m\pi + \pi), (1/2)(4m\pi + 3\pi)]$, we have $\cos(t) \leq 0$. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi + \pi), (1/2)(4m\pi + 3\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have $\cos[\beta f(\mathbf{x})] \leq 0$. We also have $[f(\mathbf{x})]^\alpha \cos[\beta f(\mathbf{x})] \leq 0$, from which we derive $C(f)(\alpha, \beta) \leq 0$.

This ends the proof. □

Therefore, based on this result, the possible values of βf can directly determine the sign of the S and C operators. Some of their more technical properties are described below.

2.3. **Series expansions.** The S and C operators can be difficult to calculate for complex functions f, and this can be a barrier to their further analysis. We partially solve this problem in the following proposition with the use of infinite series expansions.

Proposition 2.7. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$, be a non-negative function. The multivariate interchange of integral and sum rule assumptions on f and the involved parameters are supposed.*

(1) *We have*

$$S(f)(\alpha, \beta) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \beta^{2k+1} \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+2k+1} d\mathbf{x}.$$

(2) *We have*

$$C(f)(\alpha, \beta) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+2k} d\mathbf{x}.$$

Proof. The multivariate interchange of integral and sum rule assumptions on f and the involved parameters are supposed, and are at the heart of the proof.

(1) A series expansion of the sine function is

$$\sin(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1}$$

for any $t \in \mathbb{R}$. Therefore, we have

$$S(f)(\alpha,\beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \sin[\beta f(\mathbf{x})] d\mathbf{x}$$
$$= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} [\beta f(\mathbf{x})]^{2k+1} \right\} d\mathbf{x}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \beta^{2k+1} \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+2k+1} d\mathbf{x}.$$

(2) A series expansion of the cosine function is

$$\cos(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k}$$

for any $t \in \mathbb{R}$. Therefore, we have

$$C(f)(\alpha,\beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \cos[\beta f(\mathbf{x})] d\mathbf{x}$$
$$= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} [\beta f(\mathbf{x})]^{2k} \right\} d\mathbf{x}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+2k} d\mathbf{x}.$$

The desired decompositions are established. □

Thanks to this proposition, the complexity of calculating the operators S and C of a function f is transposed to the calculation of an integral of the form $\int_{\mathcal{X}} [f(\mathbf{x})]^{\gamma} d\mathbf{x}$, where $\gamma \in \mathbb{R}$, which can be trivial in some cases, or involved special integral functions (gamma function, beta function, etc.).

For the determination of the S and C operators of some classical functions, we refer to Sections 4 and 5, respectively.

2.4. **Summation relationships.** Some comprehensive summation relationships between the S and C operators are shown in the result below.

Proposition 2.8. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$, be a non-negative function.*

(1) *If α denotes a non-negative integer, and $\beta \neq 0$, the following relation holds:*

$$S\left(f + \frac{\pi}{2\beta}\right)(\alpha,\beta) = \sum_{k=0}^{\alpha} \binom{\alpha}{k} \left(\frac{\pi}{2\beta}\right)^{\alpha-k} C(f)(k,\beta),$$

where $\binom{\alpha}{k} = \alpha!/[k!(\alpha-k)!]$ for $k = 0,\ldots,\alpha$.

(2) *If α denotes a non-negative integer, and $\beta \neq 0$, the following relation is true:*

$$C\left(f - \frac{\pi}{2\beta}\right)(\alpha,\beta) = \sum_{k=0}^{\alpha} \binom{\alpha}{k} \left(-\frac{\pi}{2\beta}\right)^{\alpha-k} S(f)(k,\beta).$$

Proof. The proof is mainly based on classical trigonometric formulas and the binomial formula.

(1) For $\beta \neq 0$, we have

$$S\left(f + \frac{\pi}{2\beta}\right)(\alpha,\beta) = \int_{\mathcal{X}} \left[f(\mathbf{x}) + \frac{\pi}{2\beta}\right]^{\alpha} \sin\left\{\beta\left[f(\mathbf{x}) + \frac{\pi}{2\beta}\right]\right\} d\mathbf{x}.$$

By using the following standard trigonometric formula: $\sin(t+\pi/2)=\cos(t)$, we have

$$\sin\left\{\beta\left[f(\mathbf{x})+\frac{\pi}{2\beta}\right]\right\}=\sin\left[\beta f(\mathbf{x})+\frac{\pi}{2}\right]=\cos[\beta f(\mathbf{x})].$$

On the other hand, since α is a non-negative integer, the standard binomial formula gives

$$\left[f(\mathbf{x})+\frac{\pi}{2\beta}\right]^{\alpha}=\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(\frac{\pi}{2\beta}\right)^{\alpha-k}[f(\mathbf{x})]^{k}.$$

Therefore, we have

$$\begin{aligned}S\left(f+\frac{\pi}{2\beta}\right)(\alpha,\beta)&=\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(\frac{\pi}{2\beta}\right)^{\alpha-k}\int_{\mathcal{X}}[f(\mathbf{x})]^{k}\cos[\beta f(\mathbf{x})]d\mathbf{x}\\&=\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(\frac{\pi}{2\beta}\right)^{\alpha-k}C(f)(k,\beta).\end{aligned}$$

(2) We use the same arguments of the previous proof; for $\beta\neq 0$, we have

$$C\left(f-\frac{\pi}{2\beta}\right)(\alpha,\beta)=\int_{\mathcal{X}}\left[f(\mathbf{x})-\frac{\pi}{2\beta}\right]^{\alpha}\cos\left\{\beta\left[f(\mathbf{x})-\frac{\pi}{2\beta}\right]\right\}d\mathbf{x}.$$

By using the following standard trigonometric formula: $\cos(t-\pi/2)=\sin(t)$, we have

$$\cos\left\{\beta\left[f(\mathbf{x})-\frac{\pi}{2\beta}\right]\right\}=\cos\left[\beta f(\mathbf{x})-\frac{\pi}{2}\right]=\sin[\beta f(\mathbf{x})].$$

On the other hand, since α is a non-negative integer, the standard binomial formula gives

$$\left[f(\mathbf{x})-\frac{\pi}{2\beta}\right]^{\alpha}=\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(-\frac{\pi}{2\beta}\right)^{\alpha-k}[f(\mathbf{x})]^{k}.$$

Therefore, we have

$$\begin{aligned}C\left(f-\frac{\pi}{2\beta}\right)(\alpha,\beta)&=\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(-\frac{\pi}{2\beta}\right)^{\alpha-k}\int_{\mathcal{X}}[f(\mathbf{x})]^{k}\sin[\beta f(\mathbf{x})]d\mathbf{x}\\&=\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(-\frac{\pi}{2\beta}\right)^{\alpha-k}S(f)(k,\beta).\end{aligned}$$

The desired relationships are obtained. □

The proposition below examines some concise expressions of $S(f)(\alpha,\beta)\pm C(f)(\alpha,\beta)$.

Proposition 2.9. *Let $\alpha\in\mathbb{R}$, $\beta\in\mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x}\in\mathcal{X}$, be a non-negative function.*

(1) *We have*

$$S(f)(\alpha,\beta)+C(f)(\alpha,\beta)=\sqrt{2}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\sin\left[\beta f(\mathbf{x})+\frac{\pi}{4}\right]d\mathbf{x}.$$

Furthermore, if α denotes a non-negative integer, and $\beta\neq 0$, the following relation holds:

$$S(f)(\alpha,\beta)+C(f)(\alpha,\beta)=\sqrt{2}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(-\frac{\pi}{4\beta}\right)^{\alpha-k}S\left(f+\frac{\pi}{4\beta}\right)(k,\beta).$$

(2) *We have*

$$S(f)(\alpha,\beta)+C(f)(\alpha,\beta)=\sqrt{2}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\cos\left[\beta f(\mathbf{x})-\frac{\pi}{4}\right]d\mathbf{x}.$$

Furthermore, if α denotes a non-negative integer, and $\beta \neq 0$, the following relation is true:

$$S(f)(\alpha,\beta)+C(f)(\alpha,\beta)=\sqrt{2}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(\frac{\pi}{4\beta}\right)^{\alpha-k}C\left(f-\frac{\pi}{4\beta}\right)(k,\beta).$$

(3) *We have*

$$S(f)(\alpha,\beta)-C(f)(\alpha,\beta)=\sqrt{2}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\sin\left[\beta f(\mathbf{x})-\frac{\pi}{4}\right]d\mathbf{x}.$$

Furthermore, if α denotes a non-negative integer, and $\beta \neq 0$, the following relation holds:

$$S(f)(\alpha,\beta)-C(f)(\alpha,\beta)=\sqrt{2}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(\frac{\pi}{4\beta}\right)^{\alpha-k}S\left(f-\frac{\pi}{4\beta}\right)(k,\beta).$$

(4) *We have*

$$S(f)(\alpha,\beta)-C(f)(\alpha,\beta)=-\sqrt{2}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\cos\left[\beta f(\mathbf{x})+\frac{\pi}{4}\right]d\mathbf{x}.$$

Furthermore, if α denotes a non-negative integer, and $\beta \neq 0$, the following relation is true:

$$S(f)(\alpha,\beta)-C(f)(\alpha,\beta)=-\sqrt{2}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(-\frac{\pi}{4\beta}\right)^{\alpha-k}C\left(f+\frac{\pi}{4\beta}\right)(k,\beta).$$

Proof.

(1) We have

$$\begin{aligned}S(f)(\alpha,\beta)+C(f)(\alpha,\beta)&=\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\sin[\beta f(\mathbf{x})]d\mathbf{x}+\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\cos[\beta f(\mathbf{x})]d\mathbf{x}\\&=\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\operatorname{cas}[\beta f(\mathbf{x})]d\mathbf{x},\end{aligned}$$

where $\operatorname{cas}(t)$ is defined in Equation (1.1). By using the following standard trigonometric formula: $\operatorname{cas}(t)=\sqrt{2}\sin(t+\pi/4)$, we have

$$\operatorname{cas}[\beta f(\mathbf{x})]=\sqrt{2}\sin\left[\beta f(\mathbf{x})+\frac{\pi}{4}\right].$$

Hence, we have

$$S(f)(\alpha,\beta)+C(f)(\alpha,\beta)=\sqrt{2}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\sin\left[\beta f(\mathbf{x})+\frac{\pi}{4}\right]d\mathbf{x}.$$

Furthermore, if α is a non-negative integer, and $\beta \neq 0$, the binomial formula gives

$$[f(\mathbf{x})]^{\alpha}=\left[f(\mathbf{x})+\frac{\pi}{4\beta}-\frac{\pi}{4\beta}\right]^{\alpha}=\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left[f(\mathbf{x})+\frac{\pi}{4\beta}\right]^{k}\left(-\frac{\pi}{4\beta}\right)^{\alpha-k}.$$

Applying it in the previous integral expression, we obtain

$$\begin{aligned}&S(f)(\alpha,\beta)+C(f)(\alpha,\beta)\\&=\sqrt{2}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(-\frac{\pi}{4\beta}\right)^{\alpha-k}\int_{\mathcal{X}}\left[f(\mathbf{x})+\frac{\pi}{4\beta}\right]^{k}\sin\left\{\beta\left[f(\mathbf{x})+\frac{\pi}{4\beta}\right]\right\}d\mathbf{x}\\&=\sqrt{2}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(-\frac{\pi}{4\beta}\right)^{\alpha-k}S\left(f+\frac{\pi}{4\beta}\right)(k,\beta).\end{aligned}$$

(2) The proof is similar to the previous proof; only the function $\operatorname{cas}[\beta f(\mathbf{x})]$ can be expressed in another way. We have
$$S(f)(\alpha,\beta)+C(f)(\alpha,\beta)=\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\operatorname{cas}[\beta f(\mathbf{x})]d\mathbf{x}.$$
Owing to the following standard trigonometric formula: $\operatorname{cas}(t)=\sqrt{2}\cos(t-\pi/4)$, we have
$$\operatorname{cas}[\beta f(\mathbf{x})]=\sqrt{2}\cos\left[\beta f(\mathbf{x})-\frac{\pi}{4}\right].$$
Hence, we can write
$$S(f)(\alpha,\beta)+C(f)(\alpha,\beta)=\sqrt{2}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\cos\left[\beta f(\mathbf{x})-\frac{\pi}{4}\right]d\mathbf{x}.$$
Furthermore, if α is a non-negative integer, and $\beta\neq 0$, the binomial formula gives
$$[f(\mathbf{x})]^{\alpha}=\left[f(\mathbf{x})-\frac{\pi}{4\beta}+\frac{\pi}{4\beta}\right]^{\alpha}=\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left[f(\mathbf{x})-\frac{\pi}{4\beta}\right]^{k}\left(\frac{\pi}{4\beta}\right)^{\alpha-k}.$$
By substituting it in the previous integral expression, we obtain
$$\begin{aligned}&S(f)(\alpha,\beta)+C(f)(\alpha,\beta)\\&=\sqrt{2}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(\frac{\pi}{4\beta}\right)^{\alpha-k}\int_{\mathcal{X}}\left[f(\mathbf{x})-\frac{\pi}{4\beta}\right]^{k}\cos\left\{\beta\left[f(\mathbf{x})-\frac{\pi}{4\beta}\right]\right\}d\mathbf{x}\\&=\sqrt{2}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(\frac{\pi}{4\beta}\right)^{\alpha-k}C\left(f-\frac{\pi}{4\beta}\right)(k,\beta).\end{aligned}$$

(3) The proof is similar to that of item 1 with the use of the following standard trigonometric formula: $\cos(t)-\sin(t)=\sqrt{2}\sin(t-\pi/4)$; therefore, details are omitted.

(4) The proof mimics that of item 2 with the use of the following standard trigonometric formula: $\cos(t)-\sin(t)=-\sqrt{2}\cos(t+\pi/4)$. We thus omit the details.

The proof ends. □

2.5. **Differential equations.** After investigations, the S and C operators satisfy numerous and interesting differential equations. Some of them are described in the next result.

Proposition 2.10. *Let $\alpha\in\mathbb{R}$, $\beta\in\mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x}\in\mathcal{X}$, be a non-negative function. The multivariate Leibniz integral rule assumptions on f and the involved parameters are supposed.*

(1) *We have*
$$\frac{\partial}{\partial\beta}S(f)(\alpha,\beta)-C(f)(\alpha+1,\beta)=0.$$
(2) *We have*
$$\frac{\partial}{\partial\beta}C(f)(\alpha,\beta)+S(f)(\alpha+1,\beta)=0.$$
(3) *We have*
$$\frac{\partial^2}{\partial\beta^2}S(f)(\alpha,\beta)+S(f)(\alpha+2,\beta)=0.$$
(4) *We have*
$$\frac{\partial^2}{\partial\beta^2}C(f)(\alpha,\beta)+C(f)(\alpha+2,\beta)=0.$$

Proof. The multivariate Leibniz integral rule assumptions on f and the involved parameters are supposed in all the developments below.

(1) Since $\partial \left\{\sin[\beta f(\mathbf{x})]\right\}/(\partial\beta) = f(\mathbf{x})\cos[\beta f(\mathbf{x})]$, we have

$$\begin{aligned}\frac{\partial}{\partial\beta}S(f)(\alpha,\beta) &= \frac{\partial}{\partial\beta}\left[\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\sin[\beta f(\mathbf{x})]d\mathbf{x}\right]\\ &= \int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\frac{\partial}{\partial\beta}\left\{\sin[\beta f(\mathbf{x})]\right\}d\mathbf{x}\\ &= \int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha+1}\cos[\beta f(\mathbf{x})]d\mathbf{x} = C(f)(\alpha+1,\beta).\end{aligned}$$

As a result, we establish that

$$\frac{\partial}{\partial\beta}S(f)(\alpha,\beta) - C(f)(\alpha+1,\beta) = 0.$$

(2) Since $\partial \left\{\cos[\beta f(\mathbf{x})]\right\}/(\partial\beta) = -f(\mathbf{x})\sin[\beta f(\mathbf{x})]$, we have

$$\begin{aligned}\frac{\partial}{\partial\beta}C(f)(\alpha,\beta) &= \frac{\partial}{\partial\beta}\left[\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\cos[\beta f(\mathbf{x})]d\mathbf{x}\right]\\ &= \int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\frac{\partial}{\partial\beta}\left\{\cos[\beta f(\mathbf{x})]\right\}d\mathbf{x}\\ &= -\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha+1}\sin[\beta f(\mathbf{x})]d\mathbf{x} = -S(f)(\alpha+1,\beta).\end{aligned}$$

Therefore, we find that

$$\frac{\partial}{\partial\beta}C(f)(\alpha,\beta) + S(f)(\alpha+1,\beta) = 0.$$

(3) Since $\partial^2 \left\{\sin[\beta f(\mathbf{x})]\right\}/(\partial\beta^2) = -[f(\mathbf{x})]^2\sin[\beta f(\mathbf{x})]$, we have

$$\begin{aligned}\frac{\partial^2}{\partial\beta^2}S(f)(\alpha,\beta) &= \frac{\partial^2}{\partial\beta^2}\left[\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\sin[\beta f(\mathbf{x})]d\mathbf{x}\right]\\ &= \int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\frac{\partial^2}{\partial\beta^2}\left\{\sin[\beta f(\mathbf{x})]\right\}d\mathbf{x}\\ &= -\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha+2}\sin[\beta f(\mathbf{x})]d\mathbf{x} = -S(f)(\alpha+2,\beta).\end{aligned}$$

Thus, we obtain

$$\frac{\partial^2}{\partial\beta^2}S(f)(\alpha,\beta) + S(f)(\alpha+2,\beta) = 0.$$

(4) Since $\partial^2 \left\{\cos[\beta f(\mathbf{x})]\right\}/(\partial\beta^2) = -[f(\mathbf{x})]^2\cos[\beta f(\mathbf{x})]$, we have

$$\begin{aligned}\frac{\partial^2}{\partial\beta^2}C(f)(\alpha,\beta) &= \frac{\partial^2}{\partial\beta^2}\left[\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\cos[\beta f(\mathbf{x})]d\mathbf{x}\right]\\ &= \int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\frac{\partial^2}{\partial\beta^2}\left\{\cos[\beta f(\mathbf{x})]\right\}d\mathbf{x}\\ &= -\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha+2}\cos[\beta f(\mathbf{x})]d\mathbf{x} = -C(f)(\alpha+2,\beta).\end{aligned}$$

As a result, we get

$$\frac{\partial^2}{\partial\beta^2}C(f)(\alpha,\beta) + C(f)(\alpha+2,\beta) = 0.$$

The stated differential equations are demonstrated. □

This result also implies that, if we have the analytical expressions of the S or C operators, then we can derive a multitude of differential equations where they are the solutions.

A list of higher order differentiation of the S and C operators with respect to the parameters α and β is detailed in the proposition below.

Proposition 2.11. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $f(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$, be a non-negative function and m be a non-negative integer. The multivariate Leibniz integral rule assumptions on f and the involved parameters are supposed.*

(1) *We have*

$$\frac{\partial^m}{\partial \beta^m} S(f)(\alpha, \beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+m} \sin\left[\beta f(\mathbf{x}) + \frac{m\pi}{2}\right] d\mathbf{x}.$$

Furthermore, if α denotes a non-negative integer, and $\beta \neq 0$, the following relation holds:

$$\frac{\partial^m}{\partial \beta^m} S(f)(\alpha, \beta) = \sum_{k=0}^{\alpha+m} \binom{\alpha+m}{k} \left(-\frac{m\pi}{2\beta}\right)^{\alpha+m-k} S\left(f + \frac{m\pi}{2\beta}\right)(k, \beta).$$

(2) *We have*

$$\frac{\partial^m}{\partial \beta^m} C(f)(\alpha, \beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+m} \cos\left[\beta f(\mathbf{x}) + \frac{m\pi}{2}\right] d\mathbf{x}.$$

Furthermore, if α denotes a non-negative integer, and $\beta \neq 0$, the following relation is true:

$$\frac{\partial^m}{\partial \beta^m} C(f)(\alpha, \beta) = \sum_{k=0}^{\alpha+m} \binom{\alpha+m}{k} \left(-\frac{m\pi}{2\beta}\right)^{\alpha+m-k} C\left(f + \frac{m\pi}{2\beta}\right)(k, \beta).$$

(3) *We have*

$$\frac{\partial^m}{\partial \alpha^m} S(f)(\alpha, \beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \{\log[f(\mathbf{x})]\}^m \sin[\beta f(\mathbf{x})] d\mathbf{x}.$$

(4) *We have*

$$\frac{\partial^m}{\partial \alpha^m} C(f)(\alpha, \beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \{\log[f(\mathbf{x})]\}^m \cos[\beta f(\mathbf{x})] d\mathbf{x}.$$

Proof. We mention that the multivariate Leibniz integral rule assumptions on f and the involved parameters are supposed in all the coming developments. This is an important aspect of the proof.

(1) Since $[\sin(t)]^{(m)} = \sin(t + m\pi/2)$, we have $\partial^m \{\sin[\beta f(\mathbf{x})]\} / (\partial \beta^m) = [f(\mathbf{x})]^m \sin[\beta f(\mathbf{x}) + m\pi/2]$. Therefore, we obtain

$$\begin{aligned}
\frac{\partial^m}{\partial \beta^m} S(f)(\alpha, \beta) &= \frac{\partial^m}{\partial \beta^m} \left[\int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \sin[\beta f(\mathbf{x})] d\mathbf{x}\right] \\
&= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \frac{\partial^m}{\partial \beta^m} \{\sin[\beta f(\mathbf{x})]\} d\mathbf{x} \\
&= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+m} \sin\left[\beta f(\mathbf{x}) + \frac{m\pi}{2}\right] d\mathbf{x}.
\end{aligned}$$

Furthermore, if α is a non-negative integer, and $\beta \neq 0$, the binomial formula gives

$$\begin{aligned}
[f(\mathbf{x})]^{\alpha+m} &= \left[f(\mathbf{x}) + \frac{m\pi}{2\beta} - \frac{m\pi}{2\beta}\right]^{\alpha+m} \\
&= \sum_{k=0}^{\alpha+m} \binom{\alpha+m}{k} \left[f(\mathbf{x}) + \frac{m\pi}{2\beta}\right]^k \left(-\frac{m\pi}{2\beta}\right)^{\alpha+m-k}.
\end{aligned}$$

Substituting it in the previous integral expression, we obtain

$$\begin{aligned}&\frac{\partial^m}{\partial\beta^m}S(f)(\alpha,\beta)\\&=\sum_{k=0}^{\alpha+m}\binom{\alpha+m}{k}\left(-\frac{m\pi}{2\beta}\right)^{\alpha+m-k}\int_{\mathcal{X}}\left[f(\mathbf{x})+\frac{m\pi}{2\beta}\right]^k\sin\left\{\beta\left[f(\mathbf{x})+\frac{m\pi}{2\beta}\right]\right\}d\mathbf{x}\\&=\sum_{k=0}^{\alpha+m}\binom{\alpha+m}{k}\left(-\frac{m\pi}{2\beta}\right)^{\alpha+m-k}S\left(f+\frac{m\pi}{2\beta}\right)(k,\beta).\end{aligned}$$

(2) Since $[\cos(t)]^{(m)}=\cos(t+m\pi/2)$, we have $\partial^m\{\cos[\beta f(\mathbf{x})]\}/(\partial\beta^m)=[f(\mathbf{x})]^m\cos[\beta f(\mathbf{x})+m\pi/2]$. Hence, we get

$$\begin{aligned}\frac{\partial^m}{\partial\beta^m}C(f)(\alpha,\beta)&=\frac{\partial^m}{\partial\beta^m}\left[\int_{\mathcal{X}}[f(\mathbf{x})]^\alpha\cos[\beta f(\mathbf{x})]d\mathbf{x}\right]\\&=\int_{\mathcal{X}}[f(\mathbf{x})]^\alpha\frac{\partial^m}{\partial\beta^m}\{\cos[\beta f(\mathbf{x})]\}d\mathbf{x}\\&=\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha+m}\cos\left[\beta f(\mathbf{x})+\frac{m\pi}{2}\right]d\mathbf{x}.\end{aligned}$$

If α is a non-negative integer, and $\beta\neq 0$, the summation expression is obtained as in the previous proof. The details are omitted.

(3) Since $\partial^m\{[f(\mathbf{x})]^\alpha\}/(\partial\alpha^m)=[f(\mathbf{x})]^\alpha\{\log[f(\mathbf{x})]\}^m$, we have

$$\begin{aligned}&\frac{\partial^m}{\partial\alpha^m}S(f)(\alpha,\beta)=\frac{\partial^m}{\partial\alpha^m}\left[\int_{\mathcal{X}}[f(\mathbf{x})]^\alpha\sin[\beta f(\mathbf{x})]d\mathbf{x}\right]\\&=\int_{\mathcal{X}}\frac{\partial^m}{\partial\alpha^m}\{[f(\mathbf{x})]^\alpha\}\sin[\beta f(\mathbf{x})]d\mathbf{x}\\&=\int_{\mathcal{X}}[f(\mathbf{x})]^\alpha\{\log[f(\mathbf{x})]\}^m\sin[\beta f(\mathbf{x})]d\mathbf{x}.\end{aligned}$$

(4) By proceeding exactly as in the previous proof, just replacing $\sin[\beta f(\mathbf{x})]$ by $\cos[\beta f(\mathbf{x})]$, we get the desired result.

