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LERAY SCHAUDER TYPE FIXED POINT THEOREMS IN RWC-BANACH ALGEBRAS AND APPLICATION TO CHANDRASEKHAR INTEGRAL EQUATIONS

KHALED BEN AMARA

ABSTRACT. In this paper, the existence of fixed point results of Leray Schauder type for the sum and the product of nonlinear operators acting on RWC-Banach algebras under weak topology is proved. Our results are formulated in terms of a sequential characterization of the RWC-Banach algebra and the De Blasi measure of weak noncompactness. Application to Chandrasekhar Integral equations is also given.

1. INTRODUCTION

Many mathematical models are expressed as operator equations of fixed point type:

$$A(x) \cdot B(x) + C(x) = x, \tag{1.1}$$

associated with some nonlinear operators A, B, and C acting on Banach algebras.
Fixed point theory offer a lot of principles and methods for the existence, uniqueness, and approximation of solutions. In [19], a mixed method combines the contraction principal and the Schauder fixed point theorem was used in order to resolve (1.1). Subsequently, many extensions and development of this method in the strong and the weak topologies are developped, see [8, 12, 13, 24].
Because the weak topology represent a natural framework of different applied problems, and since the sequential weak continuity property of the product of two sequentially weakly continuous operators is not always conserved, Ben Amar et al in [8] have introduced a new class of Banach algebras satisfying certain sequential condition, called condition $(\mathcal{P})$. This condition now represent a key to overcome the last cited problem of sequential weak continuity and it plays an important role to solve a large class of nonlinear integro-differential equations; see, for example, [8, 10, 12, 23, 24, 26]. The Leray Schauder alternative, one of the most important tools in nonlinear analysis. Many fixed point theorems of Leray Schauder type are obtained on Banach algebras satisfying the condition $(\mathcal{P})$, see [2, 7, 9, 11].

Recently, in [6] Banaś and Olszowy have introduced the class of RWC-Banach algebras which generalizes that of Banach algebras satisfying the condition $(\mathcal{P})$, see [13, Example 2.1 and Example 2.2] and [6, Example 2.1].
In this paper, we establish Leray Schauder type fixed point results for (1.1) where A, B and C are three nonlinear operators acting on a closed, convex subset of a RWC-Banach algebra. This extends, improves and generalizes the above cited works. Our results are applied to prove the existence of a continuous

DEPARTMENT OF MATHEMATICS. FACULTY OF SCIENCES OF SFAX, UNIVERSITY OF SFAX, SFAX, 3000, TUNISIA
E-mail address: khaled.benamara.etud@fss.usf.tn.
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solution for the Chandrasekhar integral equation:

$$x(t) = (\Xi_1 x)(t)\left[\left(\gamma(t) + \int_0^{\sigma(t)} \frac{t}{t+s}\kappa(s, x(\mu(s)))ds\right)\cdot u\right] + (\Xi_2 x)(t), \tag{1.2}$$

for $t \in J$, where $J = [0, 1]$, X is a RWC-Banach algebra, and $u \neq 0$ is a fixed vector in X.
When $\gamma = 0, \sigma_1 = \mu = 1$, $\kappa(t, x) = \lambda\phi(t)\log(1 + |x|)$, $\Xi_1(x) = x$ and $\Xi_2(x) \equiv u \equiv 1$, Equation (4.1) reduced to the following Chandrasekhar quadratic integral equation

$$x(t) = 1 + x(t)\int_0^1 \frac{t\lambda}{t+s}\phi(s)\log(1 + |x(s)|)\,ds, t \in J. \tag{1.3}$$

Equation (1.3) was studied and developed in [4, 15, 17, 21].
This paper is organized as follows. Section 2 is devoted to recall useful preliminary results. In Section 3, we establish new variants Leray Schauder alternative for the fixed point of (1.1), which extends several works to RWC-Banach algebras. In Section 4, we use our theoretical results to study the existence of a continuous solution of the Chandrasekhar integral equation (1.2) in the Banach algebra $C(J, X)$, where X is a RWC-Banach algebra.