The proof is complete. □

In addition to summation and differential equations, an advantage of the S and C operators is that they generate original and precise inequalities.

Some of them are determined in the next section.

3. Mathematical inequalities

The inequalities derived from the S and C operators can be of different natures. In this section, we distinguish between basic, convex-type, and Hölder-type inequalities.

3.1. **Basic inequalities.** Some basic upper bounds for $|S(f)(\alpha,\beta)|$ and $|S(f)(\alpha,\beta)|$ are presented in the result below.

Proposition 3.1. *Let $\alpha\in\mathbb{R}$, $\beta\in\mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x}\in\mathcal{X}$, be a non-negative function.*

(1) *We have*

$$|S(f)(\alpha,\beta)|\leq\min\left\{\int_{\mathcal{X}}[f(\mathbf{x})]^\alpha d\mathbf{x},|\beta|\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha+1}d\mathbf{x}\right\}.$$

(2) *We have*

$$|C(f)(\alpha,\beta)|\leq\int_{\mathcal{X}}[f(\mathbf{x})]^\alpha d\mathbf{x}.$$

Proof.

(1) By applying the triangle inequality and the fact that $|\sin(t)|\leq \min(1, |t|)$ for any $t \in \mathbb{R}$, we obviously get

$$\begin{aligned}|S(f)(\alpha, \beta)| &\leq \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} |\sin[\beta f(\mathbf{x})]| d\mathbf{x} \\ &\leq \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \min[1, |\beta| f(\mathbf{x})] d\mathbf{x} \\ &\leq \min\left\{\int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} d\mathbf{x}, |\beta| \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+1} d\mathbf{x}\right\}.\end{aligned}$$

(2) It follows from the triangle inequality and $|\cos(t)|\leq 1$ for any $t \in \mathbb{R}$ that

$$|C(f)(\alpha, \beta)|\leq \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} |\cos[\beta f(\mathbf{x})]| d\mathbf{x} \leq \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} d\mathbf{x}.$$

This ends the proof. □

Bounds for the absolute value difference of the S or C operators of two functions are examined in the result below.

Proposition 3.2. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, and $f(\mathbf{x})$ and $g(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$, be non-negative functions.*

(1) *We have*

$$\begin{aligned}&|S(f)(\alpha, \beta) - S(g)(\alpha, \beta)| \\ &\leq \int_{\mathcal{X}} |[f(\mathbf{x})]^{\alpha} - [g(\mathbf{x})]^{\alpha}| \min[1, |\beta| f(\mathbf{x})] d\mathbf{x} + |\beta| \int_{\mathcal{X}} [g(\mathbf{x})]^{\alpha} |f(\mathbf{x}) - g(\mathbf{x})| d\mathbf{x}.\end{aligned}$$

(2) *We have*

$$\begin{aligned}&|C(f)(\alpha, \beta) - C(g)(\alpha, \beta)| \\ &\leq \int_{\mathcal{X}} |[f(\mathbf{x})]^{\alpha} - [g(\mathbf{x})]^{\alpha}| d\mathbf{x} + |\beta| \int_{\mathcal{X}} [g(\mathbf{x})]^{\alpha} |f(\mathbf{x}) - g(\mathbf{x})| d\mathbf{x}.\end{aligned}$$

Proof.

(1) We can write

$$\begin{aligned}S(f)(\alpha, \beta) - S(g)(\alpha, \beta) &= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \sin[\beta f(\mathbf{x})] d\mathbf{x} - \int_{\mathcal{X}} [g(\mathbf{x})]^{\alpha} \sin[\beta g(\mathbf{x})] d\mathbf{x} \\ = \int_{\mathcal{X}} \{[f(\mathbf{x})]^{\alpha} - [g(\mathbf{x})]^{\alpha}\} &\sin[\beta f(\mathbf{x})] d\mathbf{x} - \int_{\mathcal{X}} [g(\mathbf{x})]^{\alpha} \{\sin[\beta g(\mathbf{x})] - \sin[\beta f(\mathbf{x})]\} d\mathbf{x}.\end{aligned}$$

The triangle inequality gives

$$\begin{aligned}&|S(f)(\alpha, \beta) - S(g)(\alpha, \beta)| \\ &\leq \int_{\mathcal{X}} |[f(\mathbf{x})]^{\alpha} - [g(\mathbf{x})]^{\alpha}| |\sin[\beta f(\mathbf{x})]| d\mathbf{x} + \int_{\mathcal{X}} [g(\mathbf{x})]^{\alpha} |\sin[\beta g(\mathbf{x})] - \sin[\beta f(\mathbf{x})]| d\mathbf{x}.\end{aligned}$$

By using the inequality $|\sin(t)|\leq \min(1, |t|)$ for $t \in \mathbb{R}$ and the fact that the sine function is 1-Lipschitz, i.e., $|\sin(t) - \sin(u)|\leq |t - u|$, for any $t \in \mathbb{R}$ and $u \in \mathbb{R}$, we obtain

$$\begin{aligned}&|S(f)(\alpha, \beta) - S(g)(\alpha, \beta)| \\ &\leq \int_{\mathcal{X}} |[f(\mathbf{x})]^{\alpha} - [g(\mathbf{x})]^{\alpha}| \min[1, |\beta| f(\mathbf{x})] d\mathbf{x} + |\beta| \int_{\mathcal{X}} [g(\mathbf{x})]^{\alpha} |f(\mathbf{x}) - g(\mathbf{x})| d\mathbf{x}.\end{aligned}$$

(2) The proof is similar to the previous proof, just replacing the sine function by the cosine function, using the inequality $|\cos(t)|\leq 1$ for $t \in \mathbb{R}$ and the fact that the cosine function is 1-Lipschitz. The details are omitted.

The desired inequalities are demonstrated. □

Based on item 1 of the above proposition, under the assumption that there exists a positive constant M such that $f(\mathbf{x}) \leq M$ and $g(\mathbf{x}) \leq M$ for any $\mathbf{x} \in \mathcal{X}$, and $\alpha \geq 1$, by using the fact that t^α, $t \in [0, M]$, has the Lipschitz property with the constant $\alpha M^{\alpha-1}$, we have

$$|S(f)(\alpha,\beta) - S(g)(\alpha,\beta)|\leq \kappa \int_{\mathcal{X}} |f(\mathbf{x}) - g(\mathbf{x})| d\mathbf{x},$$

where $\kappa = M^{\alpha-1} \{\alpha \min[1, |\beta|M] + |\beta|M\}$. Such a Lipschitz inequality can be used to investigate the convergence of sequences of functions through the S operator. An analogous result is satisfied for the C operator.

The next result shows that, under some assumptions, we have a clear order of the S and C operators.

Proposition 3.3. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$, be a non-negative function.*

(1) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2(m\pi - 3\pi/8), 2(m\pi + \pi/8)]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$S(f)(\alpha, \beta) \leq C(f)(\alpha, \beta).$$

(2) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2(m\pi + \pi/8), 2m\pi + \pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$C(f)(\alpha, \beta) \leq S(f)(\alpha, \beta).$$

Proof.

(1) For any integer m and $t \in [2(m\pi - 3\pi/8), 2(m\pi + \pi/8)]$, we have $\sin(t) \leq \cos(t)$. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2(m\pi - 3\pi/8), 2(m\pi + \pi/8)]$ for any $\mathbf{x} \in \mathcal{X}$, we have $\sin[\beta f(\mathbf{x})] \leq \cos[\beta f(\mathbf{x})]$, which implies that

$$S(f)(\alpha, \beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \sin[\beta f(\mathbf{x})] d\mathbf{x} \leq \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \cos[\beta f(\mathbf{x})] d\mathbf{x} = C(f)(\alpha, \beta).$$

(2) Conversely, for any integer m and $t \in [2(m\pi + \pi/8), 2m\pi + \pi]$, we have $\cos(t) \leq \sin(t)$. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2(m\pi + \pi/8), 2m\pi + \pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have $\cos[\beta f(\mathbf{x})] \leq \sin[\beta f(\mathbf{x})]$, which implies that $C(f)(\alpha, \beta) \leq S(f)(\alpha, \beta)$.

This ends the proof. □

Technical inequalities on the S and C operators are demonstrated in the next proposition, which is mainly based on well-known trigonometric inequalities.

Proposition 3.4. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$, be a non-negative function. Under the assumption that $\beta f(\mathbf{x}) \in [0, \pi/2]$ for any $\mathbf{x} \in \mathcal{X}$, the inequalities below are fulfilled.*

(1) *We have*

$$S(f)(\alpha, \beta) \geq \frac{2\beta}{\pi} \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+1} d\mathbf{x}.$$

(2) *We have*

$$S(f)(\alpha, \beta) \leq \frac{\beta}{3} \left[2 \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+1} d\mathbf{x} + C(f)(\alpha + 1, \beta) \right].$$

(3) *We have*

$$C(f)(\alpha,\beta) \geq \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} d\mathbf{x} - \frac{2\beta}{\pi}\int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+1} d\mathbf{x}.$$

Proof. Each of the inequalities is based on a known trigonometric inequality, recalled before the developments.

(1) The Jordan inequality ensures that $\sin(t) \geq (2/\pi)t$ for any $t \in [0, \pi/2]$ (see [37]). As a result, under the assumption that $\beta f(\mathbf{x}) \in [0, \pi/2]$ for any $\mathbf{x} \in \mathcal{X}$, we get

$$\sin[\beta f(\mathbf{x})] \geq \frac{2\beta}{\pi} f(\mathbf{x}).$$

Therefore, we have

$$S(f)(\alpha,\beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \sin[\beta f(\mathbf{x})] d\mathbf{x} \geq \frac{2\beta}{\pi}\int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} f(\mathbf{x}) d\mathbf{x} = \frac{2\beta}{\pi}\int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+1} d\mathbf{x}.$$

(2) The Cusa-Huygens inequality states that $\sin(t) \leq (1/3)t[2 + \cos(t)]$ for any $t \in [0, \pi/2]$ (see [20]). Thanks to it, under the assumption that $\beta f(\mathbf{x}) \in [0, \pi/2]$ for any $\mathbf{x} \in \mathcal{X}$, we have

$$\sin[\beta f(\mathbf{x})] \leq \frac{\beta}{3} f(\mathbf{x}) \left\{2 + \cos[\beta f(\mathbf{x})]\right\},$$

so

$$\begin{aligned} S(f)(\alpha,\beta) &= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \sin[\beta f(\mathbf{x})] d\mathbf{x} \\ &\leq \frac{\beta}{3}\left[2\int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} f(\mathbf{x}) d\mathbf{x} + \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} f(\mathbf{x}) \cos[\beta f(\mathbf{x})] d\mathbf{x}\right] \\ &= \frac{\beta}{3}\left[2\int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+1} d\mathbf{x} + C(f)(\alpha+1,\beta)\right]. \end{aligned}$$

(3) The Kober inequality ensures that $\cos(t) \geq 1 - (2/\pi)t$ for any $t \in [0, \pi/2]$ (see [27]). Under the assumption that $\beta f(\mathbf{x}) \in [0, \pi/2]$ for any $\mathbf{x} \in \mathcal{X}$, we get

$$\cos[\beta f(\mathbf{x})] \geq 1 - \frac{2\beta}{\pi} f(\mathbf{x}).$$

Therefore, we have

$$\begin{aligned} C(f)(\alpha,\beta) &= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \cos[\beta f(\mathbf{x})] d\mathbf{x} \geq \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \left[1 - \frac{2\beta}{\pi} f(\mathbf{x})\right] d\mathbf{x} \\ &= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} d\mathbf{x} - \frac{2\beta}{\pi}\int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+1} d\mathbf{x}. \end{aligned}$$

The proof is complete. □

The inequalities in items 1 and 3 of this proposition will be examined in the applications in Section 6. For a given f, this can lead to sharp and innovative inequalities.

Some refined inequalities involving the S and C operators are described in the proposition below.

Proposition 3.5. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$, be a non-negative function. Under the assumption that $\beta f(\mathbf{x}) \in [0, \pi]$ for any $\mathbf{x} \in \mathcal{X}$, the inequalities below hold.*

(1) *We have*

$$S(f)(\alpha,\beta) \geq \beta C(f)\left(\alpha+1, \frac{\beta}{\sqrt{3}}\right).$$

(2) *We have*

$$S(f)(\alpha,\beta) \leq \beta C(f)\left(\alpha+1, \frac{\beta}{2}\right).$$

Proof.

(1) The following trigonometric inequality holds: $\sin(t) \geq t\cos[t/\sqrt{3}]$ for any $t \in [0, \pi]$ (see [21] and [29]). Therefore, under the assumption $\beta f(\mathbf{x}) \in [0, \pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have

$$\sin[\beta f(\mathbf{x})] \geq \beta f(\mathbf{x}) \cos\left[\frac{\beta}{\sqrt{3}} f(\mathbf{x})\right],$$

which implies that

$$\begin{aligned} S(f)(\alpha, \beta) &= \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \sin[\beta f(\mathbf{x})] d\mathbf{x} \geq \beta \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha f(\mathbf{x}) \cos\left[\frac{\beta}{\sqrt{3}} f(\mathbf{x})\right] d\mathbf{x} \\ &= \beta C(f)\left(\alpha + 1, \frac{\beta}{\sqrt{3}}\right). \end{aligned}$$

(2) In a complementary manner, the following trigonometric inequality holds: $\sin(t) \leq t\cos(t/2)$ for any $t \in [0, \pi]$ (see [21] and [29]). Therefore, under the assumption $\beta f(\mathbf{x}) \in [0, \pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have

$$\sin[\beta f(\mathbf{x})] \leq \beta f(\mathbf{x}) \cos\left[\frac{\beta}{2} f(\mathbf{x})\right],$$

which implies that

$$\begin{aligned} S(f)(\alpha, \beta) &= \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \sin[\beta f(\mathbf{x})] \leq \beta \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha f(\mathbf{x}) \cos\left[\frac{\beta}{2} f(\mathbf{x})\right] d\mathbf{x} \\ &= \beta C(f)\left(\alpha + 1, \frac{\beta}{2}\right). \end{aligned}$$

This ends the proof. □

The proposition below determines inequalities concerning the functions $|S(f)(\alpha, \beta) \pm C(f)(\alpha, \beta)|$, $[S(f)(\alpha, \beta)]^2 + [C(f)(\alpha, \beta)]^2$ and $S(f)(\alpha, \beta) + C(f)(\alpha, \beta)$.

Proposition 3.6. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$, be a non-negative function.*

(1) *We have*

$$\begin{aligned} &|S(f)(\alpha, \beta) + C(f)(\alpha, \beta)| \\ &\leq \sqrt{2} \min\left\{\int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}, |\beta| \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+1} d\mathbf{x} + \frac{\pi}{4} \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}\right\}. \end{aligned}$$

(2) *We have*

$$\begin{aligned} &|S(f)(\alpha, \beta) - C(f)(\alpha, \beta)| \\ &\leq \sqrt{2} \min\left\{\int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}, |\beta| \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+1} d\mathbf{x} + \frac{\pi}{4} \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}\right\}. \end{aligned}$$

(3) *We have*

$$[S(f)(\alpha, \beta)]^2 + [C(f)(\alpha, \beta)]^2 \leq \int_{\mathcal{X}} [f(\mathbf{x})]^{2\alpha} d\mathbf{x}.$$

(4) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi + \pi/2]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$\int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x} \leq S(f)(\alpha, \beta) + C(f)(\alpha, \beta).$$

Proof.

(1) By using item 1 of Proposition 2.9, we have

$$S(f)(\alpha,\beta)+C(f)(\alpha,\beta)=\sqrt{2}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\sin\left[\beta f(\mathbf{x})+\frac{\pi}{4}\right]d\mathbf{x}.$$

By applying the triangle inequality and the fact that $|\sin(t)|\leq\min(1,|t|)$ for any $t\in\mathbb{R}$, we get

$$\begin{aligned}|S(f)(\alpha,\beta)+C(f)(\alpha,\beta)|&\leq\sqrt{2}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\left|\sin\left[\beta f(\mathbf{x})+\frac{\pi}{4}\right]\right|d\mathbf{x}\\ &\leq\sqrt{2}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\min\left[1,\left|\beta f(\mathbf{x})+\frac{\pi}{4}\right|\right]d\mathbf{x}\\ &\leq\sqrt{2}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\min\left[1,|\beta|f(\mathbf{x})+\frac{\pi}{4}\right]d\mathbf{x}\\ &\leq\sqrt{2}\min\left\{\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}d\mathbf{x},|\beta|\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha+1}d\mathbf{x}+\frac{\pi}{4}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}d\mathbf{x}\right\}.\end{aligned}$$

(2) We use similar arguments to the previous proof. Thus, by using item 3 of Proposition 2.9, we have

$$S(f)(\alpha,\beta)-C(f)(\alpha,\beta)=\sqrt{2}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\sin\left[\beta f(\mathbf{x})-\frac{\pi}{4}\right]d\mathbf{x}.$$

Owing to the triangle inequality and the fact that $|\sin(t)|\leq\min(1,|t|)$ for any $t\in\mathbb{R}$, we get

$$\begin{aligned}|S(f)(\alpha,\beta)-C(f)(\alpha,\beta)|&\leq\sqrt{2}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\left|\sin\left[\beta f(\mathbf{x})-\frac{\pi}{4}\right]\right|d\mathbf{x}\\ &\leq\sqrt{2}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\min\left[1,\left|\beta f(\mathbf{x})-\frac{\pi}{4}\right|\right]d\mathbf{x}\\ &\leq\sqrt{2}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\min\left[1,|\beta|f(\mathbf{x})+\frac{\pi}{4}\right]d\mathbf{x}\\ &\leq\sqrt{2}\min\left\{\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}d\mathbf{x},|\beta|\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha+1}d\mathbf{x}+\frac{\pi}{4}\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}d\mathbf{x}\right\}.\end{aligned}$$

The same upper bound as the previous item is derived.

(3) By using the integral version of the Cauchy-Schwarz inequality, we obtain

$$[S(f)(\alpha,\beta)]^2=\left[\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\sin[\beta f(\mathbf{x})]d\mathbf{x}\right]^2\leq\int_{\mathcal{X}}[f(\mathbf{x})]^{2\alpha}\left\{\sin[\beta f(\mathbf{x})]\right\}^2d\mathbf{x}$$

and

$$[C(f)(\alpha,\beta)]^2=\left[\int_{\mathcal{X}}[f(\mathbf{x})]^{\alpha}\cos[\beta f(\mathbf{x})]d\mathbf{x}\right]^2\leq\int_{\mathcal{X}}[f(\mathbf{x})]^{2\alpha}\left\{\cos[\beta f(\mathbf{x})]\right\}^2d\mathbf{x}.$$

As a result, since $\{\sin[\beta f(\mathbf{x})]\}^2+\{\cos[\beta f(\mathbf{x})]\}^2=1$, we have

$$\begin{aligned}[S(f)(\alpha,\beta)]^2+[C(f)(\alpha,\beta)]^2&\leq\int_{\mathcal{X}}[f(\mathbf{x})]^{2\alpha}\left\{\sin[\beta f(\mathbf{x})]\right\}^2d\mathbf{x}+\int_{\mathcal{X}}[f(\mathbf{x})]^{2\alpha}\left\{\cos[\beta f(\mathbf{x})]\right\}^2d\mathbf{x}\\ &=\int_{\mathcal{X}}[f(\mathbf{x})]^{2\alpha}\left[\left\{\sin[\beta f(\mathbf{x})]\right\}^2+\left\{\cos[\beta f(\mathbf{x})]\right\}^2\right]d\mathbf{x}\\ &=\int_{\mathcal{X}}[f(\mathbf{x})]^{2\alpha}d\mathbf{x}.\end{aligned}$$

(4) For any integer m and $t\in[2m\pi,2m\pi+\pi/2]$, we have $\sin(t)\in[0,1]$ and $\cos(t)\in[0,1]$, so $[\sin(t)]^2\leq\sin(t)$ and $[\cos(t)]^2\leq\cos(t)$. Hence, under the assumption that there exists an integer m such that

$\beta f(\mathbf{x}) \in [2m\pi, 2m\pi + \pi/2]$ for any $\mathbf{x} \in \mathcal{X}$, we have $\{\sin[\beta f(\mathbf{x})]\}^2 \leq \sin[\beta f(\mathbf{x})]$ and $\{\cos[\beta f(\mathbf{x})]\}^2 \leq \cos[\beta f(\mathbf{x})]$. This, combined with $\{\sin[\beta f(\mathbf{x})]\}^2 + \{\cos[\beta f(\mathbf{x})]\}^2 = 1$, gives

$$\begin{aligned} S(f)(\alpha,\beta) + C(f)(\alpha,\beta) &= \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \sin[\beta f(\mathbf{x})] d\mathbf{x} + \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \cos[\beta f(\mathbf{x})] d\mathbf{x} \\ &\geq \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \{\sin[\beta f(\mathbf{x})]\}^2 d\mathbf{x} + \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \{\cos[\beta f(\mathbf{x})]\}^2 d\mathbf{x} \\ &= \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \left[\{\sin[\beta f(\mathbf{x})]\}^2 + \{\cos[\beta f(\mathbf{x})]\}^2\right] d\mathbf{x} \\ &= \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}. \end{aligned}$$

The indicated inequalities are proved. □

A simple consequence of this proposition and the known properties of the S and C operators is that, if there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi + \pi/2]$ for any $\mathbf{x} \in \mathcal{X}$, we have

$$\int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x} \leq S(f)(\alpha,\beta) + C(f)(\alpha,\beta) \leq 2\int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}.$$

In this case, we have precise control over what we call the cas operator, which can be expressed as

$$C_{as}(f)(\alpha,\beta) = S(f)(\alpha,\beta) + C(f)(\alpha,\beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \operatorname{cas}[\beta f(\mathbf{x})] d\mathbf{x}.$$

It can be viewed as a variant of the Hartley operator that may merit more attention.

3.2. **Convex-type inequalities.** Another interest of the S and C operators is to satisfy diverse kinds of convex-type inequalities. In particular, in the next result, we show that the trigonometric function can be extracted from the integral to provide upper or lower bounds for these operators. The key tool in the proof is the integral version of the Jensen inequality.

Proposition 3.7. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$, be a non-negative function.*

(1) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi - \pi, 2m\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have*
$$S(f)(\alpha,\beta) \geq \left\{\int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}\right\} \sin\left\{\beta \frac{1}{\int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}} \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+1} d\mathbf{x}\right\}.$$

(2) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi + \pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have*
$$S(f)(\alpha,\beta) \leq \left\{\int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}\right\} \sin\left\{\beta \frac{1}{\int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}} \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+1} d\mathbf{x}\right\}.$$

(3) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi + \pi), (1/2)(4m\pi + 3\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have*
$$C(f)(\alpha,\beta) \geq \left\{\int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}\right\} \cos\left\{\beta \frac{1}{\int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}} \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+1} d\mathbf{x}\right\}.$$

(4) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi - \pi), (1/2)(4m\pi + \pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have*
$$C(f)(\alpha,\beta) \leq \left\{\int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}\right\} \cos\left\{\beta \frac{1}{\int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}} \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+1} d\mathbf{x}\right\}.$$

Proof.

(1) For any integer m and $t \in [2m\pi - \pi, 2m\pi]$, we have $\sin(t) \leq 0$, implying that $[\sin(t)]'' = -\sin(t) \geq 0$. Hence, $\sin(t)$ is convex for such t. Furthermore, the function $g_\alpha(\mathbf{x}) = [f(\mathbf{x})]^\alpha / \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}$ is a valid probability density function because $g_\alpha(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \mathcal{X}$ and $\int_{\mathcal{X}} g_\alpha(\mathbf{x}) d\mathbf{x} = 1$. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi - \pi, 2m\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we can apply the integral version of the Jensen inequality with the convex sine function and the measure $\nu(A) = \int_A g_\alpha(\mathbf{x}) d\mathbf{x}$, $A \subseteq \mathcal{X}$. Hence, we have

$$\begin{aligned} S(f)(\alpha,\beta) &= \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \sin[\beta f(\mathbf{x})] d\mathbf{x} \\ &= \left\{ \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x} \right\} \int_{\mathcal{X}} \sin[\beta f(\mathbf{x})] g_\alpha(\mathbf{x}) d\mathbf{x} \\ &\geq \left\{ \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x} \right\} \sin\left[\int_{\mathcal{X}} \beta f(\mathbf{x}) g_\alpha(\mathbf{x}) d\mathbf{x} \right] \\ &= \left\{ \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x} \right\} \sin\left\{ \beta \frac{1}{\int_{\mathcal{X}} [f(\mathbf{x})]^\alpha d\mathbf{x}} \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha+1} d\mathbf{x} \right\}. \end{aligned}$$

(2) Conversely, for any integer m and $t \in [2m\pi, 2m\pi + \pi]$, we have $\sin(t) \geq 0$, implying that $[\sin(t)]'' = -\sin(t) \leq 0$. Hence, $\sin(t)$ is concave for such t. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi + \pi]$ for any $\mathbf{x} \in \mathcal{X}$, the integral version of the Jensen inequality applied in the concave case gives the reversed inequality in the previous item.