2. Preliminary Results

Let X be a Banach space with the norm $\|\cdot\|$. We denote by B_r the closed ball of X centered at θ with radius r. Here θ is the zero element X. We write $co(M)$, and $\overline{co}(M)$ to denote the convex hull and the closed convex hull of a subset $M \subset X$, respectively. The symbol $\rightarrow$ (resp. $\rightharpoonup$) stands to denote the strong (resp. the weak) convergence in X.

Definition 2.1. Let $A : X \longrightarrow X$. We say that A is $\mathcal{D}$-Lipschitzian, if there exists a continuous nondecreasing function $\Psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\psi(0) = 0$ such that

$$\|Ax - Ay\| \leq \Psi(\|x - y\|)$$

for all $x, y \in X$. If $\Psi(r) < r$ for $r > 0$ and Ψ is not necessarily nondecreasing, we say that A is a nonlinear contraction mapping.

Definition 2.2. Let $A : X \longrightarrow X$.

(1) We say that A is weakly sequentially continuous, if for every sequence $(x_n)_n \subset X$ such that $x_n \rightharpoonup x$, we have $A(x_n) \rightharpoonup A(x)$.
(2) We say that A is weakly compact, if $A(M)$ is relatively weakly compact, for every bounded subset M of X.
(3) We say that A is ww-compact, if A is continuous and for every sequence $(x_n)_n \subset X$ such that $x_n \rightharpoonup x$, we have $(A(x_n))_n$ has a weakly convergent subsequence.
(4) We say that A is strongly continuous, if for every sequence $(x_n)_n \subset X$ such that $x_n \rightharpoonup x$, we have $A(x_n) \rightarrow A(x)$.

Notice that the ww-compact operators are not necessarily weakly sequentially continuous. In addition, the concepts of ww-compact and strongly continuous mappings arise naturally in the study of integral and partial differential equations, see [20, 25, 28, 29].

In [18], De Blasi introduced the notion of the measure of weak noncompactness in the following way:

$$\beta(M) := \inf\{\varepsilon > 0;\ \text{there exist a weakly compact set } K \text{ and } \varepsilon > 0 : M \subset K + B_\varepsilon\}$$

for all $M \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ is the set of all continuous subset of X.
This measure satisfies several useful properties such as:

(1) $\beta(M_1 + M_2) \leq \beta(M_1) + \beta(M_2)$ for all $M_1, M_2 \in \mathcal{B}(X)$.
(2) $\beta(M_1 \cup M_2) = \max(\beta(M_1), \beta(M_2))$ for all $M_1, M_2 \in \mathcal{B}(X)$.
(3) $\beta(\alpha M) = \alpha\,\beta(M)$ for all $\alpha > 0$ and $M \in \mathcal{B}(X)$.

Definition 2.3. An operator $A : X \longrightarrow X$ is said to be β-condensing or condensing with respect to β, if A is bounded, i.e. maps bounded subsets into bounded ones, and

$$\beta\left(A(M)\right) < \beta(M)$$

for any bounded subset M of X such that $\beta(M) > 0$.

Definition 2.4. Let $A : X \to X$.

(1) We say that A satisfies the condition $(\mathcal{H}_1)$ if for any weakly convergent sequence $(x_n)_n \subset D(A) \subset X$, the sequence $\left(A(x_n)\right)_n$ has a strongly convergent subsequence.
(2) We say that A satisfies the condition $(\mathcal{H}_2)$ if for any weakly convergent sequence $(x_n) \subset D(A) \subset X$, $\left(A(x_n)\right)_n$ has a weakly convergent subsequence in X.

Remark 2.5. Notice that the condition $(\mathcal{H}_1)$ does not imply the compactness of A even if A is a linear operator. Moreover, the condition $(\mathcal{H}_1)$ or $(\mathcal{H}_2)$ does not implies the weak continuity of the operator.

Lemma 2.6. *[1] If $A : X \to X$ is a $\mathcal{D}$-Lipschitzian operator with $\mathcal{D}$-function φ such that A satisfies the condition $(\mathcal{H}_2)$, then we have*

$$\beta\left(A(M)\right) \leq \varphi\left(\beta(M)\right)$$

for all bounded subset M of X.