(3) For any integer m and $t \in [(1/2)(4m\pi + \pi), (1/2)(4m\pi + 3\pi)]$, we have $\cos(t) \leq 0$, implying that $[\cos(t)]'' = -\cos(t) \geq 0$. Hence, $\cos(t)$ is convex for such t. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi + \pi), (1/2)(4m\pi + 3\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, the integral version of the Jensen inequality applied in the convex case give a similar result as the one in item 1 with $\cos[\beta f(\mathbf{x})]$ instead of $\sin[\beta f(\mathbf{x})]$, providing the desired result.

(4) Conversely, for any integer m and $t \in [(1/2)(4m\pi - \pi), (1/2)(4m\pi + \pi)]$, we have $\cos(t) \geq 0$, implying that $[\sin(t)]'' = -\cos(t) \leq 0$. Hence, $\cos(t)$ is concave for such t. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi - \pi), (1/2)(4m\pi + \pi)]$ for any $\mathbf{x} \in \mathcal{X}$, the integral version of the Jensen inequality applied in the concave case gives the reversed inequality in the previous item.

The desired inequalities are established. □

This proposition will be crucial in the applications in Section 6; for a given f, it can lead to sharp and innovative trigonometric inequalities.

Other convex-type inequalities satisfied by the S and C operators with respect to f are determined in the next proposition in the case $\alpha = 0$.

Proposition 3.8. *Let $\beta \in \mathbb{R}$, $\mu \in [0,1]$, and $f(\mathbf{x})$ and $g(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$, be non-negative functions.*

(1) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi - \pi, 2m\pi]$ and $\beta g(\mathbf{x}) \in [2m\pi - \pi, 2m\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$S[\mu f + (1-\mu)g](0,\beta) \leq \mu S(f)(0,\beta) + (1-\mu) S(g)(0,\beta).$$

(2) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi + \pi]$ and $\beta g(\mathbf{x}) \in [2m\pi, 2m\pi + \pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$\mu S(f)(0,\beta) + (1-\mu) S(g)(0,\beta) \leq S[\mu f + (1-\mu)g](0,\beta).$$

(3) *Under the assumption that there exists an integer* m *such that* $\beta f(\mathbf{x}) \in [(1/2)(4m\pi+\pi),(1/2)(4m\pi+3\pi)]$ *and* $\beta g(\mathbf{x}) \in [(1/2)(4m\pi+\pi),(1/2)(4m\pi+3\pi)]$ *for any* $\mathbf{x} \in \mathcal{X}$*, we have*

$$C[\mu f+(1-\mu)g](0,\beta) \leq \mu C(f)(0,\beta)+(1-\mu)C(g)(0,\beta).$$

(4) *Under the assumption that there exists an integer* m *such that* $\beta f(\mathbf{x}) \in [(1/2)(4m\pi-\pi),(1/2)(4m\pi+\pi)]$ *and* $\beta g(\mathbf{x}) \in [(1/2)(4m\pi-\pi),(1/2)(4m\pi+\pi)]$ *for any* $\mathbf{x} \in \mathcal{X}$*, we have*

$$\mu C(f)(0,\beta)+(1-\mu)C(g)(0,\beta) \leq C[\mu f+(1-\mu)g](0,\beta).$$

Proof. We re-use the convex and concave properties of the sine and cosine functions as described in the proof of Proposition 3.7.

(1) For any integer m and $t \in [2m\pi-\pi,2m\pi]$, $\sin(t)$ is convex. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi-\pi,2m\pi]$ and $\beta g(\mathbf{x}) \in [2m\pi-\pi,2m\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have

$$\begin{aligned}\sin\{\beta[\mu f+(1-\mu)g](\mathbf{x})\} &= \sin[\mu\beta f(\mathbf{x})+(1-\mu)\beta g(\mathbf{x})] \\ &\leq \mu\sin[\beta f(\mathbf{x})]+(1-\mu)\sin[\beta g(\mathbf{x})].\end{aligned}$$

As a result, we have

$$\begin{aligned}S[\mu f+(1-\mu)g](0,\beta) &= \int_{\mathcal{X}} \sin\{\beta[\mu f+(1-\mu)g](\mathbf{x})\}d\mathbf{x} \\ &\leq \mu\int_{\mathcal{X}} \sin[\beta f(\mathbf{x})]d\mathbf{x}+(1-\mu)\int_{\mathcal{X}} \sin[\beta g(\mathbf{x})]d\mathbf{x} \\ &= \mu S(f)(0,\beta)+(1-\mu)S(g)(0,\beta).\end{aligned}$$

(2) Conversely, for any integer m and $t \in [2m\pi,2m\pi+\pi]$, $\sin(t)$ is concave. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi,2m\pi+\pi]$ and $\beta g(\mathbf{x}) \in [2m\pi,2m\pi+\pi]$ for any $\mathbf{x} \in \mathcal{X}$, the classical concave property followed by an integration gives the reversed inequality in the previous item.

(3) For any integer m and $t \in [(1/2)(4m\pi+\pi),(1/2)(4m\pi+3\pi)]$, $\cos(t)$ is convex. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi+\pi),(1/2)(4m\pi+3\pi)]$ and $\beta g(\mathbf{x}) \in [(1/2)(4m\pi+\pi),(1/2)(4m\pi+3\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, a similar inequality to item 1 is obtained with $\cos[\beta f(\mathbf{x})]$ instead of $\sin[\beta f(\mathbf{x})]$, giving the desired inequality.

(4) Conversely, for any integer m and $t \in [(1/2)(4m\pi-\pi),(1/2)(4m\pi+\pi)]$, $\cos(t)$ is concave. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi-\pi),(1/2)(4m\pi+\pi)]$ and $\beta g(\mathbf{x}) \in [(1/2)(4m\pi-\pi),(1/2)(4m\pi+\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, the classical concave property followed by an integration produces the reversed inequality in the previous item.

This ends the proof. □

A more general result below is presented, but under some additional assumptions on the domains of βf and βg.

Proposition 3.9. *Let* $\alpha \in \mathbb{R}$*,* $\beta \in \mathbb{R}$*,* $\mu \in [0,1]$*, and* $f(\mathbf{x})$ *and* $g(\mathbf{x})$*,* $\mathbf{x} \in \mathcal{X}$*, be non-negative functions.*

(1) *Under the assumptions that there exists a positive integer* m *such that* $\beta f(\mathbf{x}) \in [(1/2)(4m\pi-\pi),2m\pi]$ *and* $\beta g(\mathbf{x}) \in [(1/2)(4m\pi-\pi),2m\pi]$ *for any* $\mathbf{x} \in \mathcal{X}$*, and* $\alpha \in [0,1]$*, we have*

$$S[\mu f+(1-\mu)g](\alpha,\beta) \leq \mu S(f)(\alpha,\beta)+(1-\mu)S(g)(\alpha,\beta).$$

(2) *Under the assumptions that there exists a non-negative integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi + \pi), 2m\pi + \pi]$ and $\beta g(\mathbf{x}) \in [(1/2)(4m\pi + \pi), 2m\pi + \pi]$ for any $\mathbf{x} \in \mathcal{X}$, and $\alpha \in [0, 1]$, we have*

$$\mu S(f)(\alpha, \beta) + (1 - \mu)S(g)(\alpha, \beta) \leq S[\mu f + (1 - \mu)g](\alpha, \beta).$$

(3) *Under the assumptions that there exists a positive integer m such that $\beta f(\mathbf{x}) \in [2m\pi - \pi, (1/2)(4m\pi - \pi)]$ and $\beta g(\mathbf{x}) \in [2m\pi - \pi, (1/2)(4m\pi - \pi)]$ for any $\mathbf{x} \in \mathcal{X}$, and $\alpha \in [0, 1]$, we have*

$$C[\mu f + (1 - \mu)g](\alpha, \beta) \leq \mu C(f)(\alpha, \beta) + (1 - \mu)C(g)(\alpha, \beta).$$

(4) *Under the assumptions that there exists a non-negative integer m such that $\beta f(\mathbf{x}) \in [2m\pi, (1/2)(4m\pi+\pi)]$ and $\beta g(\mathbf{x}) \in [2m\pi, (1/2)(4m\pi + \pi)]$ for any $\mathbf{x} \in \mathcal{X}$, and $\alpha \in [0, 1]$, we have*

$$\mu C(f)(\alpha, \beta) + (1 - \mu)C(g)(\alpha, \beta) \leq C[\mu f + (1 - \mu)g](\alpha, \beta).$$

Proof.

(1) Let us consider the function $h(t) = t^\alpha \sin(t)$ for any $t \geq 0$. Under the considered assumptions on f and βf, it is clear that $\beta > 0$. Therefore, we can write

$$S(f)(\alpha, \beta) = \frac{1}{\beta^a} \int_{\mathcal{X}} [\beta f(\mathbf{x})]^\alpha \sin[\beta f(\mathbf{x})] d\mathbf{x} = \frac{1}{\beta^a} \int_{\mathcal{X}} h[\beta f(\mathbf{x})] d\mathbf{x}.$$

Let us now study the two first derivatives of h and their signs. We have

$$h'(t) = t^{\alpha-1}[\alpha \sin(t) + \ t \cos(t)]$$

and

$$h''(t) = t^{\alpha-2}\{2\alpha t \cos(t) - [\alpha(1 - \alpha) + t^2] \sin(t)\}.$$

For any positive integer m, $t \in [(1/2)(4m\pi - \pi), 2m\pi]$, and $\alpha \in [0, 1]$, we have $t \geq 0$, $\cos(t) \geq 0$, $\sin(t) \leq 0$ and $\alpha(1 - \alpha) \geq 0$, implying that $h''(t) \geq 0$. Hence, $h(t)$ is convex for such t. Therefore, under the assumptions that there exists a positive integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi-\pi), 2m\pi]$ and $\beta g(\mathbf{x}) \in [(1/2)(4m\pi - \pi), 2m\pi]$ for any $\mathbf{x} \in \mathcal{X}$, and $\alpha \in [0, 1]$, we have

$$\begin{aligned} h\{\beta[\mu f + (1 - \mu)g](\mathbf{x})\} &= h[\mu\beta f(\mathbf{x}) + (1 - \mu)\beta g(\mathbf{x})] \\ &\leq \mu h[\beta f(\mathbf{x})] + (1 - \mu)h[\beta g(\mathbf{x})]. \end{aligned}$$

As a result, we have

$$\begin{aligned} S[\mu f + (1 - \mu)g](\alpha, \beta) &= \frac{1}{\beta^a} \int_{\mathcal{X}} h\{\beta[\mu f + (1 - \mu)g](\mathbf{x})\} d\mathbf{x} \\ &\leq \mu \frac{1}{\beta^a} \int_{\mathcal{X}} h[\beta f(\mathbf{x})] d\mathbf{x} + (1 - \mu) \frac{1}{\beta^a} \int_{\mathcal{X}} h[\beta g(\mathbf{x})] d\mathbf{x} \\ &= \mu S(f)(\alpha, \beta) + (1 - \mu)S(g)(\alpha, \beta). \end{aligned}$$

(2) Similarly to the previous proof, for any non-negative integer m, $t \in [(1/2)(4m\pi + \pi), 2m\pi + \pi]$ and $\alpha \in [0, 1]$, we have $t \geq 0$, $\cos(t) \leq 0$, $\sin(t) \geq 0$ and $\alpha(1 - \alpha) \geq 0$, implying that $h''(t) \leq 0$. Hence, $h(t)$ is concave for such t. Therefore, under the assumptions that there exists a non-negative integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi + \pi), 2m\pi + \pi]$ and $\beta g(\mathbf{x}) \in [(1/2)(4m\pi + \pi), 2m\pi + \pi]$ for any $\mathbf{x} \in \mathcal{X}$, and $\alpha \in [0, 1]$, the concave property gives the reversed inequality in the previous item.

(3) Following the spirit of the proof of item 1, let us consider the function $\ell(t) = t^\alpha \cos(t)$ in such a way that

$$C(f)(\alpha, \beta) = \frac{1}{\beta^a} \int_{\mathcal{X}} \ell[\beta f(\mathbf{x})] d\mathbf{x}.$$

Then we have

$$\ell'(t) = t^{\alpha-1}[\alpha \cos(t) - t \sin(t)]$$

and

$$\ell''(t) = t^{\alpha-2}\{-2\alpha t\sin(t) - [\alpha(1-\alpha)+t^2]\cos(t)\}.$$

For any positive integer m, $t \in [2m\pi-\pi,(1/2)(4m\pi-\pi)]$, and $\alpha \in [0,1]$, we have $t \geq 0$, $\cos(t) \leq 0$, $\sin(t) \leq 0$ and $\alpha(1-\alpha) \geq 0$, implying that $\ell''(t) \geq 0$. Hence, $\ell(t)$ is convex for such t. Therefore, under the assumptions that there exists a positive integer m such that $\beta f(\mathbf{x}) \in [2m\pi-\pi,(1/2)(4m\pi-\pi)]$ and $\beta g(\mathbf{x}) \in [2m\pi-\pi,(1/2)(4m\pi-\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, and $\alpha \in [0,1]$, a similar inequality in item 1 is obtained with ℓ instead of h, giving the desired result.

(4) Similarly, for any non-negative integer m, $t \in [2m\pi,(1/2)(4m\pi+\pi)]$, and $\alpha \in [0,1]$, we have $t \geq 0$, $\cos(t) \geq 0$, $\sin(t) \geq 0$ and $\alpha(1-\alpha) \geq 0$, implying that $\ell''(t) \leq 0$. Hence, $\ell(t)$ is concave for such t. Therefore, under the assumptions that there exists a non-negative integer m such that $\beta f(\mathbf{x}) \in [2m\pi,(1/2)(4m\pi+\pi)]$ and $\beta g(\mathbf{x}) \in [2m\pi,(1/2)(4m\pi+\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, and $\alpha \in [0,1]$, the concave property gives the reversed inequality in the previous item.

The desired convex-type inequalities are demonstrated. □

The next result investigates convex-type inequalities for the S and C operators with respect to the parameter β for the case $\alpha = 0$.

Proposition 3.10. *Let $\beta_1 \in \mathbb{R}$, $\beta_2 \in \mathbb{R}$, $\mu \in [0,1]$, and $f(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$ be a non-negative function.*

(1) *Under the assumption that there exists an integer m such that $\beta_1 f(\mathbf{x}) \in [2m\pi-\pi,2m\pi]$ and $\beta_2 f(\mathbf{x}) \in [2m\pi-\pi,2m\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$S(f)[0,\mu\beta_1+(1-\mu)\beta_2] \leq \mu S(f)(0,\beta_1) + (1-\mu)S(f)(0,\beta_2).$$

(2) *Under the assumption that there exists an integer m such that $\beta_1 f(\mathbf{x}) \in [2m\pi,2m\pi+\pi]$ and $\beta_2 f(\mathbf{x}) \in [2m\pi,2m\pi+\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$\mu S(f)(0,\beta_1) + (1-\mu)S(f)(0,\beta_2) \leq S(f)[0,\mu\beta_1+(1-\mu)\beta_2].$$

(3) *Under the assumption that there exists an integer m such that $\beta_1 f(\mathbf{x}) \in [(1/2)(4m\pi+\pi),(1/2)(4m\pi+3\pi)]$ and $\beta_2 f(\mathbf{x}) \in [(1/2)(4m\pi+\pi),(1/2)(4m\pi+3\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$C(f)[0,\mu\beta_1+(1-\mu)\beta_2] \leq \mu C(f)(0,\beta_1) + (1-\mu)C(f)(0,\beta_2).$$

(4) *Under the assumption that there exists an integer m such that $\beta_1 f(\mathbf{x}) \in [(1/2)(4m\pi-\pi),(1/2)(4m\pi+\pi)]$ and $\beta_2 f(\mathbf{x}) \in [(1/2)(4m\pi-\pi),(1/2)(4m\pi+\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$\mu C(f)(0,\beta_1) + (1-\mu)C(f)(0,\beta_2) \leq C(f)[0,\mu\beta_1+(1-\mu)\beta_2].$$

Proof. The arguments are identical to those used in the proof of Proposition 3.8. Let us focus mainly on the first item. For any integer m and $t \in [2m\pi-\pi,2m\pi]$, $\sin(t)$ is convex. Therefore, under the assumption that there exists an integer m such that $\beta_1 f(\mathbf{x}) \in [2m\pi-\pi,2m\pi]$ and $\beta_2 f(\mathbf{x}) \in [2m\pi-\pi,2m\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have

$$\begin{aligned}\sin\{[\mu\beta_1+(1-\mu)\beta_2]f(\mathbf{x})\} &= \sin[\mu\beta_1 f(\mathbf{x}) + (1-\mu)\beta_2 f(\mathbf{x})] \\ &\leq \mu\sin[\beta_1 f(\mathbf{x})] + (1-\mu)\sin[\beta_2 f(\mathbf{x})].\end{aligned}$$

As a result, we have

$$\begin{aligned}S(f)[0,\mu\beta_1+(1-\mu)\beta_2] &= \int_{\mathcal{X}} \sin\{[\mu\beta_1+(1-\mu)\beta_2]f(\mathbf{x})\}d\mathbf{x}\\ &\le \mu\int_{\mathcal{X}} \sin[\beta_1 f(\mathbf{x})]d\mathbf{x} + (1-\mu)\int_{\mathcal{X}} \sin[\beta_2 f(\mathbf{x})]d\mathbf{x}\\ &= \mu S(f)(0,\beta_1)+(1-\mu)S(f)(0,\beta_2).\end{aligned}$$

In some sense, the proof is identical to the one in Proposition 3.8, it is enough to put $\beta = \beta_1$ and $g = (\beta_2/\beta_1)f$. With this mind, we directly demonstrate items 2, 3 and 4. This ends the proof. □

The next proposition is analogous to the previous one, but with respect to the parameter α.

Proposition 3.11. *Let $\alpha_1 \in \mathbb{R}$, $\alpha_2 \in \mathbb{R}$, $\mu \in [0,1]$, $\beta \in \mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$, be a non-negative function.*

(1) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi+\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$S(f)[\mu\alpha_1+(1-\mu)\alpha_2,\beta] \le \mu S(f)(\alpha_1,\beta)+(1-\mu)S(f)(\alpha_2,\beta).$$

(2) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi-\pi, 2m\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$\mu S(f)(\alpha_1,\beta)+(1-\mu)S(f)(\alpha_2,\beta) \le S(f)[\mu\alpha_1+(1-\mu)\alpha_2,\beta].$$

(3) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi-\pi), (1/2)(4m\pi+\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$C(f)[\mu\alpha_1+(1-\mu)\alpha_2,\beta] \le \mu C(f)(\alpha_1,\beta)+(1-\mu)C(f)(\alpha_2,\beta).$$

(4) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi+\pi), (1/2)(4m\pi+3\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$\mu C(f)(\alpha_1,\beta)+(1-\mu)C(f)(\alpha_2,\beta) \le C(f)[\mu\alpha_1+(1-\mu)\alpha_2,\beta].$$

Proof.

(1) It follows from item 3 of Proposition 2.11 applied with $m = 2$ that

$$\frac{\partial^2}{\partial\alpha^2}S(f)(\alpha,\beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha\{\log[f(\mathbf{x})]\}^2 \sin[\beta f(\mathbf{x})]d\mathbf{x}.$$

It is obvious that $[f(\mathbf{x})]^\alpha\{\log[f(\mathbf{x})]\}^2 \ge 0$ and, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi+\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have $\sin[\beta f(\mathbf{x})] \ge 0$. As a result, we have

$$\frac{\partial^2}{\partial\alpha^2}S(f)(\alpha,\beta) \ge 0,$$

implying that $S(f)(\alpha,\beta)$ is convex with respect to α. A consequence of this convex property is the desired inequality, i.e.,

$$S(f)[\mu\alpha_1+(1-\mu)\alpha_2,\beta] \le \mu S(f)(\alpha_1,\beta)+(1-\mu)S(f)(\alpha_2,\beta).$$

(2) Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi-\pi, 2m\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have $\sin[\beta f(\mathbf{x})] \le 0$. By using the same arguments as those in the previous proof, we get

$$\frac{\partial^2}{\partial\alpha^2}S(f)(\alpha,\beta) \le 0,$$

implying that $S(f)(\alpha,\beta)$ is concave with respect to α. The reversed inequality in the previous item is directly obtained.

(3) It follows from item 4 of Proposition 2.11 applied with $m = 2$ that

$$\frac{\partial^2}{\partial \alpha^2} C(f)(\alpha, \beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \{\log[f(\mathbf{x})]\}^2 \cos[\beta f(\mathbf{x})] d\mathbf{x}.$$

It is obvious that $[f(\mathbf{x})]^{\alpha}\{\log[f(\mathbf{x})]\}^2 \geq 0$ and, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi - \pi), (1/2)(4m\pi + \pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have $\cos[\beta f(\mathbf{x})] \geq 0$. As a result, we have

$$\frac{\partial^2}{\partial \alpha^2} C(f)(\alpha, \beta) \geq 0,$$

implying that $C(f)(\alpha, \beta)$ is convex with respect to α, so

$$C(f)[\mu\alpha_1 + (1-\mu)\alpha_2, \beta] \leq \mu C(f)(\alpha_1, \beta) + (1-\mu) C(f)(\alpha_2, \beta).$$

(4) Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi+\pi), (1/2)(4m\pi+3\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have $\cos[\beta f(\mathbf{x})] \leq 0$. By using the same arguments as those in the previous proof, we get

$$\frac{\partial^2}{\partial \alpha^2} C(f)(\alpha, \beta) \leq 0,$$

implying that $C(f)(\alpha, \beta)$ is concave with respect to α. The reversed inequality in the previous item is established.

The proof is complete. □

The result below is about convex-type convex inequalities based on the power versions of the S and C operators.

Proposition 3.12. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $f(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$, be a non-negative function.*

(1) *Under the assumptions that $\alpha \geq 1$ and that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi + \pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$S(f)(\alpha, \beta) \geq [S(f)(0, \beta)]^{1-\alpha} [S(f)(1, \beta)]^{\alpha}.$$

(2) *Under the assumptions that $\alpha \in [0, 1]$ and that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi + \pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$S(f)(\alpha, \beta) \leq [S(f)(0, \beta)]^{1-\alpha} [S(f)(1, \beta)]^{\alpha}.$$

(3) *Under the assumptions that $\alpha \geq 1$ and that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi - \pi), (1/2)(4m\pi + \pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$C(f)(\alpha, \beta) \geq [C(f)(0, \beta)]^{1-\alpha} [C(f)(1, \beta)]^{\alpha}.$$

(4) *Under the assumptions that $\alpha \in [0, 1]$ and that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi - \pi), (1/2)(4m\pi + \pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$C(f)(\alpha, \beta) \leq [C(f)(0, \beta)]^{1-\alpha} [C(f)(1, \beta)]^{\alpha}.$$

Proof. The integral version of the Jensen inequality is again at the center of the proof.

(1) For any $t \geq 0$ and $\alpha \geq 1$, it is clear that $(t^{\alpha})'' = \alpha(\alpha - 1)t^{\alpha-2} \geq 0$, implying that t^{α} is convex. Moreover, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi + \pi]$ for any $\mathbf{x} \in \mathcal{X}$ implying that $\sin[\beta f(\mathbf{x})] \geq 0$, the function $h_{\beta}(\mathbf{x}) = \sin[\beta f(\mathbf{x})] / \int_{\mathcal{X}} \sin[\beta f(\mathbf{x})] d\mathbf{x} = \sin[\beta f(\mathbf{x})] / S(f)(0, \beta)$ is a valid probability density function because $h_{\beta}(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \mathcal{X}$ and

$\int_{\mathcal{X}} h_\beta(\mathbf{x})d\mathbf{x} = 1$. Therefore, the integral version of the Jensen inequality applied with the measure $\nu(B) = \int_B h_\beta(\mathbf{x})d\mathbf{x}$, $B \subseteq \mathcal{X}$, gives

$$\begin{aligned} S(f)(\alpha,\beta) &= \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha \sin[\beta f(\mathbf{x})]d\mathbf{x} = S(f)(0,\beta) \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha h_\beta(\mathbf{x})d\mathbf{x} \\ &\geq S(f)(0,\beta) \left[\int_{\mathcal{X}} f(\mathbf{x})h_\beta(\mathbf{x})d\mathbf{x}\right]^\alpha \\ &= S(f)(0,\beta) \left\{\frac{1}{S(f)(0,\beta)} \int_{\mathcal{X}} [f(\mathbf{x})] \sin[\beta f(\mathbf{x})]d\mathbf{x}\right\}^\alpha \\ &= [S(f)(0,\beta)]^{1-\alpha} [S(f)(1,\beta)]^\alpha . \end{aligned}$$

(2) For any $t \geq 0$ and $\alpha \in [0,1]$, it is clear that $(t^\alpha)'' = \alpha(\alpha-1)t^{\alpha-2} \leq 0$. Hence, t^α is concave for such t. Therefore, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi+\pi]$ for any $\mathbf{x} \in \mathcal{X}$ implying that $\sin[\beta f(\mathbf{x})] \geq 0$, the integral version of the Jensen inequality applied in the concave case gives the reversed inequality in the previous item.

(3) Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi-\pi),(1/2)(4m\pi+\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have $\cos[\beta f(\mathbf{x})] \geq 0$ and the function $\ell_\beta(\mathbf{x}) = \cos[\beta f(\mathbf{x})]/\int_{\mathcal{X}} \cos[\beta f(\mathbf{x})]d\mathbf{x} = \cos[\beta f(\mathbf{x})]/C(f)(0,\beta)$ is a probability density function. So, we can apply the integral version of the Jensen inequality with the measure $\tau(B) = \int_B \ell_\beta(\mathbf{x})d\mathbf{x}$, $B \subseteq \mathcal{X}$. The desired results follow the lines of the proof of item 1 with ℓ_β instead of h_β.

(4) The proof combines the arguments of those in items 2 and 3; we omit the details.

The proof ends. □

Thanks to this result, if $S(f)(0,\beta)$, $C(f)(0,\beta)$, $S(f)(1,\beta)$ and $C(f)(1,\beta)$ are calculable, we have direct lower or upper bounds of the possibly more complex associated S and C operators, depending on whether $\alpha \geq 1$ or $\alpha \in [0,1]$.

3.3. **Hölder-type inequalities.** Inequalities of Hölder-type involving the S and C operators are exhibited in the next result.

Proposition 3.13. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\theta \geq 1$, $\mu \in [0,1]$, and $f(\mathbf{x})$, $\mathbf{x} \in \mathcal{X}$, be a non-negative function.*

(1) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi+\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$S(f)(\alpha,\beta) \leq [S(f)(\alpha\theta\mu,\beta)]^{1/\theta} \left\{S(f)\left[\frac{\alpha\theta(1-\mu)}{\theta-1},\beta\right]\right\}^{1-1/\theta}.$$

(2) *Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi-\pi),(1/2)(4m\pi+\pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have*

$$C(f)(\alpha,\beta) \leq [C(f)(\alpha\theta\mu,\beta)]^{1/\theta} \left\{C(f)\left[\frac{\alpha\theta(1-\mu)}{\theta-1},\beta\right]\right\}^{1-1/\theta}.$$

Proof.

(1) Under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [2m\pi, 2m\pi+\pi]$ for any $\mathbf{x} \in \mathcal{X}$, we have $\sin[\beta f(\mathbf{x})] \geq 0$. In light of this, by the decompositions $\alpha = \alpha\mu + (1-\mu)\alpha$ and $1 = 1/\theta + (1-1/\theta)$, the integral version of the Hölder inequality applied with the parameter $p = \theta \geq 1$

gives

$$\begin{aligned} S(f)(\alpha,\beta) &= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha} \sin[\beta f(\mathbf{x})] d\mathbf{x}. \\ &= \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha\mu} \{\sin[\beta f(\mathbf{x})]\}^{1/\theta} [f(\mathbf{x})]^{(1-\mu)\alpha} \{\sin[\beta f(\mathbf{x})]\}^{1-1/\theta} d\mathbf{x} \\ &\leq \left\{ \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha\theta\mu} \sin[\beta f(\mathbf{x})] d\mathbf{x} \right\}^{1/\theta} \left\{ \int_{\mathcal{X}} [f(\mathbf{x})]^{\alpha\theta(1-\mu)/(\theta-1)} \sin[\beta f(\mathbf{x})] d\mathbf{x} \right\}^{1-1/\theta} \\ &= [S(f)(\alpha\theta\mu,\beta)]^{1/\theta} \left\{ S(f) \left[\frac{\alpha\theta(1-\mu)}{\theta-1}, \beta \right] \right\}^{1-1/\theta}. \end{aligned}$$

(2) Just noticing that, under the assumption that there exists an integer m such that $\beta f(\mathbf{x}) \in [(1/2)(4m\pi - \pi), (1/2)(4m\pi + \pi)]$ for any $\mathbf{x} \in \mathcal{X}$, we have $\cos[\beta f(\mathbf{x})] \geq 0$, so we can replace the sine function by the cosine function in the previous proof. The details are omitted.

The proof is complete. □

Let us notice that, by taking $\mu = 1$ and $\theta = 1/\alpha$ with $\alpha \geq 1$, items 1 and 2 of Proposition 3.13 correspond to items 2 and 4 of Proposition 3.12, respectively. Thus, it can be viewed as a generalization in a certain case.

4. Some explicit formulas for the S operator

This section is devoted to some explicit formulas for the S operator, taking the usual functions for f. For the sake of simplicity, we always suppose that $\beta > 0$. Indeed, when the S (or C) operator is considered, the cases $\beta < 0$ and $\beta = 0$ can be easily derived by using items 2 and 3 of Proposition 2.5, respectively. We emphasize the cases $n = 1$ and $\alpha = 0$, $n = 1$ and $\alpha = 1$, $n = 1$ and varying α, cases $n = 2$ and $\alpha = 0$, and some more general cases.

4.1. **Cases $n = 1$ and $\alpha = 0$.** First, let us consider the cases $n = 1$ and $\alpha = 0$.

For $f(x) = x$, $x \in [0,1]$, the calculus of the S operator gives

$$S(f)(0,\beta) = \int_{[0,1]} \sin(\beta x) dx = \frac{1}{\beta}[1 - \cos(\beta)].$$

The same result holds for $f(x) = 1 - x$, $x \in [0,1]$.

By taking $f(x) = x^2$, $x \in [0,1]$, we obtain

$$S(f)(0,\beta) = \int_{[0,1]} \sin(\beta x^2) dx = \sqrt{\frac{\pi}{2\beta}} S_{\star} \left[\sqrt{\frac{2\beta}{\pi}} \right],$$

where $S_{\star}(x) = \int_{[0,x]} \sin(\pi t^2/2) dt$ is the Fresnel S integral. For $f(x) = (1-x)^2$, $x \in [0,1]$, the same formula is found.

For $f(x) = \sqrt{x}$, $x \in [0,1]$, we find that

$$S(f)(0,\beta) = \int_{[0,1]} \sin[\beta \sqrt{x}] dx = \frac{2}{\beta^2}[\sin(\beta) - \beta\cos(\beta)].$$

This is also true for $f(x) = \sqrt{1-x}$, $x \in [0,1]$.

By selecting $f(x) = \exp(x)$, $x \in [0,1]$, the calculus of the S operator gives

$$S(f)(0,\beta) = \int_{[0,1]} \sin[\beta \exp(x)] dx = S_o[\beta \exp(1)] - S_o(\beta),$$

where $S_o(x) = \int_{[0,x]} [\sin(t)/t] dt$ is the sine integral. The same result holds for $f(x) = \exp(1-x)$, $x \in [0,1]$.

For $f(x) = -\log(x)$, $x \in (0, 1]$, we obtain

$$S(f)(0,\beta) = \int_{[0,1]} \sin[-\beta \log(x)]dx = \frac{\beta}{\beta^2 + 1}.$$

For $f(x) = -\log(1 - x)$, $x \in [0, 1)$, the same formula is found.

By choosing $f(x) = 1/x$, $x \in (0, 1]$, we have

$$S(f)(0,\beta) = \int_{[0,1]} \sin\left(\frac{\beta}{x}\right) dx = \sin(\beta) - \beta C_o(\beta),$$

where $C_o(x) = -\int_{[x,\infty)} [\cos(t)/t]dt$ is the cosine integral. The same result holds for $f(x) = 1/(1 - x)$, $x \in [0, 1)$.

For $f(x) = 1/(1 + x)$, $x \in [0, 1]$, we get

$$S(f)(0,\beta) = \int_{[0,1]} \sin\left(\frac{\beta}{1+x}\right) dx = \beta \left[C_o(\beta) - C_o\left(\frac{\beta}{2}\right)\right] + 2\sin\left(\frac{\beta}{2}\right) - \sin(\beta).$$

This is also true for $f(x) = 1/(2 - x)$, $x \in [0, 1]$.

For $f(x) = 1/x^2$, $x \in (0, 1]$, we have

$$S(f)(0,\beta) = \int_{[0,1]} \sin\left(\frac{\beta}{x^2}\right) dx = \sqrt{\frac{\beta\pi}{2}} \left\{1 - 2C_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]\right\} + \sin(\beta),$$

where $C_\star(x) = \int_{[0,x]} \cos(\pi t^2/2)dt$ is the Fresnel C integral. For $f(x) = 1/(1 - x)^2$, $x \in [0, 1)$, the same formula is found.

By considering $f(x) = x(1 - x)$, $x \in [0, 1]$, we obtain

$$\begin{aligned} S(f)(0,\beta) &= \int_{[0,1]} \sin[\beta x(1-x)]dx \\ &= \sqrt{\frac{2\pi}{\beta}} \left\{C_\star\left[\sqrt{\frac{\beta}{2\pi}}\right] \sin\left(\frac{\beta}{4}\right) - S_\star\left[\sqrt{\frac{\beta}{2\pi}}\right] \cos\left(\frac{\beta}{4}\right)\right\}. \end{aligned}$$

For $f(x) = \arcsin(x)$, $x \in [0, 1]$, we get

$$S(f)(0,\beta) = \int_{[0,1]} \sin[\beta \arcsin(x)]dx = \frac{1}{1-\beta^2}\left[\sin\left(\frac{\beta\pi}{2}\right) - \beta\right].$$

The same result holds for $f(x) = \arcsin(1 - x)$, $x \in [0, 1]$.

For $f(x) = \arccos(x)$, $x \in [0, 1]$, we have

$$S(f)(0,\beta) = \int_{[0,1]} \sin[\beta \arccos(x)]dx = \frac{\beta}{1-\beta^2} \cos\left(\frac{\beta\pi}{2}\right).$$

For $f(x) = \arccos(1 - x)$, $x \in [0, 1)$, the same formula is found.

4.2. **Cases $n = 1$ and $\alpha = 1$.** Let us consider the cases $n = 1$ and $\alpha = 1$.

By selecting $f(x) = x$, $x \in [0, 1]$, we establish that

$$S(f)(1,\beta) = \int_{[0,1]} x \sin(\beta x)dx = \frac{1}{\beta^2}[\sin(\beta) - \beta\cos(\beta)].$$

For $f(x) = 1 - x$, $x \in [0, 1)$, the same formula is found.

By taking $f(x) = x^2$, $x \in [0, 1]$, the calculus of the S operator gives

$$S(f)(1,\beta) = \int_{[0,1]} x^2 \sin(\beta x^2)dx = \frac{1}{4}\left[\frac{\sqrt{2\pi}}{\beta^{3/2}} C_\star\left(\sqrt{\frac{2\beta}{\pi}}\right) - \frac{2\cos(\beta)}{\beta}\right].$$

This is also true for $f(x) = (1 - x)^2$, $x \in [0, 1]$.

For $f(x) = \sqrt{x}$, $x \in [0,1]$, we have

$$\begin{aligned} S(f)(1,\beta) &= \int_{[0,1]} \sqrt{x}\sin[\beta\sqrt{x}]dx \\ &= \frac{2}{\beta^3}\left[2\beta\sin(\beta) - (\beta^2-2)\cos(\beta) - 2\right]. \end{aligned}$$

We get the same result for $f(x) = \sqrt{1-x}$, $x \in [0,1]$.

For $f(x) = \exp(x)$, $x \in [0,1]$, the following S operator is obtained:

$$S(f)(1,\beta) = \int_{[0,1]} \exp(x)\sin[\beta\exp(x)]dx = \frac{1}{\beta}\{\cos(\beta) - \cos[\beta\exp(1)]\}.$$

The same result holds for $f(x) = \exp(1-x)$, $x \in [0,1]$.

By choosing $f(x) = -\log(x)$, $x \in (0,1]$, we get

$$S(f)(1,\beta) = \int_{[0,1]} [-\log(x)]\sin[-\beta\log(x)]dx = \frac{2\beta}{(\beta^2+1)^2}.$$

This is also true for $f(x) = -\log(1-x)$, $x \in [0,1)$.

For $f(x) = 1/x$, $x \in (0,1]$, we obtain

$$S(f)(1,\beta) = \int_{[0,1]} \frac{1}{x}\sin\left(\frac{\beta}{x}\right)dx = \frac{1}{2}[\pi - 2S_o(\beta)].$$

The same result holds for $f(x) = 1/(1-x)$, $x \in [0,1)$.

For $f(x) = 1/x^2$, $x \in (0,1]$, we have

$$S(f)(1,\beta) = \int_{[0,1]} \frac{1}{x^2}\sin\left(\frac{\beta}{x^2}\right)dx = \frac{1}{2}\sqrt{\frac{\pi}{2\beta}}\left\{1 - 2S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]\right\}.$$

This is also true for $f(x) = 1/(1-x)^2$, $x \in [0,1)$.

By considering $f(x) = 1/(1+x)$, $x \in [0,1]$, we obtain

$$S(f)(1,\beta) = \int_{[0,1]} \frac{1}{1+x}\sin\left(\frac{\beta}{1+x}\right)dx = S_o(\beta) - S_o\left(\frac{\beta}{2}\right).$$

This formula is also valid for $f(x) = 1/(2-x)$, $x \in [0,1]$.

For $f(x) = \arcsin(x)$, $x \in [0,1]$, the following S operator is calculated:

$$\begin{aligned} S(f)(1,\beta) &= \int_{[0,1]} \arcsin(x)\sin[\beta\arcsin(x)]dx \\ &= -\frac{1}{2(1-\beta^2)^2}\left[(\beta^2-1)\pi\sin\left(\frac{\beta\pi}{2}\right) + 4\beta\cos\left(\frac{\beta\pi}{2}\right)\right]. \end{aligned}$$

This is also true for $f(x) = \arcsin(1-x)$, $x \in [0,1]$.

For $f(x) = \arccos(x)$, $x \in [0,1]$, we have

$$\begin{aligned} S(f)(1,\beta) &= \int_{[0,1]} \arccos(x)\sin[\beta\arccos(x)]dx \\ &= \frac{1}{(1-\beta^2)^2}\left[(1+\beta^2)\sin\left(\frac{\beta\pi}{2}\right) - \frac{\beta}{2}(\beta^2-1)\pi\cos\left(\frac{\beta\pi}{2}\right) - 2\beta\right]. \end{aligned}$$

We get the same result for $f(x) = \arccos(1-x)$, $x \in [0,1]$.

4.3. **Cases $n = 1$ and varying α.** We now focus on the cases $n = 1$ and varying α. Also, due to the effect of α, most of the expressions of the S operator are in infinite series form, with reference to Proposition 2.7.

By selecting $f(x) = x^m$, $x \in [0, 1]$, with $m(\alpha + 1) > -1$, we obtain

$$S(f)(\alpha, \beta) = \int_{[0,1]} x^{m\alpha} \sin(\beta x^m) dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \beta^{2k+1} \frac{1}{m(\alpha + 2k + 1) + 1},$$

provided that convergence is achieved.

This formula is also valid for $f(x) = (1 - x)^m$, $x \in [0, 1]$.

By taking the more general function $f(x) = x^m(1 - x)^s$, $x \in [0, 1]$, with $m(\alpha+1) > -1$ and $s(\alpha+1) > -1$, we obtain

$$S(f)(\alpha, \beta) = \int_{[0,1]} x^{m\alpha}(1 - x)^{s\alpha} \sin[\beta x^m(1 - x)^s]dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \beta^{2k+1} B[m(\alpha + 2k + 1) + 1, s(\alpha + 2k + 1) + 1],$$

where $B(t, v) = \int_0^1 x^{t-1}(1 - x)^{v-1}dx$ is the standard beta function, provided that convergence is achieved.

For $f(x) = [-\log(x)]^m$, $x \in (0, 1]$, with $m(\alpha + 1) > -1$, the following S operator is computed:

$$S(f)(\alpha, \beta) = \int_{[0,1]} [-\log(x)]^{m\alpha} \sin\{\beta[-\log(x)]^m\}dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \beta^{2k+1}\Gamma[m(\alpha + 2k + 1) + 1],$$

where $\Gamma(t) = \int_{[0,\infty)} x^{t-1} \exp(-x)dx$ is the standard gamma integral function, provided that convergence is achieved. For $f(x) = [-\log(1 - x)]^m$, $x \in (0, 1]$, the same formula is found.

For $f(x) = [-x \log(x)]^m$, $x \in (0, 1]$, with $m(\alpha + 1) > -1$, we get

$$S(f)(\alpha, \beta) = \int_{[0,1]} [-x \log(x)]^{m\alpha} \sin\{\beta[-x \log(x)]^m\}dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \beta^{2k+1} \frac{1}{[m(\alpha + 2k + 1) + 1]^{m(\alpha+2k+1)+1}} \Gamma[m(\alpha + 2k + 1) + 1],$$

provided that convergence is achieved. For $f(x) = [-(1 - x) \log(1 - x)]^m$, $x \in (0, 1]$, the same formula is found.

By choosing $f(x) = \exp(mx)$, $x \in [0, 1]$ with $m \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, we find that

$$S(f)(\alpha, \beta) = \int_{[0,1]} \exp(m\alpha x) \sin[\beta \exp(mx)]dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \beta^{2k+1} \frac{1}{m(\alpha + 2k + 1)} \{\exp[m(\alpha + 2k + 1)] - 1\},$$

provided that convergence is achieved. This is also true for $f(x) = \exp[m(1 - x)]$, $x \in [0, 1]$.

4.4. **Cases $n = 2$ and $\alpha = 0$.** We now emphasize the cases $n = 2$ and $\alpha = 0$.

For $f(x_1, x_2) = x_1 + x_2$, $(x_1, x_2) \in [0, 1]^2$, the calculus of the S operator gives

$$S(f)(0, \beta) = \int_{[0,1]^2} \sin[\beta(x_1 + x_2)]dx_1dx_2 = \frac{2}{\beta^2} \sin(\beta)[1 - \cos(\beta)],$$

or, eventually, $S(f)(0,\beta) = (2/\beta)\,\mathrm{sinc}(\beta)[1-\cos(\beta)]$, where $\mathrm{sinc}(\beta) = \sin(\beta)/\beta$ is the sine cardinal, or $S(f)(0,\beta) = (4/\beta)\,\mathrm{sinc}(\beta)\{\sin[\beta/2]\}^2$ or $S(f)(0,\beta) = \beta\,\mathrm{sinc}(\beta)\{\mathrm{sinc}[\beta/2]\}^2$. For $f(x_1,x_2) = 1-x_1+x_2$, $f(x_1,x_2) = x_1+1-x_2$ and $f(x_1,x_2) = 2-x_1-x_2$, $(x_1,x_2) \in [0,1]^2$, the same formula is found.

By taking $f(x_1,x_2) = x_1+x_2$, for $x_2 \le x_1$, $(x_1,x_2) \in [0,1]^2$, and $f(x_1,x_2) = 0$ otherwise, the following S operator is obtained:

$$S(f)(0,\beta) = \int_{[0,1]}\int_{[0,x_1]} \sin[\beta(x_1+x_2)]dx_2dx_1 = \frac{1}{\beta^2}\sin(\beta)[1-\cos(\beta)],$$

or, eventually, $S(f)(0,\beta) = [\mathrm{sinc}(\beta)/\beta][1-\cos(\beta)]$.

This is also true for $f(x_1,x_2) = 1-x_1+x_2$, for $x_2 \le 1-x_1$, $(x_1,x_2) \in [0,1]^2$, and $f(x_1,x_2) = 0$ otherwise, $f(x_1,x_2) = 1+x_1-x_2$, for $1-x_2 \le x_1$, $(x_1,x_2) \in [0,1]^2$, and $f(x_1,x_2) = 0$ otherwise, and $f(x_1,x_2) = 2-x_1-x_2$, for $x_1 \le x_2$, $(x_1,x_2) \in [0,1]^2$, and $f(x_1,x_2) = 0$ otherwise.

For $f(x_1,x_2) = |x_1-x_2|$, $(x_1,x_2) \in [0,1]^2$, we get

$$S(f)(0,\beta) = \int_{[0,1]^2} \sin[\beta|x_1-x_2|]dx_1dx_2 = \frac{2}{\beta^2}[\beta-\sin(\beta)],$$

or, eventually, $S(f)(0,\beta) = (2/\beta)[1-\mathrm{sinc}(\beta)]$.

The same result holds for $f(x_1,x_2) = |1-x_1-x_2|$ and $f(x_1,x_2) = |x_1-1+x_2|$, $(x_1,x_2) \in [0,1]^2$.

By considering $f(x_1,x_2) = (x_1-x_2)^2$, $(x_1,x_2) \in [0,1]^2$, we have

$$S(f)(0,\beta) = \int_{[0,1]^2} \sin[\beta(x_1-x_2)^2]dx_1dx_2 = \frac{1}{\beta}\left[\sqrt{2\beta\pi}S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]-1+\cos(\beta)\right].$$

This formula is also valid for $f(x_1,x_2) = (1-x_1-x_2)^2$ and $f(x_1,x_2) = (x_1-1+x_2)^2$, $(x_1,x_2) \in [0,1]^2$.

For $f(x_1,x_2) = x_1^2+x_2^2$, $(x_1,x_2) \in [0,1]^2$, we have

$$S(f)(0,\beta) = \int_{[0,1]^2} \sin[\beta(x_1^2+x_2^2)]dx_1dx_2 = \frac{\pi}{\beta}C_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right].$$

For $f(x_1,x_2) = (1-x_1)^2+x_2^2$, $f(x_1,x_2) = x_1^2+(1-x_2)^2$ and $f(x_1,x_2) = (1-x_1)^2+(1-x_2)^2$, $(x_1,x_2) \in [0,1]^2$, the same formula is found.

By choosing $f(x_1,x_2) = \sqrt{x_1}+\sqrt{x_2}$, $(x_1,x_2) \in [0,1]^2$, we have

$$\begin{aligned} S(f)(0,\beta) &= \int_{[0,1]^2} \sin\left\{\beta[\sqrt{x_1}+\sqrt{x_2}]\right\}dx_1dx_2 \\ &= \frac{1}{\beta^4}\left[-8\sin(\beta)-4(\beta^2-1)\sin(2\beta)+8\beta\cos(\beta)-8\beta\cos(2\beta)\right]. \end{aligned}$$

This formula is also obtained for $f(x_1,x_2) = \sqrt{1-x_1}+\sqrt{x_2}$, $f(x_1,x_2) = \sqrt{x_1}+\sqrt{1-x_2}$ and $f(x_1,x_2) = \sqrt{1-x_1}+\sqrt{1-x_2}$, $(x_1,x_2) \in [0,1]^2$.

For $f(x_1,x_2) = x_1x_2$, $(x_1,x_2) \in [0,1]^2$, the following S operator is obtained:

$$S(f)(0,\beta) = \int_{[0,1]^2} \sin(\beta x_1x_2)dx_1dx_2 = \frac{1}{\beta}[\gamma+\log(\beta)-C_o(\beta)],$$

where γ denotes the Euler-Mascheroni constant. The same result holds for $f(x_1,x_2) = (1-x_1)x_2$, $f(x_1,x_2) = x_1(1-x_2)$ and $f(x_1,x_2) = (1-x_1)(1-x_2)$, $(x_1,x_2) \in [0,1]^2$.

For $f(x_1,x_2) = x_1/x_2$, $(x_1,x_2) \in (0,1]^2$, we have

$$S(f)(0,\beta) = \int_{[0,1]^2} \sin\left(\beta\frac{x_1}{x_2}\right)dx_1dx_2 = -\frac{1}{2\beta}[\beta^2C_o(\beta)-1-\beta\sin(\beta)+\cos(\beta)].$$

This is also true for $f(x_1,x_2) = (1-x_1)/x_2$, $f(x_1,x_2) = x_1/(1-x_2)$ and $f(x_1,x_2) = (1-x_1)/(1-x_2)$, $(x_1,x_2) \in (0,1)^2$.

By selecting $f(x_1, x_2) = -\log(x_1x_2)$, $(x_1, x_2) \in (0, 1]^2$, we find that

$$S(f)(0,\beta) = \int_{[0,1]^2} \sin\{\beta[-\log(x_1x_2)]\}\, dx_1dx_2 = \frac{2\beta}{(1+\beta^2)^2}.$$

For $f(x_1, x_2) = -\log[(1-x_1)x_2]$, $f(x_1, x_2) = -\log[x_1(1-x_2)]$ and $f(x_1, x_2) = -\log[(1-x_1)(1-x_2)]$, $(x_1, x_2) \in (0, 1)^2$, the same formula is found.

4.5. **Some more general cases.** Some more general cases than the previous ones are now examined.