Let X be a Banach algebra. For any arbitrary nonempty sets M and N of a Banach algebra X, we put

$$M \cdot N = \{x \cdot y;\ x \in M,\ y \in N\}$$

and if M is bounded we put

$$\|M\| = \sup_{x \in M} \|x\|.$$

Definition 2.7. We say that a Banach algebra X is weakly compact (WC-Banach algebra for short) if the product of two arbitrary weakly compact subsets of X is weakly compact.

An equivalence of the concepts of weak compactness of Banach algebra and Banach algebras satisfying the condition $(\mathcal{P})$ was proved by J. Banas, and L. Olszowy in [6].

Definition 2.8. [6] A Banach algebra X is said to be relatively weakly compact (RWC-Banach algebra for short) if the product of two arbitrary relatively weakly compact subsets of X is relatively weakly compact.

Remark 2.9. Notice that, every WC-Banach algebra or equivalently every Banach algebra satisfying the condition $(\mathcal{P})$ is a RWC-Banach algebra, and from the Kakutani Theorem it follows that every reflexive Banach algebra is a RWC-Banach algebra. In addition, the space $L_1(J;X)$ of all Bochner integrable functions $f : J \to X$ endowed with $\|\cdot\|_1$-norm and the convolution product multiplication, (resp. the space $C(K,X)$ of all continuous functions from a Hausdorff compact space K to X is a RWC-Banach algebra whenever X is reflexive (resp. X is a RWC-Banach algebra), see [13].

A characterization of the RWC-Banach algebra was investigated in [13].

Definition 2.10. Let X be a Banach algebra. We say that X satisfies a sequential condition $(\overline{\mathcal{P}})$ if for any weakly convergent sequences $(x_n)_n$ and $(y_n)_n$ of X, the sequence $(x_n \cdot y_n)_n$ has a weakly convergent subsequence.

A sequential characterization of the RWC-Banach algebras was established as follows.

Theorem 2.11. *[13] A Banach algebra X is a RWC-Banach algebra, if and only, if X satisfies the condition $(\overline{\mathcal{P}})$.*

Lemma 2.12. *[13] Let X be a RWC-Banach algebra. Then, we have*

$$\beta(M \cdot N) \leq \|M\|\beta(N) + \|N\|\beta(M) + \beta(M)\beta(N)$$

for all bounded subsets M and N of X.

3. Leray Schauder Fixed Point Results for Hybrid Operator Equation in RWC-Banach Algebras

In this Section, we establish a nonlinear alternative of Leray Schauder type of the fixed point theorems for the sum and the product of three nonlinear operators.

Lemma 3.1. *Let Ω be a nonempty, closed and convex subset of a Banach algebra X, and let U be a weakly open subset of Ω, and such that $0 \in U$. Let $A, C : X \longrightarrow X$ and $B : \overline{U^w} \longrightarrow X$ such that*

1. *A and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions Φ_A and Φ_C respectively.*
2. *$B(\overline{U^w})$ is bounded.*
3. *$\|B(\overline{U^w})\| \, \Phi_A(r) + \Phi_C(r) < r, r > 0$.*

Then, for every $y \in \overline{U^w}$ there exist an unique x such that $x = A(x) \cdot y + C(x)$.

Proof. It suffices to prove that, for each $y \in B(\overline{U^w})$, the fixed point problem

$$x = A(x) \cdot y + C(x)$$

has a unique solution in X. To do this, let $x_1, x_2 \in X$. We have

$$\begin{aligned} &\|A(x_1) \cdot y - A(x_2) \cdot y + C(x_1) - C(x_2)\| \\ &\leq \|A(x_1) - A(x_2)\| \, \|y\| + \|C(x_1) - C(x_2)\| \\ &\leq \|B(\overline{U^w})\| \, \Phi_A(\|x_1 - x_2\|) + \Phi_C(\|x_1 - x_2\|). \end{aligned}$$