By taking $f(x_1, x_2, x_3) = -\log(x_1x_2x_3)$, $(x_1, x_2, x_3) \in (0, 1]^3$, we obtain

$$S(f)(0,\beta) = \int_{[0,1]^3} \sin\{\beta[-\log(x_1x_2x_3)]\}\, dx_1dx_2dx_3 = \frac{\beta(3-\beta^2)}{(1+\beta^2)^3}.$$

We find the same result for $f(x_1, x_2, x_3) = -\log[(1-x_1)x_2x_3]$, $f(x_1, x_2, x_3) = -\log[x_1(1-x_2)x_3]$, $f(x_1, x_2, x_3) = -\log[x_1x_2(1-x_3)]$, $f(x_1, x_2, x_3) = -\log[(1-x_1)(1-x_2)x_3]$, $f(x_1, x_2, x_3) = -\log[x_1(1-x_2)(1-x_3)]$, $f(x_1, x_2, x_3) = -\log[(1-x_1)x_2(1-x_3)]$ and $f(x_1, x_2, x_3) = -\log[(1-x_1)(1-x_2)(1-x_3)]$, $(x_1, x_2, x_3) \in (0, 1)^3$.

For $f(\mathbf{x}) = \prod_{i=1}^n x_i^{m_i}$, $\mathbf{x} \in \mathcal{X}$, with $m_i(\alpha+1) > -1$ for any $i = 1, \ldots, n$, we have

$$\begin{aligned} S(f)(\alpha,\beta) &= \int_{\mathcal{X}} \left[\prod_{i=1}^n x_i^{m_i}\right]^{\alpha} \sin\left[\beta \prod_{i=1}^n x_i^{m_i}\right] d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\beta^{2k+1} \int_{\mathcal{X}} \left[\prod_{i=1}^n x_i^{m_i}\right]^{\alpha+2k+1} d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\beta^{2k+1} \prod_{i=1}^n \int_{[0,1]} x_i^{m_i(\alpha+2k+1)} dx_i \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\beta^{2k+1} \prod_{i=1}^n \frac{1}{m_i(\alpha+2k+1)+1}, \end{aligned}$$

provided that convergence is achieved.

For $f(\mathbf{x}_*) = \prod_{i=1}^n (x_i^*)^{m_i}$ with $x_i^* \in \{x_i, 1-x_i\}$ for each $i = 1, \ldots, n$, $\mathbf{x} \in \mathcal{X}$, the same formula holds.

By considering $f(\mathbf{x}) = \prod_{i=1}^n [-\log(x_i)]^{m_i}$, $\mathbf{x} \in \mathcal{X}$, with $m_i(\alpha+1) > -1$ for any $i = 1, \ldots, n$, we get

$$\begin{aligned} S(f)(\alpha,\beta) &= \int_{\mathcal{X}} \left\{\prod_{i=1}^n [-\log(x_i)]^{m_i}\right\}^{\alpha} \sin\left\{\beta \prod_{i=1}^n [-\log(x_i)]^{m_i}\right\} d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\beta^{2k+1} \int_{\mathcal{X}} \left[\prod_{i=1}^n [-\log(x_i)]^{m_i}\right]^{\alpha+2k+1} d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\beta^{2k+1} \prod_{i=1}^n \int_{[0,1]} [-\log(x_i)]^{m_i(\alpha+2k+1)} dx_i \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\beta^{2k+1} \prod_{i=1}^n \Gamma[m_i(\alpha+2k+1)+1], \end{aligned}$$

provided that convergence is achieved. We find the same result for $f(\mathbf{x}) = \prod_{i=1}^n [-\log(x_i^*)]^{m_i}$ with $x_i^* \in \{x_i, 1-x_i\}$ for each $i = 1, \ldots, n$, $\mathbf{x} \in \mathcal{X}$.

For $f(\mathbf{x}) = \prod_{i=1}^{n}[-x_i \log(x_i)]^{m_i}$, $\mathbf{x} \in \mathcal{X}$, with $m_i(\alpha+1) > -1$ for any $i = 1, \ldots, n$, we have

$$\begin{aligned} S(f)(\alpha,\beta) &= \int_{\mathcal{X}} \left\{ \prod_{i=1}^{n}[-x_i \log(x_i)]^{m_i} \right\}^{\alpha} \sin\left\{ \beta \prod_{i=1}^{n}[-x_i \log(x_i)]^{m_i} \right\} d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\beta^{2k+1} \int_{\mathcal{X}} \left[\prod_{i=1}^{n}[-x_i \log(x_i)]^{m_i} \right]^{\alpha+2k+1} d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\beta^{2k+1} \prod_{i=1}^{n} \int_{[0,1]} [-x_i \log(x_i)]^{m_i(\alpha+2k+1)} dx_i \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\beta^{2k+1} \prod_{i=1}^{n} \frac{1}{[m_i(\alpha+2k+1)+1]^{m_i(\alpha+2k+1)+1}} \Gamma[m_i(\alpha+2k+1)+1], \end{aligned}$$

provided that convergence is achieved.

The same formula holds for $f(\mathbf{x}) = \prod_{i=1}^{n}[-x_i^* \log(x_i^*)]^{m_i}$ with $x_i^* \in \{x_i, 1-x_i\}$ for each $i = 1, \ldots, n$, $\mathbf{x} \in \mathcal{X}$.

By considering $f(\mathbf{x}) = \exp\left(\sum_{i=1}^{n} m_i x_i\right) = \prod_{i=1}^{n} \exp(m_i x_i)$, $\mathbf{x} \in \mathcal{X}$, with $m_i \in \mathbb{R}$ for any $i = 1, \ldots, n$, and $\alpha \in \mathbb{R}$, the following S operator is obtained:

$$\begin{aligned} S(f)(\alpha,\beta) &= \int_{\mathcal{X}} \left[\prod_{i=1}^{n} \exp(m_i x_i) \right]^{\alpha} \sin\left[\beta \prod_{i=1}^{n} \exp(m_i x_i) \right] d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\beta^{2k+1} \int_{\mathcal{X}} \left[\prod_{i=1}^{n} \exp(m_i x_i) \right]^{\alpha+2k+1} d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\beta^{2k+1} \prod_{i=1}^{n} \int_{[0,1]} \exp[x_i m_i(\alpha+2k+1)] dx_i \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\beta^{2k+1} \prod_{i=1}^{n} \frac{1}{m_i(\alpha+2k+1)} \left\{ \exp[m_i(\alpha+2k+1)] - 1 \right\}, \end{aligned}$$

provided that convergence is achieved.

We find the same result for $f(\mathbf{x}) = \exp\left(\sum_{i=1}^{n} m_i x_i^*\right)$ with $x_i^* \in \{x_i, 1-x_i\}$ for each $i = 1, \ldots, n$, $\mathbf{x} \in \mathcal{X}$.

5. Some explicit formulas for the C operator

By taking inspiration from the above section, we now examine some explicit formulas for the C operator. For the sake of simplicity, we always suppose that $\beta > 0$.

5.1. Cases $n = 1$ and $\alpha = 0$.

First, let us consider the cases $n = 1$ and $\alpha = 0$.

By taking $f(x) = x$, $x \in [0,1]$, the following C operator is established:

$$C(f)(0,\beta) = \int_{[0,1]} \cos(\beta x) dx = \operatorname{sinc}(\beta).$$

The same result holds for $f(x) = 1 - x$, $x \in [0,1]$.

For $f(x) = x^2$, $x \in [0,1]$, we get

$$C(f)(0,\beta) = \int_{[0,1]} \cos(\beta x^2) dx = \sqrt{\frac{\pi}{2\beta}} C_\star \left[\sqrt{\frac{2\beta}{\pi}} \right].$$

For $f(x) = (1-x)^2$, $x \in [0,1]$, the same formula is found.

For $f(x) = \sqrt{x}$, $x \in [0,1]$, we have

$$C(f)(0,\beta) = \int_{[0,1]} \cos[\beta\sqrt{x}]dx = \frac{2}{\beta^2}[\beta\sin(\beta) + \cos(\beta) - 1].$$

This is also true for $f(x) = \sqrt{1-x}$, $x \in [0,1]$.

By selecting $f(x) = \exp(x)$, $x \in [0,1]$, we get

$$C(f)(0,\beta) = \int_{[0,1]} \cos[\beta\exp(x)]dx = C_o[\beta\exp(1)] - C_o(\beta).$$

The same result holds for $f(x) = \exp(1-x)$, $x \in [0,1]$.

For $f(x) = -\log(x)$, $x \in (0,1]$, we have

$$C(f)(0,\beta) = \int_{[0,1]} \cos[-\beta\log(x)]dx = \frac{1}{\beta^2+1}.$$

For $f(x) = -\log(1-x)$, $x \in [0,1)$, the same formula is found.

By choosing $f(x) = 1/x$, $x \in (0,1]$, the following C operator is obtained:

$$C(f)(0,\beta) = \int_{[0,1]} \cos\left(\frac{\beta}{x}\right)dx = \beta\left[S_o(\beta) - \frac{\pi}{2}\right] + \cos(\beta).$$

The same result holds for $f(x) = 1/(1-x)$, $x \in [0,1)$.

For $f(x) = 1/(1+x)$, $x \in [0,1]$, we get

$$\begin{aligned} C(f)(0,\beta) &= \int_{[0,1]} \cos\left(\frac{\beta}{1+x}\right)dx \\ &= \beta\left[S_o\left(\frac{\beta}{2}\right) - S_o(\beta)\right] + 2\cos\left(\frac{\beta}{2}\right) - \cos(\beta). \end{aligned}$$

This is also true for $f(x) = 1/(2-x)$, $x \in [0,1]$.

By selecting $f(x) = 1/x^2$, $x \in (0,1]$, we have

$$C(f)(0,\beta) = \int_{[0,1]} \cos\left(\frac{\beta}{x^2}\right)dx = \sqrt{\frac{\beta\pi}{2}}\left\{2S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right] - 1\right\} + \cos(\beta).$$

For $f(x) = 1/(1-x)^2$, $x \in [0,1)$, the same formula is found.

For $f(x) = x(1-x)$, $x \in [0,1]$, we have

$$\begin{aligned} C(f)(0,\beta) &= \int_{[0,1]} \cos[\beta x(1-x)]dx \\ &= \sqrt{\frac{2\pi}{\beta}}\left\{C_\star\left[\sqrt{\frac{\beta}{2\pi}}\right]\cos\left(\frac{\beta}{4}\right) + S_\star\left[\sqrt{\frac{\beta}{2\pi}}\right]\sin\left(\frac{\beta}{4}\right)\right\}. \end{aligned}$$

For $f(x) = \arcsin(x)$, $x \in [0,1]$, the calculus of the C operator gives

$$C(f)(0,\beta) = \int_{[0,1]} \cos[\beta\arcsin(x)]dx = \frac{1}{1-\beta^2}\cos\left(\frac{\beta\pi}{2}\right).$$

The same result holds for $f(x) = \arcsin(1-x)$, $x \in [0,1]$.

For $f(x) = \arccos(x)$, $x \in [0,1]$, we have

$$C(f)(0,\beta) = \int_{[0,1]} \cos[\beta\arccos(x)]dx = \frac{1}{1-\beta^2}\left[1 - \beta\sin\left(\frac{\beta\pi}{2}\right)\right].$$

For $f(x) = \arccos(1-x)$, $x \in [0,1)$, the same formula is found.

5.2. **Cases $n = 1$ and $\alpha = 1$.** Let us consider the cases $n = 1$ and $\alpha = 1$.

For $f(x) = x$, $x \in [0,1]$, we obtain

$$C(f)(1,\beta) = \int_{[0,1]} x\cos(\beta x)dx = \frac{1}{\beta^2}[\beta\sin(\beta) + \cos(\beta) - 1].$$

For $f(x) = 1-x$, $x \in [0,1)$, the same formula is found.

By choosing $f(x) = x^2$, $x \in [0,1]$, we get

$$C(f)(1,\beta) = \int_{[0,1]} x^2\cos(\beta x^2)dx = \frac{1}{4}\left\{2\,\mathrm{sinc}(\beta) - \frac{\sqrt{2\pi}}{\beta^{3/2}}S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]\right\}.$$

This is also true for $f(x) = (1-x)^2$, $x \in [0,1]$.

For $f(x) = \sqrt{x}$, $x \in [0,1]$, we find that

$$C(f)(1,\beta) = \int_{[0,1]} \sqrt{x}\cos[\beta\sqrt{x}]dx = \frac{2}{\beta^3}\left[2\beta\cos(\beta) + (\beta^2-2)\sin(\beta)\right].$$

We get the same result for $f(x) = \sqrt{1-x}$, $x \in [0,1]$.

For $f(x) = \exp(x)$, $x \in [0,1]$, the calculus of the C operator gives

$$C(f)(1,\beta) = \int_{[0,1]} \exp(x)\cos[\beta\exp(x)]dx = \frac{1}{\beta}\{\sin[\beta\exp(1)] - \sin(\beta)\}.$$

The same result holds for $f(x) = \exp(1-x)$, $x \in [0,1]$.

By considering $f(x) = -\log(x)$, $x \in (0,1]$, we have

$$C(f)(1,\beta) = \int_{[0,1]} [-\log(x)]\cos[-\beta\log(x)]dx = \frac{1-\beta^2}{(1+\beta^2)^2}.$$

This is also true for $f(x) = -\log(1-x)$, $x \in [0,1)$.

For $f(x) = 1/x$, $x \in (0,1]$, we obtain

$$C(f)(1,\beta) = \int_{[0,1]} \frac{1}{x}\cos\left(\frac{\beta}{x}\right)dx = -C_o(\beta).$$

The same result holds for $f(x) = 1/(1-x)$, $x \in [0,1)$.

By selecting $f(x) = 1/x^2$, $x \in (0,1]$, we have

$$C(f)(1,\beta) = \int_{[0,1]} \frac{1}{x^2}\cos\left(\frac{\beta}{x^2}\right)dx = \frac{1}{2}\sqrt{\frac{\pi}{2\beta}}\left\{1 - 2C_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]\right\}.$$

This is also true for $f(x) = 1/(1-x)^2$, $x \in [0,1)$.

Based on $f(x) = 1/(1+x)$, $x \in [0,1]$, the calculus of the C operator gives

$$C(f)(1,\beta) = \int_{[0,1]} \frac{1}{1+x}\cos\left(\frac{\beta}{1+x}\right)dx = C_o(\beta) - C_o\left(\frac{\beta}{2}\right).$$

This formula is also valid for $f(x) = 1/(2-x)$, $x \in [0,1]$.

For $f(x) = \arccos(x)$, $x \in [0,1]$, we have

$$\begin{aligned}C(f)(1,\beta) &= \int_{[0,1]} \arccos(x)\cos[\beta\arccos(x)]dx \\ &= \frac{1}{(1-\beta^2)^2}\left[(1+\beta^2)\cos\left(\frac{\beta\pi}{2}\right) + \frac{\beta}{2}(\beta^2-1)\pi\sin\left(\frac{\beta\pi}{2}\right)\right].\end{aligned}$$

We get the same result for $f(x) = \arccos(1-x)$, $x \in [0,1]$.

5.3. **Cases $n = 1$ and varying α.** We now focus on the cases $n = 1$ and varying α. Also, due to the effect of α, most of the expressions of the C operator are in infinite series form, with reference to Proposition 2.7.

By taking $f(x) = x^m$, $x \in [0, 1]$, with $m\alpha > -1$, we obtain

$$C(f)(\alpha,\beta) = \int_{[0,1]} x^{m\alpha} \cos(\beta x^m) dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \frac{1}{m(\alpha + 2k) + 1},$$

provided that convergence is achieved. This formula is also valid for $f(x) = (1 - x)^m$, $x \in [0, 1]$.

By considering the more general function $f(x) = x^m(1 - x)^s$, $x \in [0, 1]$, with $m(\alpha + 1) > -1$ and $s(\alpha + 1) > -1$, we obtain

$$\begin{aligned} C(f)(\alpha,\beta) &= \int_{[0,1]} x^{m\alpha}(1 - x)^{s\alpha} \cos[\beta x^m(1 - x)^s]dx \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} B[m(\alpha + 2k) + 1, s(\alpha + 2k) + 1], \end{aligned}$$

provided that convergence is achieved.

For $f(x) = [-\log(x)]^m$, $x \in (0, 1]$, with $m\alpha > -1$, we have

$$\begin{aligned} C(f)(\alpha,\beta) &= \int_{[0,1]} [-\log(x)]^{m\alpha} \cos\{\beta[-\log(x)]^m\}dx \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \Gamma[m(\alpha + 2k) + 1], \end{aligned}$$

provided that convergence is achieved. For $f(x) = [-\log(1 - x)]^m$, $x \in (0, 1]$, the same formula is found.

By choosing $f(x) = [-x \log(x)]^m$, $x \in (0, 1]$, with $m\alpha > -1$, we establish that

$$\begin{aligned} C(f)(\alpha,\beta) &= \int_{[0,1]} [-x\log(x)]^{m\alpha} \cos\{\beta[-x\log(x)]^m\}dx \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \frac{1}{[m(\alpha + 2k) + 1]^{m(\alpha+2k)+1}} \Gamma[m(\alpha + 2k) + 1], \end{aligned}$$

provided that convergence is achieved. For $f(x) = [-(1 - x) \log(1 - x)]^m$, $x \in (0, 1]$, the same formula is found.

For $f(x) = \exp(mx)$, $x \in [0, 1]$ with $m \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} C(f)(\alpha,\beta) &= \int_{[0,1]} \exp(m\alpha x) \cos[\beta \exp(mx)]dx \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \frac{1}{m(\alpha + 2k)} \{\exp[m(\alpha + 2k)] - 1\}, \end{aligned}$$

provided that convergence is achieved. This is also true for $f(x) = \exp[m(1 - x)]$, $x \in [0, 1]$.

5.4. **Cases $n = 2$ and $\alpha = 0$.** We now emphasize the cases $n = 2$ and $\alpha = 0$.

By selecting $f(x_1, x_2) = x_1 + x_2$, $(x_1, x_2) \in [0, 1]^2$, the calculus of the C operator gives

$$C(f)(0,\beta) = \int_{[0,1]^2} \cos[\beta(x_1 + x_2)]dx_1dx_2 = \frac{4}{\beta^2} \cos(\beta) \left[\sin\left(\frac{\beta}{2}\right)\right]^2,$$

or, eventually, $C(f)(0, \beta) = \cos(\beta)[\text{sinc}(\beta/2)]^2$.

For $f(x_1, x_2) = 1 - x_1 + x_2$, $f(x_1, x_2) = x_1 + 1 - x_2$ and $f(x_1, x_2) = 2 - x_1 - x_2$, $(x_1, x_2) \in [0, 1]^2$, the same formula is found.

Let us now consider the case $f(x_1, x_2) = x_1 + x_2$, for $x_2 \leq x_1$, $(x_1, x_2) \in [0,1]^2$, and $f(x_1, x_2) = 0$ otherwise. In this case, an important detail must be mentioned. Since $\cos(0) = 1$, if the support of f is not entirely $\mathcal{X}$, say Ξ, then we have

$$C(f)(0, \beta) = \int_{\Xi} \cos[\beta f(\mathbf{x})] d\mathbf{x} + \int_{\mathcal{X} \setminus \Xi} d\mathbf{x}.$$

Based on this remark, we have

$$\begin{aligned} C(f)(0, \beta) &= \int_{[0,1]} \int_{[0,x_1]} \cos[\beta(x_1 + x_2)] dx_2 dx_1 + \int_{[0,1]} \int_{[x_1,1]} dx_2 dx_1 \\ &= \frac{1}{\beta^2} \left\{ [\sin(\beta)]^2 + \cos(\beta) - 1 \right\} + \frac{1}{2}. \end{aligned}$$

This is also true for $f(x_1, x_2) = 1 - x_1 + x_2$, for $x_2 \leq 1 - x_1$, $(x_1, x_2) \in [0,1]^2$, and $f(x_1, x_2) = 0$ otherwise, $f(x_1, x_2) = 1 + x_1 - x_2$, for $1 - x_2 \leq x_1$, $(x_1, x_2) \in [0,1]^2$, and $f(x_1, x_2) = 0$ otherwise, and $f(x_1, x_2) = 2 - x_1 - x_2$, for $x_1 \leq x_2$, $(x_1, x_2) \in [0,1]^2$, and $f(x_1, x_2) = 0$ otherwise.

Based on $f(x_1, x_2) = |x_1 - x_2|$, $(x_1, x_2) \in [0,1]^2$, we get

$$C(f)(0, \beta) = \int_{[0,1]^2} \cos[\beta |x_1 - x_2|] dx_1 dx_2 = \frac{2}{\beta^2} [1 - \cos(\beta)].$$

The same result holds for $f(x_1, x_2) = |1 - x_1 - x_2|$ and $f(x_1, x_2) = |x_1 - 1 + x_2|$, $(x_1, x_2) \in [0,1]^2$.

For $f(x_1, x_2) = (x_1 - x_2)^2$, $(x_1, x_2) \in [0,1]^2$, we have

$$C(f)(0, \beta) = \int_{[0,1]^2} \cos[\beta(x_1 - x_2)^2] dx_1 dx_2 = \frac{1}{\beta} \left\{ \sqrt{2\beta\pi} C_\star \left[\sqrt{\frac{2\beta}{\pi}} \right] - \sin(\beta) \right\}.$$

This formula is also valid for $f(x_1, x_2) = (1 - x_1 - x_2)^2$ and $f(x_1, x_2) = (x_1 - 1 + x_2)^2$, $(x_1, x_2) \in [0,1]^2$.

By considering $f(x_1, x_2) = x_1^2 + x_2^2$, $(x_1, x_2) \in [0,1]^2$, we obtain

$$\begin{aligned} C(f)(0, \beta) &= \int_{[0,1]^2} \cos[\beta(x_1^2 + x_2^2)] dx_1 dx_2 \\ &= \frac{\pi}{2\beta} \left[\left\{ C_\star \left[\sqrt{\frac{2\beta}{\pi}} \right] \right\}^2 - \left\{ S_\star \left[\sqrt{\frac{2\beta}{\pi}} \right] \right\}^2 \right]. \end{aligned}$$

For $f(x_1, x_2) = (1 - x_1)^2 + x_2^2$, $f(x_1, x_2) = x_1^2 + (1 - x_2)^2$ and $f(x_1, x_2) = (1 - x_1)^2 + (1 - x_2)^2$, $(x_1, x_2) \in [0,1]^2$, the same formula is found.

For $f(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$, $(x_1, x_2) \in [0,1]^2$, we have

$$\begin{aligned} C(f)(0, \beta) &= \int_{[0,1]^2} \cos\left\{ \beta[\sqrt{x_1} + \sqrt{x_2}] \right\} dx_1 dx_2 \\ &= \frac{4}{\beta^4} \left\{ 2\beta[\sin(2\beta) - \sin(\beta)] - 2[1 - \cos(\beta)] \cos(\beta) - \beta^2 \cos(2\beta) \right\}. \end{aligned}$$

This formula is also obtained for $f(x_1, x_2) = \sqrt{1 - x_1} + \sqrt{x_2}$, $f(x_1, x_2) = \sqrt{x_1} + \sqrt{1 - x_2}$ and $f(x_1, x_2) = \sqrt{1 - x_1} + \sqrt{1 - x_2}$, $(x_1, x_2) \in [0,1]^2$.

For $f(x_1, x_2) = x_1 x_2$, $(x_1, x_2) \in [0,1]^2$, the calculus of the C operator gives

$$C(f)(0, \beta) = \int_{[0,1]^2} \cos(\beta x_1 x_2) dx_1 dx_2 = \frac{1}{\beta} S_o(\beta).$$

The same result holds for $f(x_1, x_2) = (1 - x_1)x_2$, $f(x_1, x_2) = x_1(1 - x_2)$ and $f(x_1, x_2) = (1 - x_1)(1 - x_2)$, $(x_1, x_2) \in [0,1]^2$.

By choosing $f(x_1, x_2) = x_1/x_2$, $(x_1, x_2) \in (0, 1]^2$, we establish that

$$\begin{aligned} C(f)(0, \beta) &= \int_{[0,1]^2} \cos\left(\beta \frac{x_1}{x_2}\right) dx_1 dx_2 \\ &= \frac{1}{2}\left[\beta S_o(\beta) - \frac{\beta}{2}\pi + \operatorname{sinc}(\beta) + \cos(\beta)\right]. \end{aligned}$$

This is also true for $f(x_1, x_2) = (1 - x_1)/x_2$, $f(x_1, x_2) = x_1/(1 - x_2)$ and $f(x_1, x_2) = (1 - x_1)/(1 - x_2)$, $(x_1, x_2) \in (0, 1)^2$.

For $f(x_1, x_2) = -\log(x_1 x_2)$, $(x_1, x_2) \in (0, 1]^2$, we have

$$C(f)(0, \beta) = \int_{[0,1]^2} \cos\{\beta[-\log(x_1 x_2)]\} dx_1 dx_2 = \frac{1 - \beta^2}{(1 + \beta^2)^2}.$$

For $f(x_1, x_2) = -\log[(1 - x_1)x_2]$, $f(x_1, x_2) = -\log[x_1(1 - x_2)]$ and $f(x_1, x_2) = -\log[(1 - x_1)(1 - x_2)]$, $(x_1, x_2) \in (0, 1)^2$, the same formula is found.

5.5. **Some more general cases.** Some more general cases that the previous ones are now examined.