Then, the operator $A(\cdot) \cdot y + C(\cdot) : X \to X$ defines a nonlinear contraction with $\mathcal{D}$-function

$$\Psi(r) = \|B(\overline{U^w})\| \, \Phi_A(r) + \Phi_C(r), r \geq 0.$$

Applying the Boyd-Wong fixed point theorem [14], there exists a unique $x_y \in X$ such that

$$A(x_y) \cdot y + C(x_y) = x_y.$$

□

Let $T : B(\overline{U^w}) \to X$ be the operator which assigns for each $y \in B(\overline{U^w})$ the element x_y, and define the operator $S : \overline{U^w} \to X$ by the formula

$$S(x) = T(B(x)) \text{ for all } x \in \overline{U^w}.$$

Now, we are going to state and prove our main results for the nonlinear operator equations (1.1).

Theorem 3.2. *Let X be a RWC-Banach algebra, $\Omega \subset X$ be a closed convex subset, $U \subset \Omega$ be a weakly open set, and such that $0 \in U$. Let $A, C : X \longrightarrow X$ be two operators satisfying Condition $(\mathcal{H}_2)$, and let $B : \overline{U^w} \longrightarrow X$ such that:*

1. *A and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions Φ_A and Φ_C respectively.*
2. *$B(\overline{U^w})$ is relatively weakly compact and $S(\overline{U^w})$ is bounded.*
3. *If $x_n \subset \overline{U^w} \rightharpoonup x$ and $S(x_n) \rightharpoonup y$, then $y = A(y) \cdot Bx + C(y)$.*
4. *$\|B(\overline{U^w})\| \, \Phi_A(r) + \Phi_C(r) < r, r > 0$.*

Then, either

($\mathcal{A}_1$) $A(x) \cdot B(x) + C(x) = x$ *has a solution, or*

($\mathcal{A}_2$) *there exist* $x \in \partial_\Omega(U)$ *and* $\lambda \in (0,1)$ *such that* $A\left(\frac{x}{\lambda}\right) \cdot B(x) + C\left(\frac{x}{\lambda}\right) = \frac{x}{\lambda}$.

Proof. First, note that the equation $A\left(\frac{x}{\lambda}\right) \cdot B(x) + C\left(\frac{x}{\lambda}\right) = \frac{x}{\lambda}$ is equivalent to that $\lambda S(x) = x$. Now, suppose that ($\mathcal{A}_2$) does not occur and S has no fixed point in $\partial_\Omega(U)$ (otherwise, we are finished since ($\mathcal{A}_1$) occurs). Then for all $x \in \partial_\Omega(U)$ and for all $\lambda \in [0,1]$, we have

$$\lambda S(x) \neq x.$$

Let

$$M = \left\{x \in \overline{U^w}, \lambda S(x) = x \text{ for some } \lambda \in [0,1]\right\}.$$

The set M is nonempty since $0 \in M$, and $M \cap \partial_\Omega(U) = \emptyset$. We first claim that M is relatively weakly compact. If it is not the case, then $\beta(M) > 0$. Since $M \subset \overline{co}(S(M) \cup \{0\})$, by the properties of the De Blasi measure we get

$$\beta(M) \leq \beta\left(\overline{co}(S(M) \cup \{0\})\right) = \beta\left(S(M)\right) < \beta(M),$$

which is absurd. We next prove that M is weakly closed. To do this, we claim first that S is weakly sequentially continuous. Indeed, let $\{x_n, n \in \mathbb{N}\}$ be a sequence of $\overline{U^w}$ such that $x_n \rightharpoonup x$. Since $B(\overline{U^w})$ is relatively weakly compact, by using Inclusion

$$S(\overline{U^w}) \subset A(S(\overline{U^w})) \cdot B(\overline{U^w}) + C(S(\overline{U^w}))$$

and the fact that X is a RWC-Banach algebra, we get

$$\begin{aligned}\beta\left(S(\overline{U^w})\right) &\leq \left\|B(\overline{U^w})\right\| \beta\left(A\left(S(\overline{U^w})\right)\right) + \beta\left(C\left(S(\overline{U^w})\right)\right)\\ &\leq \left\|B(\overline{U^w})\right\| \Phi_A\left(\beta\left(S(\overline{U^w})\right)\right) + \Phi_C\left(\beta\left(S(\overline{U^w})\right)\right).\end{aligned}$$

Using a contradiction argument, we obtain that $S(\overline{U^w})$ is relatively weakly compact, and so $\left(S\left(x_n\right)\right)_n$ has a subsequence, say $\left(S\left(x_{n_k}\right)\right)_k$, converges weakly to some $y \in X$. By hypothesis, we have $y = A(y) \cdot B(x) + C(y)$, or equivalently $y = S(x)$. Now, we will prove that the whole sequence $\{S(x_n), n \in \mathbb{N}\}$ converges weakly to $S(x)$. Otherwise, there exists a subsequence $\left(x_{n_j}\right)_j$ of $\{x_n, n \in \mathbb{N}\}$ and a weak neighborhood V of $S(x)$ such that

$$S\left(x_{n_j}\right) \notin V, \text{ for all } j \in \mathbb{N}.$$

Proceeding as above, we conclude the existence of a subsequence $\left(x_{n_{j_k}}\right)_k$ such that $S\left(x_{n_{j_k}}\right) \rightharpoonup S(x)$, which is a contradiction and consequently the claim is proved.

Now, let $x \in \overline{M^w}$. Taking into account the fact that $\overline{M^w}$ is weakly compact, in view of the Eberlein-Smulian Theorem, there exists a sequence $(x_n)_n$ such that $x_n \rightharpoonup x$. Notice that for each integer $n \in \mathbb{N}$, there is $\lambda_n \in [0,1]$ such that

$$x_n = \lambda_n S(x_n).$$

By extracting a subsequence, if necessary, we assume that

$$\lambda_n \to \lambda \in [0,1].$$

Taking into account the sequential weak continuity of S and letting $n \longrightarrow \infty$, we get $\lambda S(x) = x$, and so $x \in M$, and consequently the set M is weakly closed.

Keeping in mind that $M \cap \partial_\Omega(U) = \emptyset$, M is weakly compact, and $\partial_\Omega(U)$ is weakly closed, since X endowed with the weak topology is a Tychonoff space, the Urysohn Theorem for the weak topology [22] ensures the existence of a weakly continuous mapping $\varphi : X \to [0,1]$ such that

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in M; \\ 0, & \text{if } x \in \partial_\Omega(U). \end{cases}$$

Consider the operator $S_1 : X \to X$ given by

$$S_1(x) = \begin{cases} \varphi(x)S(x), & \text{if } x \in \overline{U^w}; \\ 0, & \text{if } x \in X \setminus \overline{U^w}. \end{cases}$$

It is clear that

$$S_1(X) \subset \overline{co}(S(\overline{U^w}) \cup \{0\}). \tag{3.1}$$

Let $C = \overline{co}(S(\overline{U^w}) \cup \{0\})$. It is easy to see that C is a convex closed subset of X, and $S_1(C) \subset C$. Let

$$L = \{V \subset C \text{ such that } \overline{co}(V) \subset V, 0 \in V, S_1(V) \subset V\}.$$

The set L is nonempty since $C \in L$. Define the set

$$O = \bigcap_{V \in L} V.$$

We can see that O is a closed convex subset of C, and $S_1(O) \subset O$. This implies that

$$\overline{co}(S_1(O) \cup \{0\}) \subset O,$$

from which we get

$$S_1(\overline{co}(S_1(O) \cup \{0\})) \subset S_1(O) \subset \overline{co}(S_1(O) \cup \{0\}).$$

Consequently,

$$\overline{co}(S_1(O) \cup \{0\}) \in L$$

and so

$$O \subset \overline{co}(S_1(O) \cup \{0\}).$$

This infer that

$$O = \overline{co}(S_1(O) \cup \{0\}) \subset \overline{co}(S(O) \cup \{0\}),$$

hence

$$\beta(O) \leq \beta(\overline{co}(S(O) \cup \{0\})) = \beta(S(O)).$$

Since $S(\overline{U^w})$ is relatively weakly compact and O is closed, we deduce that O is weakly compact.
The Arino-Gautier-Penot fixed point theorem [5], implies that there exists $u \in O$ such that $S_1(u) = u$. Now $u \in \overline{U^w}$ since $0 \in \overline{U^w}$. Hence, $\varphi(u)S(u) = u$, and so $S(u) = u$. □

In the following Lemma, we prove that under the conditions of Lemma 3.1, if B is strongly continuous then S also is.