By taking $f(x_1, x_2, x_3) = -\log(x_1 x_2 x_3)$, $(x_1, x_2, x_3) \in (0, 1]^3$, we have

$$C(f)(0, \beta) = \int_{[0,1]^3} \cos\{\beta[-\log(x_1 x_2 x_3)]\} dx_1 dx_2 dx_3 = \frac{1 - 3\beta^2}{(1 + \beta^2)^3}.$$

We find the same result for $f(x_1, x_2, x_3) = -\log[(1 - x_1)x_2 x_3]$, $f(x_1, x_2, x_3) = -\log[x_1(1 - x_2)x_3]$, $f(x_1, x_2, x_3) = -\log[x_1 x_2(1 - x_3)]$, $f(x_1, x_2, x_3) = -\log[(1 - x_1)(1 - x_2)x_3]$, $f(x_1, x_2, x_3) = -\log[x_1(1 - x_2)(1 - x_3)]$, $f(x_1, x_2, x_3) = -\log[(1 - x_1)x_2(1 - x_3)]$ and $f(x_1, x_2, x_3) = -\log[(1 - x_1)(1 - x_2)(1 - x_3)]$, $(x_1, x_2, x_3) \in (0, 1)^3$.

For $f(\mathbf{x}) = \prod_{i=1}^n x_i^{m_i}$, $\mathbf{x} \in \mathcal{X}$, with $m_i\alpha > -1$ for any $i = 1, \ldots, n$, we have

$$\begin{aligned} C(f)(\alpha, \beta) &= \int_{\mathcal{X}} \left[\prod_{i=1}^n x_i^{m_i}\right]^\alpha \cos\left[\beta \prod_{i=1}^n x_i^{m_i}\right] d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\beta^{2k} \int_{\mathcal{X}} \left[\prod_{i=1}^n x_i^{m_i}\right]^{\alpha+2k} d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\beta^{2k} \prod_{i=1}^n \int_{[0,1]} x_i^{m_i(\alpha+2k)} dx_i \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\beta^{2k} \prod_{i=1}^n \frac{1}{m_i(\alpha + 2k) + 1}, \end{aligned}$$

provided that convergence is achieved. By selecting $f(\mathbf{x}_*) = \prod_{i=1}^n (x_i^*)^{m_i}$ with $x_i^* \in \{x_i, 1 - x_i\}$ for each $i = 1, \ldots, n$, $\mathbf{x} \in \mathcal{X}$, the same formula holds.

By choosing $f(\mathbf{x}) = \prod_{i=1}^{n} [-\log(x_i)]^{m_i}$, $\mathbf{x} \in \mathcal{X}$, with $m_i\alpha > -1$ for any $i = 1, \ldots, n$, we have

$$\begin{aligned} C(f)(\alpha,\beta) &= \int_{\mathcal{X}} \left\{ \prod_{i=1}^{n} [-\log(x_i)]^{m_i} \right\}^{\alpha} \cos\left\{ \beta \prod_{i=1}^{n} [-\log(x_i)]^{m_i} \right\} d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \beta^{2k} \int_{\mathcal{X}} \left[\prod_{i=1}^{n} [-\log(x_i)]^{m_i} \right]^{\alpha+2k} d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \prod_{i=1}^{n} \int_{[0,1]} [-\log(x_i)]^{m_i(\alpha+2k)} dx_i \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \prod_{i=1}^{n} \Gamma[m_i(\alpha+2k)+1], \end{aligned}$$

provided that convergence is achieved.

We find the same result for $f(\mathbf{x}) = \prod_{i=1}^{n} [-\log(x_i^*)]^{m_i}$ with $x_i^* \in \{x_i, 1-x_i\}$ for each $i = 1, \ldots, n$, $\mathbf{x} \in \mathcal{X}$.

For $f(\mathbf{x}) = \prod_{i=1}^{n} [-x_i \log(x_i)]^{m_i}$, $\mathbf{x} \in \mathcal{X}$, with $m_i\alpha > -1$ for any $i = 1, \ldots, n$, the calculus of the C operator gives

$$\begin{aligned} C(f)(\alpha,\beta) &= \int_{\mathcal{X}} \left\{ \prod_{i=1}^{n} [-x_i \log(x_i)]^{m_i} \right\}^{\alpha} \cos\left\{ \beta \prod_{i=1}^{n} [-x_i \log(x_i)]^{m_i} \right\} d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \int_{\mathcal{X}} \left[\prod_{i=1}^{n} [-x_i \log(x_i)]^{m_i} \right]^{\alpha+2k} d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \prod_{i=1}^{n} \int_{[0,1]} [-x_i \log(x_i)]^{m_i(\alpha+2k)} dx_i \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \prod_{i=1}^{n} \frac{1}{[m_i(\alpha+2k)+1]^{m_i(\alpha+2k)+1}} \Gamma[m_i(\alpha+2k)+1], \end{aligned}$$

provided that convergence is achieved.

The same formula holds for $f(\mathbf{x}) = \prod_{i=1}^{n} [-x_i^* \log(x_i^*)]^{m_i}$ with $x_i^* \in \{x_i, 1-x_i\}$ for each $i = 1, \ldots, n$, $\mathbf{x} \in \mathcal{X}$.

For $f(\mathbf{x}) = \exp\left(\sum_{i=1}^{n} m_i x_i\right) = \prod_{i=1}^{n} \exp(m_i x_i)$, $\mathbf{x} \in \mathcal{X}$, with $m_i \in \mathbb{R}$ for any $i = 1, \ldots, n$, and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} C(f)(\alpha,\beta) &= \int_{\mathcal{X}} \left[\prod_{i=1}^{n} \exp(m_i x_i) \right]^{\alpha} \cos\left[\beta \prod_{i=1}^{n} \exp(m_i x_i) \right] d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \int_{\mathcal{X}} \left[\prod_{i=1}^{n} \exp(m_i x_i) \right]^{\alpha+2k} d\mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \prod_{i=1}^{n} \int_{[0,1]} \exp[x_i m_i(\alpha+2k)] dx_i \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \prod_{i=1}^{n} \frac{1}{m_i(\alpha+2k)} \left\{ \exp[m_i(\alpha+2k)] - 1 \right\}, \end{aligned}$$

provided that convergence is achieved.

We find the same result for $f(\mathbf{x}) = \exp\left(\sum_{i=1}^{n} m_i x_i^*\right)$ with $x_i^* \in \{x_i, 1-x_i\}$ for each $i = 1, \ldots, n$, $\mathbf{x} \in \mathcal{X}$.

Some expressions in this section will be at the center of the proof of the new trigonometric inequalities examined in the next section.

6. Applications to inequalities

Some inequalities derived from the operators S and C and their characteristics are now presented. For reasons of simplicity and compliance with the format of standard trigonometric inequalities in the literature, we consider $\beta \in [0, \pi/2]$ or $\beta \in [0, 1]$; the intervals may be possibly refined if necessary for most of the inequalities presented. From a technical point of view, we exclusively use the results of Proposition 3.4 for the left inequality and those of Proposition 3.7 for the right inequality. Of course, other previously proven results can be used; this is a subjective choice motivated by the generated inequalities, which deserve to be communicated thanks to their apparent sharpness. To provide a clear visual check, figures illustrating these inequalities are offered.

6.1. **Inequalities derived from the S operator.** The result below examines inequalities centered around the function $1 - \cos(\beta)$, with maximum details in the proof as a first example of application of the S operator.

Proposition 6.1. *For any $\beta \in [0, \pi/2]$, the following inequalities hold:*

$$\frac{\beta^2}{\pi} \leq 1 - \cos(\beta) \leq \beta \sin\left(\frac{\beta}{2}\right).$$

Proof. We consider the function $f(x) = x$, $x \in [0, 1]$. Then the S operator of f at $\alpha = 0$ is given as

$$S(f)(0, \beta) = \int_{[0,1]} \sin(\beta x) dx = \frac{1}{\beta}[1 - \cos(\beta)].$$

We are now able to apply most of the theory to the S operator.

First, since $\beta \in [0, \pi/2]$, for any $x \in [0, 1]$, we have $\beta f(x) = \beta x \in [0, \pi/2]$. Therefore, by applying item 2 of Proposition 3.7 with $n = 1$ and $\alpha = 0$, we get

$$\begin{aligned} S(f)(0, \beta) &\leq \left\{\int_{[0,1]} [f(x)]^0 dx\right\} \sin\left\{\beta \frac{1}{\int_{[0,1]} [f(x)]^0 dx} \int_{[0,1]} [f(x)]^{0+1} dx\right\} \\ &= \sin\left[\beta \int_{[0,1]} f(x) dx\right] = \sin\left[\beta \int_{[0,1]} x dx\right) = \sin\left(\frac{\beta}{2}\right). \end{aligned}$$

We thus obtain

$$1 - \cos(\beta) \leq \beta \sin\left(\frac{\beta}{2}\right).$$

On the other hand, by using item 1 of Proposition 3.4 with $n = 1$ and $\alpha = 0$, we have

$$\begin{aligned} S(f)(0, \beta) &\geq \frac{2\beta}{\pi} \int_{[0,1]} [f(x)]^{0+1} dx = \frac{2\beta}{\pi} \int_{[0,1]} f(x) dx \\ &= \frac{2\beta}{\pi} \int_{[0,1]} x dx = \frac{2\beta}{\pi} \times \frac{1}{2} = \frac{\beta}{\pi}. \end{aligned}$$

We deduce that

$$\frac{\beta^2}{\pi} \leq 1 - \cos(\beta).$$

This ends the proof by combining the two obtained inequalities. □

The right inequality in the proposition also appears in [34, Theorem 11], also established via integration techniques.

From the demonstrated inequalities, for any $\beta \in [0, \pi/2]$,

- the inequality $\cos(\beta) + \beta \sin(\beta/2) \geq 1$ holds, which does not seem to be so evident at first glance,

- we also have the following inequalities centered around $\cos(\beta)$:

$$1-\beta\sin\left(\frac{\beta}{2}\right)\leq\cos(\beta)\leq 1-\frac{\beta^2}{\pi}.$$

We do not claim that it is optimal in the sharpness sense, but it has a certain originality with regard to the involved polynomial and trigonometric functions.

The proposition below presents inequalities centered around the function $\sin(\beta\pi/2)-\beta$.

Proposition 6.2. *For any $\beta\in[0,1]$, the following inequalities hold:*

$$\left(1-\frac{2}{\pi}\right)\beta(1-\beta^2)\leq\sin\left(\frac{\beta\pi}{2}\right)-\beta\leq(1-\beta^2)\sin\left[\frac{\beta}{2}(\pi-2)\right].$$

Proof. We consider the function $f(x)=\arcsin(x)$, $x\in[0,1]$. Then the S operator of f at $\alpha=0$ is given as

$$S(f)(0,\beta)=\int_{[0,1]}\sin[\beta\arcsin(x)]dx=\frac{1}{1-\beta^2}\left[\sin\left(\frac{\beta\pi}{2}\right)-\beta\right].$$

Since $\beta\in[0,1]$, for any $x\in[0,1]$, we have $\beta f(x)=\beta\arcsin(x)\in[0,\pi/2]$. Therefore, by applying item 2 of Proposition 3.7 with $n=1$ and $\alpha=0$ and using $\int_{[0,1]}\arcsin(x)dx=(1/2)(\pi-2)$, we obtain

$$S(f)(0,\beta)\leq\sin\left[\beta\int_{[0,1]}f(x)dx\right]=\sin\left[\beta\int_{[0,1]}\arcsin(x)dx\right]=\sin\left[\frac{\beta}{2}(\pi-2)\right].$$

Hence, we have

$$\sin\left(\frac{\beta\pi}{2}\right)-\beta\leq(1-\beta^2)\sin\left[\frac{\beta}{2}(\pi-2)\right].$$

On the other hand, by using item 1 of Proposition 3.4 with $n=1$ and $\alpha=0$, we have

$$\begin{aligned}S(f)(0,\beta)&\geq\frac{2\beta}{\pi}\int_{[0,1]}f(x)dx=\frac{2\beta}{\pi}\int_{[0,1]}\arcsin(x)dx\\&=\frac{2\beta}{\pi}\times\frac{1}{2}(\pi-2)=\beta\left(1-\frac{2}{\pi}\right).\end{aligned}$$

We deduce that

$$\left(1-\frac{2}{\pi}\right)\beta(1-\beta^2)\leq\sin\left(\frac{\beta\pi}{2}\right)-\beta.$$

The desired results are established. □

The inequalities in the above proposition imply that, for any $\beta\in[0,1]$, we have

$$\left(1-\frac{2}{\pi}\right)\beta(1-\beta^2)+\beta\leq\sin\left(\frac{\beta\pi}{2}\right)\leq(1-\beta^2)\sin\left[\frac{\beta}{2}(\pi-2)\right]+\beta.$$

In particular, the left inequality gives an improvement of the Jordan inequality; since $(1-2/\pi)\beta(1-\beta^2)\geq 0$, we clearly have $\sin(\beta\pi/2)\geq\beta$. However, it is not competitive with sophisticated versions, such as those established in [38] and [31].

Remark 6.3. *In item 1 of Proposition 3.4, the Jordan inequality is the main tool for the proof. Thus, our improved Jordan inequality can be applied to a more precise lower bound. In some sense, the S (or C) operator is able to improve itself with well-selected functions in progressive calculus steps.*

Figure 1 illustrates the above proposition by considering the function

$$g(\beta)=\sin\left(\frac{\beta\pi}{2}\right)-\beta-\left(1-\frac{2}{\pi}\right)\beta(1-\beta^2), \tag{6.1}$$

showing how it is both positive and close to the $y = 0$ axis, and the function

$$h(\beta) = \sin\left(\frac{\beta\pi}{2}\right) - \beta - (1-\beta^2)\sin\left[\frac{\beta}{2}(\pi - 2)\right], \tag{6.2}$$

showing how it is both negative and close to the same axis.

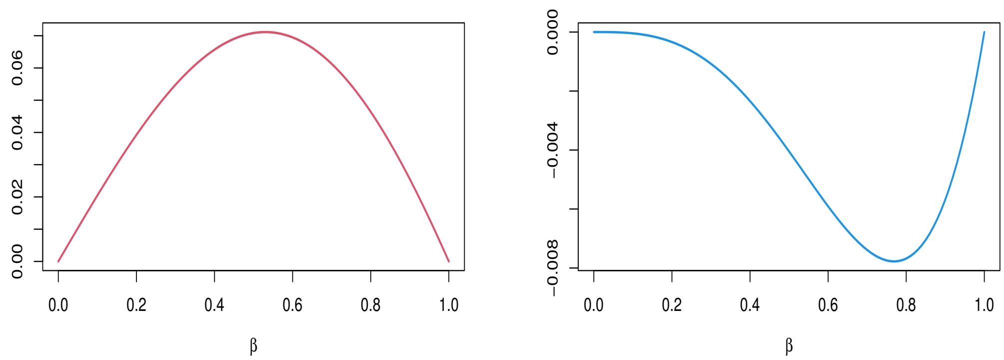

FIGURE 1. Illustration of (left) $g(\beta) \geq 0$, $\beta \in [0,1]$, where $g(\beta)$ is given in Equation (6.1), and (right) the fact that $h(\beta) \geq 0$, $\beta \in [0,1]$, where $h(\beta)$ is given in Equation (6.2)

The precision of $h(\beta)$ is notable.

The next proposition contains inequalities based on the function $\beta\cos(\beta\pi/2)$.

Proposition 6.4. *For any $\beta \in [0,1]$, the following inequalities hold:*

$$\frac{2}{\pi}\beta(1-\beta^2) \leq \beta\cos\left(\frac{\beta\pi}{2}\right) \leq (1-\beta^2)\sin(\beta).$$

Proof. We consider the function $f(x) = \arccos(x)$, $x \in [0,1]$. Then the S operator of f at $\alpha = 0$ is calculated as

$$S(f)(0,\beta) = \int_{[0,1]} \sin[\beta\arccos(x)]dx = \frac{\beta}{1-\beta^2}\cos\left(\frac{\beta\pi}{2}\right).$$

Since $\beta \in [0,1]$, for any $x \in [0,1]$, it is clear that $\beta f(x) = \beta\arccos(x) \in [0,\pi/2]$. It follows from item 2 of Proposition 3.7 with $n = 1$ and $\alpha = 0$ and $\int_{[0,1]}\arccos(x)dx = 1$ that

$$S(f)(0,\beta) \leq \sin\left[\beta\int_{[0,1]} f(x)dx\right] = \sin\left[\beta\int_{[0,1]}\arccos(x)dx\right] = \sin(\beta).$$

We thus obtain

$$\beta\cos\left(\frac{\beta\pi}{2}\right) \leq (1-\beta^2)\sin(\beta).$$

On the other hand, by using item 1 of Proposition 3.4 with $n = 1$ and $\alpha = 0$, we have

$$S(f)(0,\beta) \geq \frac{2\beta}{\pi}\int_{[0,1]} f(x)dx = \frac{2\beta}{\pi}\int_{[0,1]}\arccos(x)dx = \frac{2\beta}{\pi}.$$

We deduce that

$$\frac{2}{\pi}\beta(1-\beta^2) \leq \beta\cos\left(\frac{\beta\pi}{2}\right).$$

This ends the proof. □

From the inequalities in this proposition, for any $\beta \in [0,1]$, we notice that

- we also have the following inequalities centered around $\cos(\beta\pi/2)$:

$$\frac{2}{\pi}(1-\beta^2) \leq \cos\left(\frac{\beta\pi}{2}\right) \leq (1-\beta^2)\,\mathrm{sinc}(\beta).$$

- a lower bound for $\mathrm{sinc}(\beta)$ is thus given as

$$\mathrm{sinc}(\beta) \geq \frac{1}{1-\beta^2}\cos\left(\frac{\beta\pi}{2}\right).$$

Sharp inequalities of $\sin(\beta) - \beta\cos(\beta)$ are described in the result below.

Proposition 6.5. *For any $\beta \in [0, \pi/2]$, the following inequalities hold:*

$$\frac{2\beta^3}{3\pi} \leq \sin(\beta) - \beta\cos(\beta) \leq \frac{\beta^2}{2}\sin\left(\frac{2\beta}{3}\right).$$

Proof. We consider the function $f(x) = x$, $x \in [0,1]$. Then the S operator of f at $\alpha = 1$ is indicated as

$$S(f)(1,\beta) = \int_{[0,1]} x\sin(\beta x)dx = \frac{1}{\beta^2}[\sin(\beta) - \beta\cos(\beta)].$$

Since $\beta \in [0,\pi/2]$, for any $x \in [0,1]$, we have $\beta f(x) = \beta x \in [0,\pi/2]$. By applying item 2 of Proposition 3.7 with $n = 1$ and $\alpha = 1$, we obtain

$$\begin{aligned} S(f)(1,\beta) &\leq \left\{\int_{[0,1]} [f(x)]^1 dx\right\} \sin\left\{\beta \frac{1}{\int_{[0,1]}[f(x)]^1 dx} \int_{[0,1]} [f(x)]^{1+1} dx\right\} \\ &= \left[\int_{[0,1]} x dx\right] \sin\left[\beta \frac{1}{\int_{[0,1]} x dx} \int_{[0,1]} x^2 dx\right] \\ &= \frac{1}{2}\sin\left(\beta \times 2 \times \frac{1}{3}\right) = \frac{1}{2}\sin\left(\frac{2\beta}{3}\right). \end{aligned}$$

We thus obtain

$$\sin(\beta) - \beta\cos(\beta) \leq \frac{\beta^2}{2}\sin\left(\frac{2\beta}{3}\right).$$

On the other hand, it follows from item 1 of Proposition 3.4 with $n = 1$ and $\alpha = 1$ that

$$S(f)(1,\beta) \geq \frac{2\beta}{\pi}\int_{[0,1]} [f(x)]^{1+1} dx = \frac{2\beta}{\pi}\int_{[0,1]} x^2 dx = \frac{2\beta}{\pi} \times \frac{1}{3} = \frac{2\beta}{3\pi}.$$

We deduce that

$$\frac{2\beta^3}{3\pi} \leq \sin(\beta) - \beta\cos(\beta).$$

The proof is complete. □

In particular, this result provides a significant improvement of the well-known inequality $\sin(\beta) - \beta\cos(\beta) \geq 0$ for any $\beta \in [0,\pi/2]$. We highlight it graphically in Figure 2 by considering the two following functions:

$$g(\beta) = \sin(\beta) - \beta\cos(\beta) - \frac{2}{3\pi}\beta^3 \tag{6.3}$$

and

$$h(\beta) = \sin(\beta) - \beta\cos(\beta) - \frac{\beta^2}{2}\sin\left(\frac{2\beta}{3}\right). \tag{6.4}$$

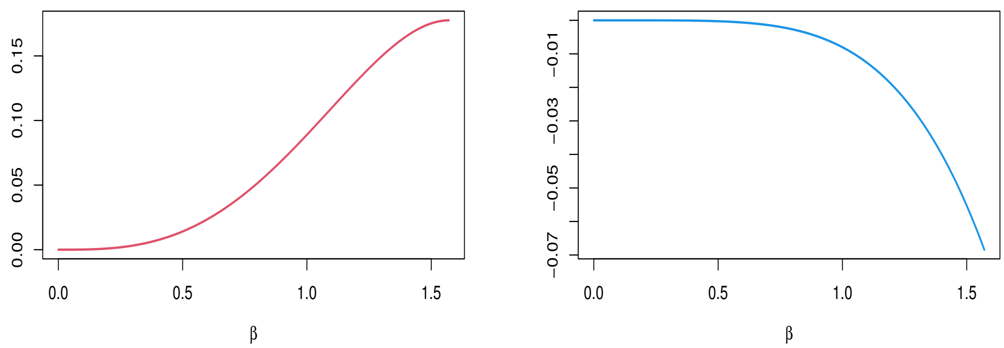

FIGURE 2. Illustration of (left) $g(\beta) \geq 0$, $\beta \in [0, \pi/2]$, where $g(\beta)$ is given in Equation (6.3), and (right) $h(\beta) \leq 0$, $\beta \in [0, \pi/2]$, where $h(\beta)$ is given in Equation (6.4)

The next proposition contains inequalities based on the function $(1 + \beta^2)\sin(\beta\pi/2) - (\beta/2)(\beta^2 - 1)\pi\cos(\beta\pi/2) - 2\beta$.