Lemma 3.3. *If B is strongly continuous, then*

(1) *S is strongly continuous, and*
(2) *if $x_n \subset \overline{U^w} \rightharpoonup x$ and $S(x_n) \rightharpoonup y$, then $y = A(y) \cdot B(x) + C(y)$.*

Proof. (1) Let $\{x_n; n \in \mathbb{N}\}$ be a sequence of $\overline{U^w}$ such that $x_n \rightharpoonup x$. Let $y_n = S(x_n)$ and $y = S(x)$. Using (3.2) we infer that

$$\begin{aligned}\|S(x_n) - S(x)\| &= \|A(S(x_n)) \cdot B(x_n) + C(S(x_n)) - A(S(x)) \cdot B(x) - C(S(x))\| \\ &\leq \|A(S(x_n)) - A(S(x))\| \, \|B(x_n)\| + \|A(S(x))\| \, \|B(x_n) - B(x)\| \\ &+ \|C(S(x_n)) - C(S(x))\| \\ &\leq \left\|B(\overline{U^w})\right\| \Phi_A(\|S(x_n) - S(x)\|) + \|A(S(\overline{U^w}))\| \, \|B(x_n) - B(x)\| \\ &+ \Phi_C(\|S(x_n) - S(x)\|).\end{aligned}$$

Consequently, we get

$$\begin{aligned}\|S(x_n) - S(x)\| \leq \left\|B(\overline{U^w})\right\| \, \Phi_A(\|S(x_n) - S(x)\|) - \Phi_C(\|S(x_n) - S(x)\|) \\ \leq \|A(S(\overline{U^w}))\| \, \|B(x_n) - B(x)\|.\end{aligned}$$

Since B is strongly continuous, then there exists $r \geq 0$ such that

$$\begin{aligned}&\lim_n \|S(x_n) - S(x)\| \\ &= \lim_n \left[\|B(\overline{U^w})\| \, \Phi_A(\|S(x_n) - S(x)\|) + \Phi_C(\|S(x_n) - S(x)\|)\right] = r.\end{aligned}$$

On the other hand, since Φ_A and Φ_C are continuous, if $r > 0$ then

$$\begin{aligned}&\lim_n \left[\left\|B(\overline{U^w})\right\| \, \Phi_A(\|S(x_n) - S(x)\|) + \Phi_C(\|S(x_n) - S(x)\|)\right] \\ &= \left(\left\|B(\overline{U^w})\right\| \, \Phi_A(r) + \Phi_C(r)\right) < r.\end{aligned}$$

This contradiction implies that $r = 0$, and consequently $S(x_{n_k}) \rightharpoonup S(x)$. By uniqueness of limit, we conclude that $y = S(x)$, or equivalently $y = A(y) \cdot B(x) + C(y)$. An application of Theorem 3.2 yields to the desired results.
(2) Let $\{x_n; n \in \mathbb{N}\}$ be a sequence of $\overline{U^w}$ such that $x_n \rightharpoonup x$ and $S(x_n) \rightharpoonup y$. Using assertion (1), we have S is strongly continuous. By uniqueness of limit, this implies that $S(x) = y$. Since $S(x) = A(S(x)) \cdot B(x) + C(S(x))$, we conclude that $y = A(y) \cdot B(x) + C(y)$. □

A combination of Lemma 3.3 together with Theorem 3.2 enables as to obtain the following result.