Proposition 6.6. *For any $\beta \in [0, 1]$, the following inequalities hold:*

$$2\left(1-\frac{2}{\pi}\right)\beta(1-\beta^2)^2 \leq (1+\beta^2)\sin\left(\frac{\beta\pi}{2}\right) - \frac{\beta}{2}(\beta^2-1)\pi\cos\left(\frac{\beta\pi}{2}\right) - 2\beta$$
$$\leq (1-\beta^2)^2\sin\left[\beta(\pi-2)\right].$$

Proof. We consider the function $f(x) = \arccos(x)$, $x \in [0, 1]$. Then the S operator of f at $\alpha = 1$ can be expressed as

$$\begin{aligned} S(f)(1,\beta) &= \int_{[0,1]} \arccos(x)\sin[\beta\arccos(x)]dx \\ &= \frac{1}{(1-\beta^2)^2}\left[(1+\beta^2)\sin\left(\frac{\beta\pi}{2}\right) - \frac{\beta}{2}(\beta^2-1)\pi\cos\left(\frac{\beta\pi}{2}\right) - 2\beta\right]. \end{aligned}$$

Since $\beta \in [0, 1]$, for any $x \in [0, 1]$, it is clear that $\beta f(x) = \beta\arccos(x) \in [0, \pi/2]$. Owing to item 2 of Proposition 3.7 with $n = 1$ and $\alpha = 1$, using $\int_{[0,1]}\arccos(x)dx = 1$ and $\int_{[0,1]}[\arccos(x)]^2dx = \pi - 2$, we get

$$\begin{aligned} S(f)(1,\beta) &\leq \left\{\int_{[0,1]}[f(x)]^1dx\right\}\sin\left\{\beta\frac{1}{\int_{[0,1]}[f(x)]^1dx}\int_{[0,1]}[f(x)]^{1+1}dx\right\} \\ &= \left[\int_{[0,1]}\arccos(x)dx\right]\sin\left\{\beta\frac{1}{\int_{[0,1]}\arccos(x)dx}\int_{[0,1]}[\arccos(x)]^2dx\right\} \\ &= \sin\left[\beta(\pi-2)\right]. \end{aligned}$$

We thus obtain

$$(1+\beta^2)\sin\left(\frac{\beta\pi}{2}\right) - \frac{\beta}{2}(\beta^2-1)\pi\cos\left(\frac{\beta\pi}{2}\right) - 2\beta \leq (1-\beta^2)^2\sin\left[\beta(\pi-2)\right].$$

On the other hand, by using item 1 of Proposition 3.4 with $n = 1$ and $\alpha = 1$, we have

$$\begin{aligned} S(f)(1,\beta) &\geq \frac{2\beta}{\pi}\int_{[0,1]}[f(x)]^{1+1}dx = \frac{2\beta}{\pi}\int_{[0,1]}[\arccos(x)]^2dx = \frac{2\beta}{\pi}\times(\pi-2) \\ &= 2\beta\left(1-\frac{2}{\pi}\right). \end{aligned}$$

We deduce that

$$2\left(1-\frac{2}{\pi}\right)\beta(1-\beta^2)^2 \leq (1+\beta^2)\sin\left(\frac{\beta\pi}{2}\right)-\frac{\beta}{2}(\beta^2-1)\pi\cos\left(\frac{\beta\pi}{2}\right)-2\beta.$$

This completes the proof. □

The obtained inequalities in the above proposition are original in form; no equivalent are in the literature to the best of our knowledge. Illustration of their sharpness is given in Figure 3, with the consideration of the following functions:

$$g(\beta)=(1+\beta^2)\sin\left(\frac{\beta\pi}{2}\right)-\frac{\beta}{2}(\beta^2-1)\pi\cos\left(\frac{\beta\pi}{2}\right)-2\beta-2\left(1-\frac{2}{\pi}\right)\beta(1-\beta^2)^2 \tag{6.5}$$

and

$$h(\beta)=(1+\beta^2)\sin\left(\frac{\beta\pi}{2}\right)-\frac{\beta}{2}(\beta^2-1)\pi\cos\left(\frac{\beta\pi}{2}\right)-2\beta-(1-\beta^2)^2\sin\left[\beta(\pi-2)\right]. \tag{6.6}$$

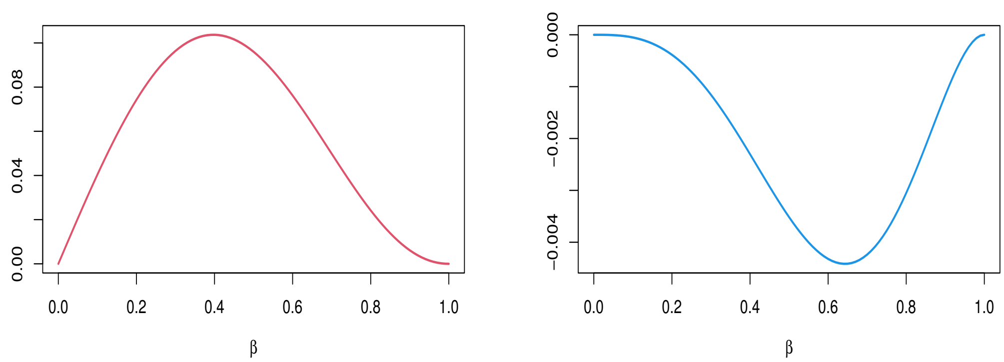

FIGURE 3. Illustration of (left) $g(\beta)\geq 0$, $\beta\in[0,1]$, where $g(\beta)$ is given in Equation (6.5), and (right) $h(\beta)\leq 0$, $\beta\in[0,1]$, where $h(\beta)$ is given in Equation (6.6)

From this figure, we observe that the left inequality is particularly sharp.

The theory on the S operator also leads to inequalities involving infinite series, as shown in the proposition below.

Proposition 6.7. *For any $\beta\in[0,\pi/2]$, $m\geq 0$ and $\alpha\geq 0$, the following inequalities hold:*

$$\frac{2\beta}{\pi[m(\alpha+1)+1]}\leq\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\beta^{2k+1}\frac{1}{m(\alpha+2k+1)+1}$$
$$\leq\frac{1}{m\alpha+1}\sin\left[\beta\frac{m\alpha+1}{m(\alpha+1)+1}\right].$$

Proof. We consider the function $f(x)=x^m$, $x\in[0,1]$, with $m\geq 0$. Then the S operator of f (with a general α) is given as

$$S(f)(\alpha,\beta)=\int_{[0,1]}x^{m\alpha}\sin(\beta x^m)dx$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\beta^{2k+1}\frac{1}{m(\alpha+2k+1)+1}.$$

Since $\beta \in [0, \pi/2]$, for any $x \in [0,1]$, we have $\beta f(x) = \beta x^m \in [0, \pi/2]$. Hence, by using item 2 of Proposition 3.7 with $n = 1$, we establish that

$$\begin{aligned} S(f)(\alpha, \beta) &\leq \left\{ \int_{[0,1]} [f(x)]^{\alpha} dx \right\} \sin \left\{ \beta \frac{1}{\int_{[0,1]} [f(x)]^{\alpha} dx} \int_{[0,1]} [f(x)]^{\alpha+1} dx \right\} \\ &= \left[\int_{[0,1]} x^{m\alpha} dx \right] \sin \left[\beta \frac{1}{\int_{[0,1]} x^{m\alpha} dx} \int_{[0,1]} x^{m(\alpha+1)} dx \right] \\ &= \frac{1}{m\alpha + 1} \sin \left[\beta \frac{m\alpha + 1}{m(\alpha+1)+1} \right]. \end{aligned}$$

We thus obtain

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \beta^{2k+1} \frac{1}{m(\alpha + 2k + 1) + 1} \leq \frac{1}{m\alpha + 1} \sin \left[\beta \frac{m\alpha + 1}{m(\alpha+1)+1} \right].$$

On the other hand, by using item 1 of Proposition 3.4 with $n = 1$, we have

$$\begin{aligned} S(f)(\alpha, \beta) &\geq \frac{2\beta}{\pi} \int_{[0,1]} [f(x)]^{\alpha+1} dx = \frac{2\beta}{\pi} \int_{[0,1]} x^{m(\alpha+1)} dx = \frac{2\beta}{\pi} \times \frac{1}{m(\alpha+1)+1} \\ &= \frac{2\beta}{\pi[m(\alpha+1)+1]}. \end{aligned}$$

We deduce that

$$\frac{2\beta}{\pi[m(\alpha+1)+1]} \leq \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \beta^{2k+1} \frac{1}{m(\alpha + 2k + 1) + 1}.$$

This ends the proof. □

Inequalities derived from the bivariate S operator are examined below.

Proposition 6.8. *For any $\beta \in [0, \pi/2]$, the following inequalities hold:*

$$\frac{\beta^2}{2\pi} \leq \gamma + \log(\beta) - C_o(\beta) \leq \beta \sin\left(\frac{\beta}{4}\right).$$

Proof. We consider the function $f(x_1, x_2) = x_1 x_2$, $(x_1, x_2) \in [0,1]^2$. Then the S operator of f at $\alpha = 0$ is determined by

$$S(f)(0, \beta) = \int_{[0,1]^2} \sin(\beta x_1 x_2) dx_1 dx_2 = \frac{1}{\beta} [\gamma + \log(\beta) - C_o(\beta)].$$

Since $\beta \in [0, \pi/2]$, for any $(x_1, x_2) \in [0,1]^2$, we have $\beta f(x_1, x_2) = \beta x_1 x_2 \in [0, \pi/2]$. By applying item 2 of Proposition 3.7 with $n = 2$ and $\alpha = 0$ and using $\int_{[0,1]^2} x_1 x_2 dx_1 dx_2 = 1/4$, we find that

$$\begin{aligned} &S(f)(0, \beta) \\ &\leq \left\{ \int_{[0,1]^2} [f(x_1, x_2)]^0 dx_1 dx_2 \right\} \sin \left\{ \beta \frac{1}{\int_{[0,1]^2} [f(x_1, x_2)]^0 dx_1 dx_2} \int_{[0,1]^2} [f(x_1, x_2)]^{0+1} dx_1 dx_2 \right\} \\ &= \sin \left[\beta \int_{[0,1]^2} f(x_1, x_2) dx_1 dx_2 \right] = \sin \left[\beta \int_{[0,1]^2} x_1 x_2 dx_1 dx_2 \right] = \sin\left(\frac{\beta}{4}\right). \end{aligned}$$

We thus obtain

$$\gamma + \log(\beta) - C_o(\beta) \leq \beta \sin\left(\frac{\beta}{4}\right).$$

On the other hand, by using item 1 of Proposition 3.4 with $n = 2$ and $\alpha = 0$, we have

$$S(f)(0,\beta) \geq \frac{2\beta}{\pi}\int_{[0,1]^2}[f(x_1,x_2)]^{0+1}dx_1dx_2 = \frac{2\beta}{\pi}\int_{[0,1]^2} x_1x_2dx_1dx_2$$
$$= \frac{2\beta}{\pi} \times \frac{1}{4} = \frac{\beta}{2\pi}.$$

We deduce that

$$\frac{\beta^2}{2\pi} \leq \gamma + \log(\beta) - C_o(\beta).$$

The stated inequalities are established. □

Figure 4 provides graphical evidence of the sharpness of these inequalities by investigating the curves of the following functions:

$$g(\beta) = \gamma + \log(\beta) - C_o(\beta) - \frac{\beta^2}{2\pi} \tag{6.7}$$

and

$$h(\beta) = \gamma + \log(\beta) - C_o(\beta) - \beta\sin\left(\frac{\beta}{4}\right). \tag{6.8}$$

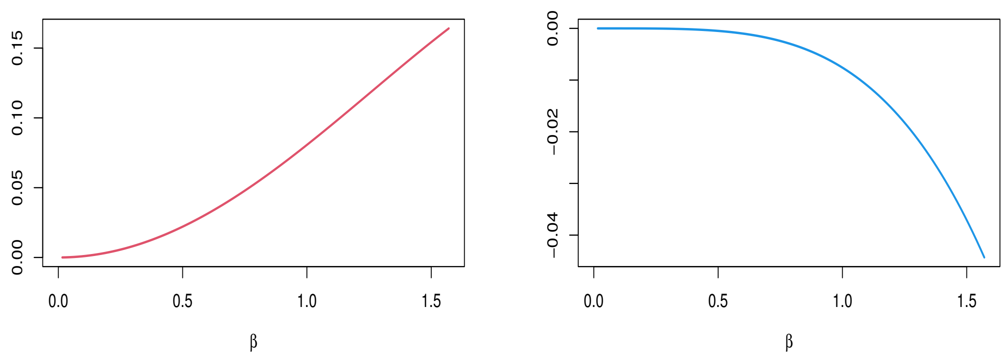

FIGURE 4. Illustration of (left) $g(\beta) \geq 0$, $\beta \in [0, \pi/2]$, where $g(\beta)$ is given in Equation (6.7), and (right) $h(\beta) \leq 0$, $\beta \in [0, \pi/2]$, where $h(\beta)$ is given in Equation (6.8)

The precision of the obtained inequalities is thus validated visually.

The next proposition contains original inequalities based on the product of the S and C Fresnel integrals.

Proposition 6.9. *For any $\beta \in [0, \pi/4]$, the following inequalities hold:*

$$\frac{4\beta^2}{3\pi^2} \leq C_\star\left[\sqrt{\frac{2\beta}{\pi}}\right] S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right] \leq \frac{\beta}{\pi}\sin\left(\frac{2\beta}{3}\right).$$

Proof. We consider the function $f(x_1,x_2) = x_1^2 + x_2^2$, $(x_1,x_2) \in [0,1]^2$. Then the S operator of f at $\alpha = 0$ is determined by

$$S(f)(0,\beta) = \int_{[0,1]^2}\sin[\beta(x_1^2+x_2^2)]dx_1dx_2 = \frac{\pi}{\beta}C_\star\left[\sqrt{\frac{2\beta}{\pi}}\right] S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right].$$

Since $\beta \in [0, \pi/4]$, for any $(x_1, x_2) \in [0,1]^2$, we have $\beta f(x_1,x_2) = \beta(x_1^2+x_2^2) \geq 0$ and $\beta f(x_1,x_2) = \beta(x_1^2+x_2^2) \leq 2\beta \leq \pi/2$, so $\beta f(x_1,x_2) \in [0,\pi/2]$. By applying item 2 of Proposition 3.7 with $n=2$ and $\alpha = 0$ and using $\int_{[0,1]^2}(x_1^2+x_2^2)dx_1dx_2 = 2/3$, we get

$$\begin{aligned} &S(f)(0,\beta)\\ &\leq \left\{\int_{[0,1]^2}[f(x_1,x_2)]^0 dx_1dx_2\right\}\sin\left\{\beta\frac{1}{\int_{[0,1]^2}[f(x_1,x_2)]^0dx_1dx_2}\int_{[0,1]^2}[f(x_1,x_2)]^{0+1}dx_1dx_2\right\}\\ &= \sin\left[\beta\int_{[0,1]^2}f(x_1,x_2)dx_1dx_2\right] = \sin\left[\beta\int_{[0,1]^2}(x_1^2+x_2^2)dx_1dx_2\right] = \sin\left(\frac{2\beta}{3}\right). \end{aligned}$$

We thus obtain

$$C_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right] \leq \frac{\beta}{\pi}\sin\left(\frac{2\beta}{3}\right).$$

On the other hand, by using item 1 of Proposition 3.4 with $n=2$ and $\alpha=0$, we have

$$\begin{aligned} S(f)(0,\beta) &\geq \frac{2\beta}{\pi}\int_{[0,1]^2}[f(x_1,x_2)]^{0+1}dx_1dx_2 = \frac{2\beta}{\pi}\int_{[0,1]^2}(x_1^2+x_2^2)dx_1dx_2\\ &= \frac{2\beta}{\pi}\times\frac{2}{3} = \frac{4\beta}{3\pi}. \end{aligned}$$

We deduce that

$$\frac{4\beta^2}{3\pi^2} \leq C_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right].$$

This ends the proof. □

Figure 5 provides graphical evidence of the sharpness of these inequalities by investigating the curves of the following functions:

$$g(\beta) = C_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right] - \frac{4\beta^2}{3\pi^2} \tag{6.9}$$

and

$$h(\beta) = C_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right] - \frac{\beta}{\pi}\sin\left(\frac{2\beta}{3}\right). \tag{6.10}$$

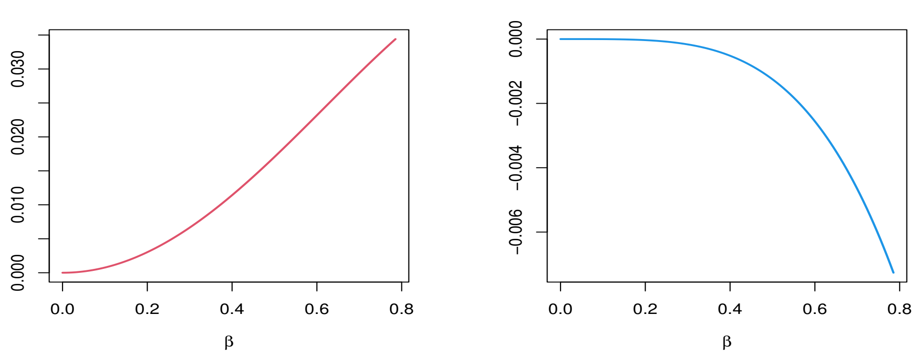

FIGURE 5. Illustration of (left) $g(\beta) \geq 0$, $\beta \in [0,\pi/4]$, where $g(\beta)$ is given in Equation (6.9), and (right) $h(\beta) \leq 0$, $\beta \in [0,\pi/4]$, where $h(\beta)$ is given in Equation (6.10)

Additional graphical tests show that these inequalities can be extended for $\beta \in [0, \pi/2]$.

6.2. **Inequalities derived from the C operator.** We now use the theory on the C operator to establish some refined trigonometric inequalities. The next proposition contains inequalities centered around the sine function.

Proposition 6.10. *For any $\beta \in [0, \pi/2]$, the following inequalities hold:*

$$\beta\left(1-\frac{\beta}{\pi}\right) \leq \sin(\beta) \leq \beta\cos\left(\frac{\beta}{2}\right).$$

Proof. We consider the function $f(x) = x$, $x \in [0, 1]$. Then the C operator of f at $\alpha = 0$ is calculated as

$$C(f)(0,\beta) = \int_{[0,1]} \cos(\beta x)dx = \frac{\sin(\beta)}{\beta},$$

or, eventually, $C(f)(0, \beta) = \mathrm{sinc}(\beta)$. Since $\beta \in [0, \pi/2]$, for any $x \in [0, 1]$, it is clear that $\beta f(x) = \beta x \in [0, \pi/2]$. By applying item 4 of Proposition 3.7 with $n = 1$ and $\alpha = 0$, we get

$$\begin{aligned} C(f)(0,\beta) &\leq \left\{\int_{[0,1]}[f(x)]^0dx\right\}\sin\left\{\beta\frac{1}{\int_{[0,1]}[f(x)]^0dx}\int_{[0,1]}[f(x)]^{0+1}dx\right\} \\ &= \cos\left[\beta\int_{[0,1]}f(x)dx\right] = \cos\left[\beta\int_{[0,1]}xdx\right] = \cos\left(\frac{\beta}{2}\right). \end{aligned}$$

We thus obtain

$$\sin(\beta) \leq \beta\cos\left(\frac{\beta}{2}\right).$$

On the other hand, by using item 3 of Proposition 3.4 with $n = 1$ and $\alpha = 0$, we have

$$\begin{aligned} C(f)(0,\beta) &\geq \int_{[0,1]}[f(x)]^0dx - \frac{2\beta}{\pi}\int_{[0,1]}[f(x)]^{0+1}dx = 1-\frac{2\beta}{\pi}\int_{[0,1]}f(x)dx \\ &= 1-\frac{2\beta}{\pi}\int_{[0,1]}xdx = 1-\frac{2\beta}{\pi}\times\frac{1}{2} = 1-\frac{\beta}{\pi}. \end{aligned}$$

We deduce that

$$\beta\left(1-\frac{\beta}{\pi}\right) \leq \sin(\beta).$$

This completes the proof. □

This proposition gives interesting complementary materials to the famous inequality $\sin(\beta) \geq \beta\cos(\beta)$ for any $\beta \in [0, \pi/2]$. In terms of the sine cardinal function, it is equivalent to, for any $\beta \in [0, \pi/2]$,

$$1-\frac{\beta}{\pi} \leq \mathrm{sinc}(\beta) \leq \cos\left(\frac{\beta}{2}\right).$$

We highlight the obtained inequalities graphically in Figure 6 by considering the two following functions:

$$g(\beta) = \sin(\beta) - \beta\left(1-\frac{\beta}{\pi}\right) \tag{6.11}$$

and

$$h(\beta) = \sin(\beta) - \beta\cos\left(\frac{\beta}{2}\right). \tag{6.12}$$

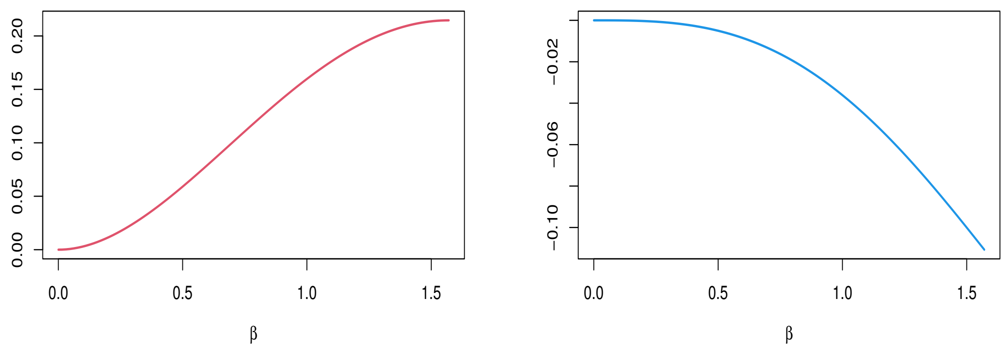

FIGURE 6. Illustration of (left) $g(\beta) \geq 0$, $\beta \in [0, \pi/2]$, where $g(\beta)$ is given in Equation (6.11), and (right) $h(\beta) \leq 0$, $\beta \in [0, \pi/2]$, where $h(\beta)$ is given in Equation (6.12)

The proposition below is about inequalities that bound the function $\cos(\beta\pi/2)$.

Proposition 6.11. *For any $\beta \in [0, 1]$, the following inequalities hold:*

$$(1-\beta^2)\left[1-\beta+\frac{2\beta}{\pi}\right] \leq \cos\left(\frac{\beta\pi}{2}\right) \leq \cos\left[\frac{\beta}{2}(\pi-2)\right].$$

Proof. We consider the function $f(x) = \arcsin(x)$, $x \in [0, 1]$. Then the C operator of f at $\alpha = 0$ is given as

$$C(f)(0, \beta) = \int_{[0,1]} \cos[\beta \arcsin(x)]dx = \frac{1}{1-\beta^2}\cos\left(\frac{\beta\pi}{2}\right).$$

Since $\beta \in [0, 1]$, for any $x \in [0, 1]$, we have $\beta f(x) = \beta \arcsin(x) \in [0, \pi/2]$. By applying item 4 of Proposition 3.7 with $n = 1$ and $\alpha = 0$, we get

$$C(f)(0, \beta) \leq \cos\left[\beta \int_{[0,1]} f(x)dx\right] = \cos\left[\beta \int_{[0,1]} \arcsin(x)dx\right] = \cos\left[\frac{\beta}{2}(\pi-2)\right].$$

We thus obtain

$$\cos\left(\frac{\beta\pi}{2}\right) \leq (1-\beta^2)\cos\left[\frac{\beta}{2}(\pi-2)\right].$$

On the other hand, by using item 3 of Proposition 3.4 with $n = 1$ and $\alpha = 0$, we have

$$C(f)(0, \beta) \geq 1 - \frac{2\beta}{\pi}\int_{[0,1]} f(x)dx = 1 - \frac{2\beta}{\pi}\int_{[0,1]} \arcsin(x)dx$$
$$= 1 - \frac{2\beta}{\pi} \times \frac{1}{2}(\pi-2) = 1-\beta+\frac{2\beta}{\pi}.$$

We deduce that

$$(1-\beta^2)\left[1-\beta+\frac{2\beta}{\pi}\right] \leq \cos\left(\frac{\beta\pi}{2}\right).$$

This ends the proof. □

The obtained inequalities are original in form. Their sharpness is illustrated in Figure 7 through the analysis of the two following functions:

$$g(\beta) = \cos\left(\frac{\beta\pi}{2}\right) - (1-\beta^2)\left[1-\beta+\frac{2\beta}{\pi}\right] \tag{6.13}$$

and

$$h(\beta) = \cos\left(\frac{\beta\pi}{2}\right) - \cos\left[\frac{\beta}{2}(\pi - 2)\right]. \tag{6.14}$$

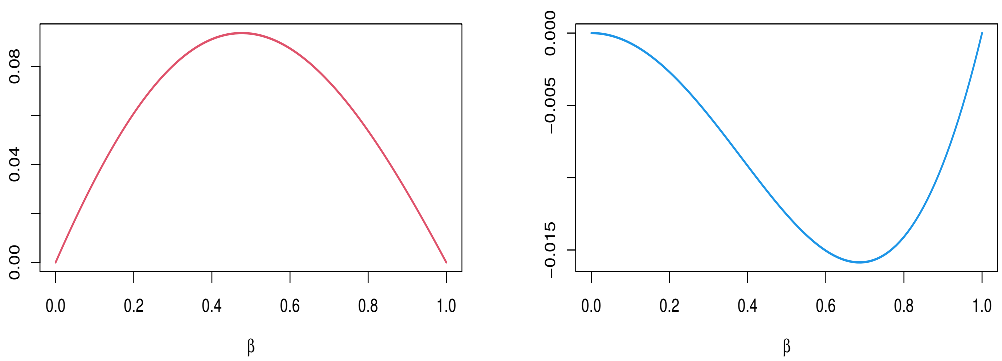

FIGURE 7. Illustration of (left) $g(\beta) \geq 0$, $\beta \in [0, 1]$, where $g(\beta)$ is given in Equation (6.13), and (right) $h(\beta) \leq 0$, $\beta \in [0, 1]$, where $h(\beta)$ is given in Equation (6.14)

This figure supports the interest of the C operator in generating sharp inequalities of original form. The main function of interest in the result below is $1 - \beta \sin(\beta\pi/2)$.

Proposition 6.12. *For any $\beta \in [0, 1]$, the following inequalities hold:*

$$(1 - \beta^2)\left(1 - \frac{2\beta}{\pi}\right) \leq 1 - \beta \sin\left(\frac{\beta\pi}{2}\right) \leq (1 - \beta^2)\cos(\beta).$$

Proof. We consider the function $f(x) = \arccos(x)$, $x \in [0, 1]$. Then the C operator of f at $\alpha = 0$ is obtained as

$$C(f)(0, \beta) = \int_{[0,1]} \cos[\beta \arccos(x)]dx = \frac{1}{1 - \beta^2}\left[1 - \beta \sin\left(\frac{\beta\pi}{2}\right)\right].$$

Since $\beta \in [0, 1]$, for any $x \in [0, 1]$, it is clear that $\beta f(x) = \beta \arccos(x) \in [0, \pi/2]$. By applying item 4 of Proposition 3.7 with $n = 1$ and $\alpha = 0$, since $\beta f(x) \in [0, \pi/2]$ for any $x \in [0, 1]$, we get

$$C(f)(0, \beta) \leq \cos\left[\beta \int_{[0,1]} f(x)dx\right] = \cos\left[\beta \int_{[0,1]} \arccos(x)dx\right] = \cos(\beta).$$

We thus obtain

$$1 - \beta \sin\left(\frac{\beta\pi}{2}\right) \leq (1 - \beta^2)\cos(\beta).$$

On the other hand, by using item 3 of Proposition 3.4 with $n = 1$ and $\alpha = 0$, we have

$$C(f)(0, \beta) \geq 1 - \frac{2\beta}{\pi} \int_{[0,1]} f(x)dx = 1 - \frac{2\beta}{\pi} \int_{[0,1]} \arccos(x)dx = 1 - \frac{2\beta}{\pi}.$$

We deduce that

$$(1 - \beta^2)\left(1 - \frac{2\beta}{\pi}\right) \leq 1 - \beta \sin\left(\frac{\beta\pi}{2}\right).$$

The desired results are established. □

The inequalities in the above proposition also imply that, for $\beta \in [0,1]$,

$$1-(1-\beta^2)\cos(\beta) \leq \beta \sin\left(\frac{\beta\pi}{2}\right) \leq 1-(1-\beta^2)\left(1-\frac{2\beta}{\pi}\right),$$

which remains new in the literature, to the best of our knowledge.

The proposition below considers the function $\beta\sin(\beta)+\cos(\beta)$ and proposes lower and upper bounds for it.

Proposition 6.13. *For any $\beta \in [0,\pi/2]$, the following inequalities hold:*

$$1+\beta^2\left(\frac{1}{2}-\frac{2\beta}{3\pi}\right) \leq \beta\sin(\beta)+\cos(\beta) \leq 1+\frac{\beta^2}{2}\cos\left(\frac{2\beta}{3}\right).$$

Proof. We consider the function $f(x)=x$, $x\in[0,1]$. Then the C operator of f at $\alpha=1$ is given as

$$C(f)(1,\beta)=\int_{[0,1]} x\cos(\beta x)dx=\frac{1}{\beta^2}[\beta\sin(\beta)+\cos(\beta)-1].$$

Since $\beta\in[0,\pi/2]$, for any $x\in[0,1]$, it is clear that $\beta f(x)=\beta x\in[0,\pi/2]$. By applying item 4 of Proposition 3.7 with $n=1$ and $\alpha=1$, we obtain

$$\begin{aligned}C(f)(1,\beta) &\leq \left\{\int_{[0,1]}[f(x)]^1dx\right\}\cos\left\{\beta\frac{1}{\int_{[0,1]}[f(x)]^1dx}\int_{[0,1]}[f(x)]^{1+1}dx\right\}\\ &=\left[\int_{[0,1]}xdx\right]\cos\left[\beta\frac{1}{\int_{[0,1]}xdx}\int_{[0,1]}x^2dx\right]\\ &=\frac{1}{2}\cos\left(\beta\times 2\times\frac{1}{3}\right)=\frac{1}{2}\cos\left(\frac{2\beta}{3}\right).\end{aligned}$$

Hence, we have

$$\beta\sin(\beta)+\cos(\beta)\leq 1+\frac{\beta^2}{2}\cos\left(\frac{2\beta}{3}\right).$$

On the other hand, by using item 3 of Proposition 3.4 with $n=1$ and $\alpha=1$, we have

$$\begin{aligned}C(f)(1,\beta) &\geq \int_{[0,1]}[f(x)]^1dx-\frac{2\beta}{\pi}\int_{[0,1]}[f(x)]^{1+1}dx\\ &=\int_{[0,1]}xdx-\frac{2\beta}{\pi}\int_{[0,1]}x^2dx=\frac{1}{2}-\frac{2\beta}{\pi}\times\frac{1}{3}=\frac{1}{2}-\frac{2\beta}{3\pi}.\end{aligned}$$

We deduce that

$$1+\beta^2\left(\frac{1}{2}-\frac{2\beta}{3\pi}\right)\leq\beta\sin(\beta)+\cos(\beta).$$

This ends the proof. □

The sharpness of the obtained inequalities is illustrated in Figure 8 by considering the two following functions:

$$g(\beta)=\beta\sin(\beta)+\cos(\beta)-1-\beta^2\left(\frac{1}{2}-\frac{2\beta}{3\pi}\right) \tag{6.15}$$

and

$$h(\beta)=\beta\sin(\beta)+\cos(\beta)-1-\frac{\beta^2}{2}\cos\left(\frac{2\beta}{3}\right). \tag{6.16}$$

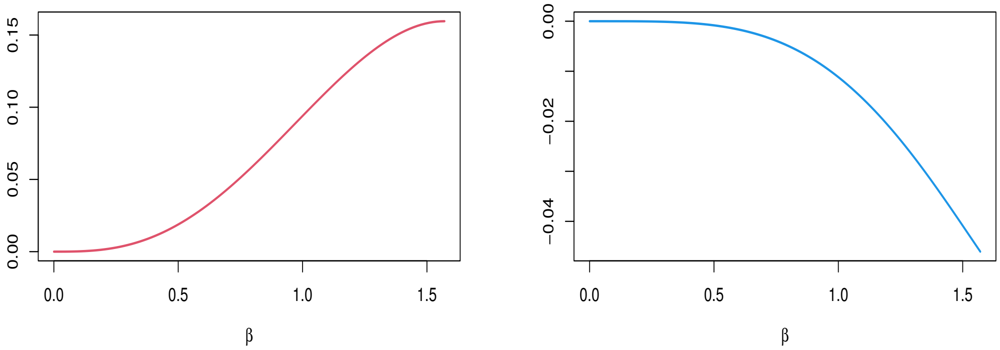

FIGURE 8. Illustration of (left) $g(\beta) \geq 0$, $\beta \in [0, \pi/2]$, where $g(\beta)$ is given in Equation (6.15), and (right) $h(\beta) \leq 0$, $\beta \in [0, \pi/2]$, where $h(\beta)$ is given in Equation (6.16)

The result below is about inequalities centered around the function $(1 + \beta^2)\cos(\beta\pi/2) + (\beta/2)(\beta^2 - 1)\pi \sin(\beta\pi/2)$.

Proposition 6.14. *For any $\beta \in [0, 1]$, the following inequalities hold:*

$$(1-\beta^2)^2\left(1 - 2\beta + \frac{4\beta}{\pi}\right) \leq (1+\beta^2)\cos\left(\frac{\beta\pi}{2}\right) + \frac{\beta}{2}(\beta^2-1)\pi\sin\left(\frac{\beta\pi}{2}\right)$$
$$\leq (1-\beta^2)^2\cos\left[\beta(\pi-2)\right].$$

Proof. We consider the function $f(x) = \arccos(x)$, $x \in [0, 1]$. Then the C operator of f at $\alpha = 1$ is calculated as

$$\begin{aligned} C(f)(1,\beta) &= \int_{[0,1]} \arccos(x)\cos[\beta\arccos(x)]dx \\ &= \frac{1}{(1-\beta^2)^2}\left[(1+\beta^2)\cos\left(\frac{\beta\pi}{2}\right) + \frac{\beta}{2}(\beta^2-1)\pi\sin\left(\frac{\beta\pi}{2}\right)\right]. \end{aligned}$$

Since $\beta \in [0, 1]$, for any $x \in [0, 1]$, it is clear that $\beta f(x) = \beta\arccos(x) \in [0, \pi/2]$. By applying item 4 of Proposition 3.7 with $n = 1$ and $\alpha = 1$, since $\beta f(x) \in [0, \pi/2]$ for any $x \in [0, 1]$, we get

$$\begin{aligned} C(f)(1,\beta) &\leq \left\{\int_{[0,1]}[f(x)]^1 dx\right\}\cos\left\{\beta\frac{1}{\int_{[0,1]}[f(x)]^1 dx}\int_{[0,1]}[f(x)]^{1+1}dx\right\} \\ &= \left[\int_{[0,1]}\arccos(x)dx\right]\cos\left\{\beta\frac{1}{\int_{[0,1]}\arccos(x)dx}\int_{[0,1]}[\arccos(x)]^2 dx\right\} \\ &= \cos\left[\beta(\pi-2)\right]. \end{aligned}$$

We thus obtain

$$(1+\beta^2)\cos\left(\frac{\beta\pi}{2}\right) + \frac{\beta}{2}(\beta^2-1)\pi\sin\left(\frac{\beta\pi}{2}\right) \leq (1-\beta^2)^2\cos\left[\beta(\pi-2)\right].$$

On the other hand, by using item 3 of Proposition 3.4 with $n = 1$ and $\alpha = 1$, we have

$$\begin{aligned} C(f)(1,\beta) &\geq \int_{[0,1]} [f(x)]^1 dx - \frac{2\beta}{\pi} \int_{[0,1]} [f(x)]^{1+1} dx \\ &= \int_{[0,1]} \arccos(x) dx - \frac{2\beta}{\pi} \int_{[0,1]} [\arccos(x)]^2 dx = 1 - \frac{2\beta}{\pi} \times (\pi - 2) \\ &= 1 - 2\beta + \frac{4\beta}{\pi}. \end{aligned}$$

We deduce that

$$(1-\beta^2)^2 \left(1 - 2\beta + \frac{4\beta}{\pi}\right) \leq (1+\beta^2)\cos\left(\frac{\beta\pi}{2}\right) + \frac{\beta}{2}(\beta^2 - 1)\pi \sin\left(\frac{\beta\pi}{2}\right).$$

This completes the proof. □

The next proposition illustrates how the theory of the C operator also results in infinite series inequalities.

Proposition 6.15. *For any $\beta \in [0, \pi/2]$, $m \geq 0$ and $\alpha \geq 0$, the following inequalities hold:*

$$\begin{aligned} &\frac{1}{m\alpha + 1} - \frac{2\beta}{\pi[m(\alpha+1)+1]} \leq \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \frac{1}{m(\alpha+2k)+1} \\ &\leq \frac{1}{m\alpha+1} \cos\left[\beta \frac{m\alpha+1}{m(\alpha+1)+1}\right]. \end{aligned}$$

Proof. We consider the function $f(x) = x^m$, $x \in [0, 1]$, with $m \geq 0$. Then the C operator of f (with a general α) is indicated as

$$C(f)(\alpha,\beta) = \int_{[0,1]} x^{m\alpha} \cos(\beta x^m) dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \frac{1}{m(\alpha+2k)+1}.$$

Since $\beta \in [0, \pi/2]$, for any $x \in [0, 1]$, it is clear that $\beta f(x) = \beta x^m \in [0, \pi/2]$. By applying item 4 of Proposition 3.7 with $n = 1$, we get

$$\begin{aligned} C(f)(\alpha,\beta) &\leq \left\{\int_{[0,1]} [f(x)]^\alpha dx\right\} \cos\left\{\beta \frac{1}{\int_{[0,1]} [f(x)]^\alpha dx} \int_{[0,1]} [f(x)]^{\alpha+1} dx\right\} \\ &= \left[\int_{[0,1]} x^{m\alpha} dx\right] \cos\left[\beta \frac{1}{\int_{[0,1]} x^{m\alpha} dx} \int_{[0,1]} x^{m(\alpha+1)} dx\right] \\ &= \frac{1}{m\alpha+1} \cos\left[\beta \frac{m\alpha+1}{m(\alpha+1)+1}\right]. \end{aligned}$$

We thus obtain

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \beta^{2k} \frac{1}{m(\alpha+2k)+1} \leq \frac{1}{m\alpha+1} \cos\left[\beta \frac{m\alpha+1}{m(\alpha+1)+1}\right].$$

On the other hand, by using item 3 of Proposition 3.4 with $n = 1$, we have

$$\begin{aligned} C(f)(\alpha,\beta) &\geq \int_{[0,1]} [f(x)]^\alpha dx - \frac{2\beta}{\pi} \int_{[0,1]} [f(x)]^{\alpha+1} dx \\ &= \int_{[0,1]} x^{m\alpha} dx - \frac{2\beta}{\pi} \int_{[0,1]} x^{m(\alpha+1)} dx \\ &= \frac{1}{m\alpha+1} - \frac{2\beta}{\pi} \times \frac{1}{m(\alpha+1)+1} \\ &= \frac{1}{m\alpha+1} - \frac{2\beta}{\pi[m(\alpha+1)+1]}. \end{aligned}$$

We deduce that

$$\frac{1}{m\alpha+1} - \frac{2\beta}{\pi[m(\alpha+1)+1]} \leq \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\beta^{2k}\frac{1}{m(\alpha+2k)+1}.$$

This ends the proof. □

Innovative inequalities based on the S and C Fresnel integrals are established in the result below.

Proposition 6.16. *For any $\beta \in [0, \pi/4]$, the following inequalities hold:*

$$\frac{2\beta}{\pi}\left(1-\frac{4\beta}{3\pi}\right) \leq \left\{C_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]\right\}^2 - \left\{S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]\right\}^2 \leq \frac{2\beta}{\pi}\cos\left(\frac{2\beta}{3}\right).$$

Proof. We consider the function $f(x_1, x_2) = x_1^2 + x_2^2$, $(x_1, x_2) \in [0, 1]^2$. Then the C operator of f at $\alpha = 0$ is calculated as

$$\begin{aligned} C(f)(0,\beta) &= \int_{[0,1]^2} \cos[\beta(x_1^2+x_2^2)]dx_1dx_2 \\ &= \frac{\pi}{2\beta}\left[\left\{C_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]\right\}^2 - \left\{S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]\right\}^2\right]. \end{aligned}$$

Since $\beta \in [0, \pi/4]$, for any $(x_1, x_2) \in [0, 1]^2$, we have $\beta f(x_1, x_2) = \beta(x_1^2 + x_2^2) \geq 0$ and $\beta f(x_1, x_2) = \beta(x_1^2 + x_2^2) \leq 2\beta \leq \pi/2$, so $\beta f(x_1, x_2) \in [0, \pi/2]$. It follows from item 4 of Proposition 3.7 with $n = 2$ and $\alpha = 0$ that

$$\begin{aligned} &C(f)(0,\beta) \\ &\leq \left\{\int_{[0,1]^2}[f(x_1,x_2)]^0 dx_1dx_2\right\}\cos\left\{\beta\frac{1}{\int_{[0,1]^2}[f(x_1,x_2)]^0dx_1dx_2}\int_{[0,1]^2}[f(x_1,x_2)]^{0+1}dx_1dx_2\right\} \\ &= \cos\left[\beta\int_{[0,1]^2} f(x_1,x_2)dx_1dx_2\right] = \cos\left[\beta\int_{[0,1]^2}(x_1^2+x_2^2)dx_1dx_2\right] = \cos\left(\beta\frac{2}{3}\right). \end{aligned}$$

We thus obtain

$$\left\{C_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]\right\}^2 - \left\{S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]\right\}^2 \leq \frac{2\beta}{\pi}\cos\left(\frac{2\beta}{3}\right).$$

On the other hand, by using item 3 of Proposition 3.4 with $n = 2$ and $\alpha = 0$, we have

$$\begin{aligned} C(f)(0,\beta) &\geq \int_{[0,1]^2}[f(x_1,x_2)]^0dx_1dx_2 - \frac{2\beta}{\pi}\int_{[0,1]^2}[f(x_1,x_2)]^{0+1}dx_1dx_2 \\ &= 1 - \frac{2\beta}{\pi}\int_{[0,1]^2}(x_1^2+x_2^2)dx_1dx_2 \\ &= 1 - \frac{2\beta}{\pi}\times\frac{2}{3} = 1 - \frac{4\beta}{3\pi}. \end{aligned}$$

We deduce that

$$\frac{2\beta}{\pi}\left(1-\frac{4\beta}{3\pi}\right) \leq \left\{C_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]\right\}^2 - \left\{S_\star\left[\sqrt{\frac{2\beta}{\pi}}\right]\right\}^2.$$

The desired inequalities are proved. □

By examining the curves of the following functions, Figure 9 offers a visual proof of the quality of these inequalities:

$$g(\beta) = \left\{ C_\star \left[\sqrt{\frac{2\beta}{\pi}} \right] \right\}^2 - \left\{ S_\star \left[\sqrt{\frac{2\beta}{\pi}} \right] \right\}^2 - \frac{2\beta}{\pi} \left(1 - \frac{4\beta}{3\pi} \right) \tag{6.17}$$

and

$$h(\beta) = \left\{ C_\star \left[\sqrt{\frac{2\beta}{\pi}} \right] \right\}^2 - \left\{ S_\star \left[\sqrt{\frac{2\beta}{\pi}} \right] \right\}^2 - \frac{2\beta}{\pi} \cos\left(\frac{2\beta}{3} \right). \tag{6.18}$$

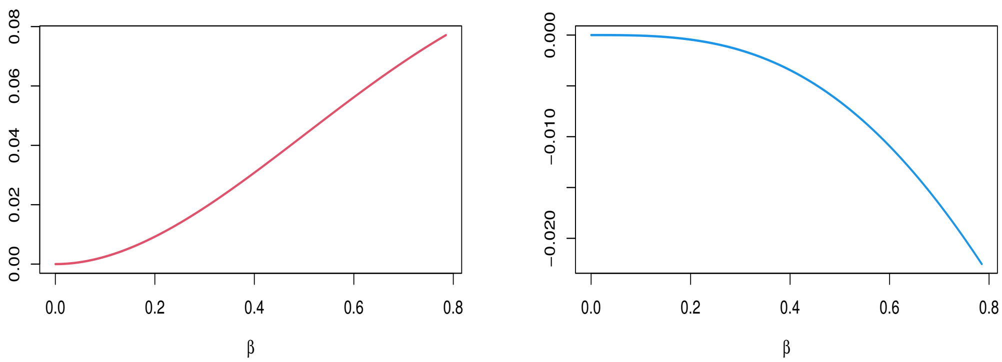

FIGURE 9. Illustration of (left) $g(\beta) \geq 0$, $\beta \in [0, \pi/4]$, where $g(\beta)$ is given in Equation (6.17), and (right) $h(\beta) \leq 0$, $\beta \in [0, \pi/4]$, where $h(\beta)$ is given in Equation (6.18)

The inequalities proved in this section form just a limited sample of the possible applications of the S and C operators; they were obtained with the application of only two general results, among all those established. So much more in this direction can be done.

7. CONCLUSION

In conclusion, integral operators provide powerful mathematical tools essential for analyzing and transforming functions, with applications in various disciplines. Although linear integral operators have received considerable attention, there is a notable gap in the exploration of multivariate nonlinear integral operators involving trigonometric functions. In this article, we emphasize two new multivariate nonlinear trigonometric integral operators. We highlight some of their promising features, such as solving complex functional and differential equations, allowing manageable series expansions, and generating sharp inequalities. As an application of some of the findings, a wide collection of original trigonometric inequalities is offered, giving key results that can be used in many fields. Graphical illustrations are given for visual evidence. New theoretical horizons are thus opened. A possible perspective is the study of the "inverse trigonometric versions" of our operator in the following general form:

$$"G(f)(\alpha, \beta) = \int_{\mathcal{X}} [f(\mathbf{x})]^\alpha g[\beta f(\mathbf{x})] d\mathbf{x}",$$

with $g(t) \in \{\arcsin(t), \arccos(t)\}$. Interesting choices also include $g(t) \in \{\text{cas}(t), \arctan(t)\}$. Finally, by addressing this under-explored area, we contribute to the understanding and use of multivariate nonlinear integral operators in analysis.

Conflict of interest statement. The author declares that there is no conflict of interests.

References

[1] A. Al-Aati, M. Hunaiber, Y. Ouideen, On triple Laplace-Aboodh-Sumudu transform and its properties with applications, J. Appl. Math. Comp. 6 (2022), 290-309.

[2] L. Angeloni, Approximation results with respect to multidimensional φ-variation for nonlinear integral operators, Z. Anal. Anwend. 32 (2013), 103-128.

[3] Y.J. Bagul, C. Chesneau, M. Kostić, The Cusa-Huygens inequality revisited, Novi Sad J. Math. 50 (2020), 149-159.

[4] C. Bardaro, J. Musielak, G. Vinti, Nonlinear integral operators and applications, Series in Nonlinear Analysis and Applications 9, Walter de Gruyter, Berlin, (2003).

[5] F.B.M. Belgacem, A.A. Karaballi, Sumudu transform fundamental properties investigations and applications, J. Appl. Math. Stoch. Anal. 2006 (2006), 91083.

[6] F.B. Belgacem, R. Silambarasan, Theory of natural transform, Math. Eng. Sci. Aerosp. 3 (2012), 99-124.

[7] N. Çetin, D. Costarelli, M. Natale, G. Vinti, Nonlinear multivariate sampling Kantorovich operators: quantitative estimates in functional spaces, Dolom. Res. Notes Approx. 15 (2022), 12-25.

[8] C. Chesneau, On a new multi-dimensional nonlinear integral operator: Theory and examples, Asian J. Math. Appl. 6 (2023), 1-22.

[9] C. Chesneau, Development of the generalized C operator: characteristics, inequalities and examples, Comm. Nonlinear Anal. to appear, 2024.

[10] C. Chesneau, A novel multivariate integral ratio operator: theory and applications including inequalities, Asian J. Math. Appl. 1 (2024), 1-37.

[11] A.R. Chouikha, On natural approaches related to classical trigonometric inequalities, Open J. Math. Sci. 7 (2023), 299-320.

[12] A.R. Chouikha, Sharp inequalities related to Wilker results, Open J. Math. Sci. 7 (2023), 19-34.

[13] R.M. Dhaigude, C. Chesneau, Y.J. Bagul, About trigonometric-polynomial bounds of sinc function, Math. Sci. Appl. E-Notes, 8 (2020), 100-104.

[14] L. Ehrenpreis, Nonlinear Fourier transform, Contemp. Math. 140 (1992), 39-48.

[15] T.M. Elzaki, The new integral transform Elzaki transform, Glob. J. Pure Appl. Math. 7 (2011), 57-64.

[16] T.M. Elzaki, A. Mousa, On the convergence of triple Elzaki transform, SN Appl. Sci. 1 (2019), 275-279.

[17] H. Faris, A. Alameer, A.H. Mohammed, New integral transformation with some of its uses, J. Data Acqui. Process. 38 (2023), 2999-3009.

[18] A. Georgieva, Application of double fuzzy Natural transform for solving fuzzy partial equations, AIP Conf. Proc. 2333 (2021), 080006.

[19] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge Mathematical Library, Cambridge: Cambridge University Press, (1988).

[20] C. Huygens, Oeuvres completes, Société Hollandaise des Sciences, Haga, 1897.

[21] K.S.K. Iyengar, B.S. Madhava Rao, T. Nanjundiah, Some trigonometrical inequalities, Half-Yearly J. Mysore Univ. B (N.S.), 6 (1945), 1-12.

[22] H. Jafari, A new general integral transform for solving integral equations, J. Adv. Res. 32 (2021), 133-138.

[23] Y. Kamimura, A nonlinear integral transform and a global inverse bifurcation theory, Ann. Sc. Norm. super. Pisa - Cl. Sci. 5 (2011), 863-911.

[24] T. Khan, S. Ahmad, G. Zaman, J. Alzabut, R. Ullah, On fractional order multiple integral transforms technique to handle three dimensional heat equation, Bound. Value Probl. 1 (2022), 16-18.

[25] A. Kilicman, H.E. Gadain, An application of double Laplace transform and double Sumudu transform, Lobachevskii J. Math. 30 (2009), 214-223.

[26] A. Kiliçman, H.E. Gadain, On the applications of Laplace and Sumudu transforms, J. Frankl. Inst. 347 (2010), 848-862.

[27] H. Kober, Approximation by integral functions in the complex domain, Trans. Amer. Math. Soc. 56 (1944), 7-31.

[28] R.P. Millane, Analytic properties of the Hartley transform, Proc. IEEE, 82 (1994), 413-428.

[29] D.S. Mitrinovic, Analytic inequalities, Springer-Verlag, Berlin, (1970).

[30] K. Narendra, P. Gallman, An iterative method for the identification of nonlinear systems using a Hammerstein model, IEEE Trans. Autom. Control. 11 (1966), 546-550.

[31] D.W. Niu, J. Cao, F. Qi, Generalizations of Jordan's inequality and concerned relations, UPB Sci. Bull. A: Appl. Math. Phys. 72 (2010), 85-98.

[32] K.J. Olejniczak, The Hartley transforms, The Transforms and Applications Handbook: Second Edition, Ed. Alexander D. Poularikas Boca Raton: CRC Press LLC, (2000).

[33] R.Z. Saadeh, B.F. Ghazal, A new approach on transforms: formable integral transform and its applications, Axioms, 10 (2021), 332.

[34] J. Sándor, On new refinements of Kober's and Jordan's trigonometric inequalities, Numb. Thy. Disc. Math. 19 (2013), 73-83.

[35] N. Sundararajan, S. Vijayachitra, Multidimensional Fourier and Hartley transforms - Are they same?, IETE J. Res. 46 (2000), 125-127.

[36] D.V. Widder, Laplace transform (PMS-6), Princeton University Press, Princeton, NJ, USA, (2015).

[37] F. Yuefeng, Proof without words, Math. Mag. 69 (1996), 126.

[38] L. Zhu, Sharpening Jordan's inequality and the Yang Le inequality, II, Appl. Math. Lett. 19 (2006), 990-994.