Corollary 3.4. *Let X be a RWC-Banach algebra, $\Omega \subset X$ be a closed convex subset, $U \subset \Omega$ be a weakly open set, and such that $0 \in U$. Let $A, C : X \longrightarrow X$ be two operators satisfying $(\mathcal{H}_2)$ and $B : \overline{U^w} \longrightarrow X$ be stronly continuous such that:*

1. *$B(\overline{U^w})$ is relatively weakly compact and $S(\overline{U^w})$ is bounded.*
2. *A and C are $\mathcal{D}$-Lipschitzian with functions Φ_A and Φ_C respectively,*
3. *$\left\|B(\overline{U^w})\right\| \, \Phi_A(r) + \Phi_C(r) < r, r > 0.$*

Then, either

- $(\mathcal{A}_1)$ *$A(x) \cdot B(x) + C(x) = x$ has a solution, or*
- $(\mathcal{A}_2)$ *there exist $x \in \partial_\Omega(U)$ and $\lambda \in (0,1)$ such that $A\left(\frac{x}{\lambda}\right) \cdot B(x) + C\left(\frac{x}{\lambda}\right) = \frac{x}{\lambda}$.*

If B satisfies Condition $(\mathcal{H}_1)$ and is weakly sequentially continuous, thus it is strongly continuous. Then, we obtain the following result.

Corollary 3.5. *Let X be a RWC-Banach algebra, $\Omega \subset X$ be a closed convex subset, $U \subset \Omega$ be a weakly open set, and such that $0 \in U$. Let $A, C : X \longrightarrow X$ be two operators satisfying $(\mathcal{H}_2)$ and $B : \overline{U^w} \longrightarrow X$ be a weakly sequentially continuous operator satisfying $(\mathcal{H}_1)$ such that*

1. *A and C are $\mathcal{D}$-Lipschitzian with functions Φ_A and Φ_C respectively,*
2. *$B(\overline{U^w})$ is relatively weakly compact and $S\left(\overline{U^w}\right)$ is bounded,*

(3) $\|B(\overline{U^w})\|\,\Phi_A(r) + \Phi_C(r) < r, r > 0.$

Then, either

($\mathcal{A}_1$) $A(x)\cdot B(x) + C(x) = x$ *has a solution, or*

($\mathcal{A}_2$) *there exist* $x \in \partial_\Omega(U)$ *and* $\lambda \in (0,1)$ *such that* $A\left(\frac{x}{\lambda}\right)\cdot B(x) + C\left(\frac{x}{\lambda}\right) = \frac{x}{\lambda}$.

If A and C are Lipschitzian, we obtain the following result.

Theorem 3.6. *Let X be a RWC-Banach algebra, $\Omega \subset X$ be a closed convex subset, $U \subset \Omega$ be a weakly open set, and such that $0 \in U$. Let $A, C : \Omega \longrightarrow X$ be nonlinear operators satisfying $(\mathcal{H}_2)$, and $B : \overline{U^w} \longrightarrow X$ be strongly continuous such that:*

(1) *A and C are Lipschitzian with constants α and γ respectively,*

(2) *$B(\overline{U^w})$ is relatively weakly compact,*

(3) $\alpha\,\|B(\overline{U^w})\| + \gamma < 1.$

Then, either

($\mathcal{A}_1$) *there exist $x \in \Omega$ such that $Ax \cdot Bx + Cx = x$, or*

($\mathcal{A}_2$) *there exist $x \in \partial_\Omega(U)$ and $0 < \lambda < 1$ such that* $A\left(\frac{x}{\lambda}\right)\cdot B(x) + C\left(\frac{x}{\lambda}\right) = \frac{x}{\lambda}$.

Proof. Arguing as in the proof of Lemma 3.1, the operator S is well defined from $\overline{U^w}$ into Ω. Let $(x_n)_n \subset \overline{U^w}$ be a weakly convergent subsequence to x, $x \in \overline{U^w}$. Put $y = S(x)$ and $y_n = S(x_n)$ for all $n \in \mathbb{N}$. Using the fact that

$$S(x) = A(S(x))\cdot B(x) + C(S(x))$$

together with assumption (i), we infer that

$$\begin{aligned}
\|y_n - y\| &= \|A(y_n)\cdot B(x_n) + C(y_n) - A(y)\cdot B(x) - C(y)\| \\
&\le \|A(y_n) - A(y)\|\,\|B(x_n)\| + \|A(y)\|\,\|B(x_n) - B(x)\| + \|C(y_n) - C(y)\| \\
&\le \|B(\overline{U^w})\|\|A(y_n) - A(y)\| + \|A(\Omega)\|\|B(x_n) - B(x)\| + \|C(y_n) - C(y)\| \\
&\le \alpha\,\|B(\overline{U^w})\|\,\|y_n - y\| + \|A(\Omega)\|\,\|B(x_n) - B(x)\| + \gamma\,\|y_n - y\|.
\end{aligned}$$

Thus,

$$\left(1 - (\alpha\,\|B(\overline{U^w})\| + \gamma)\right)\,\|y_n - y\| \le \|A(\Omega)\|\,\|B(x_n) - B(x)\|,$$

which implies that

$$\|y_n - y\| \le \frac{\|A(\Omega)\|}{1 - (\alpha\,\|B(\overline{U^w})\| + \gamma)}\|B(x_n) - B(x)\|.$$

Hence, $y_n \longrightarrow y$, by using the fact that B is strongly continuous, and in particular, S is weakly sequentially continuous. On the other hand, we can claim that $S(\overline{U^w})$ is bounded. In fact, since

$$S(x) = A(S(x))\cdot B(x) + C(S(x)),\ \text{for all } x \in \overline{U^w},$$

then

$$\begin{aligned}
\|S(x) - S(x_0)\| &\le \|A(S(x))\cdot B(x) - A(S(x_0))\cdot B(x_0)\| + \|C(S(x)) - C(S(x_0))\| \\
&\le \|A(S(x))\cdot B(x) - A(S(x_0))\cdot B(x)\| \\
&+ \|A(S(x_0))\cdot B(x) - A(S(x_0))\cdot B(x_0)\| + \|C(S(x)) - C(S(x_0))\| \\
&\le \|B(\overline{U^w})\|\alpha\|S(x) - S(x_0)\| + \|A(S(x_0))\|\|B(x) - B(x_0)\| \\
&+ \gamma\|S(x) - S(x_0)\|.
\end{aligned}$$

Consequently,

$$\|S(x) - S(x_0)\| \le \frac{\|A(S(x_0))\|\|B(x) - B(x_0)\|}{(1 - \|B(\overline{U^w})\|\alpha - \gamma)},$$

which implies that

$$\|S(x)\| \leq \|S(x) - S(x_0)\| + \|S(x_0)\| \leq \frac{\|A(S(x_0))\|\|B(x) - B(x_0)\|}{(1 - \|B(\overline{U^w})\|\alpha - \gamma)} + \|S(x_0)\|.$$

Hence,

$$\|S(\overline{U^w})\| \leq \frac{2\|A(S(x_0))\|\|B(\overline{U^w})\|}{(1 - \|B(\overline{U^w})\|\alpha - \gamma)} + \|S(x_0)\|.$$

Now proceeding essentially as in the proof of Theorem 3.2, we can obtain the desired result. □

Remark 3.7. Theorem 3.6 extend Corollary 2.4 in [27] to RWC-Banach algebras and relaxing the sequential weak continuity on A and C by assuming that A and C satisfy only Condition $(\mathcal{H}_2)$.

Remark 3.8. If X is a Dunford-Pettis space, the result of the above theorem remains true if we replace the assumption (2) by "B is a weakly compact linear operator on X."

Theorem 3.9. *Let X be a RWC-Banach algebra, $\Omega \subset X$ be a closed convex subset, $U \subset \Omega$ be a weakly open set, and such that $0 \in U$. Let $A, C : X \longrightarrow X$ and let $B : \overline{U^w} \longrightarrow X$ be nonlinear operators satisfying Condition $(\mathcal{H}_2)$ such that:*

(1) *A and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions Φ_A and Φ_C respectively,*
(2) *$B(\overline{U^w})$ is bounded and $\|B(\overline{U^w})\| \, \Phi_A(r) + \Phi_C(r) < r, r > 0$,*
(3) *$S : \overline{U^w} \to X$ is condensing with respect to the measure β, and $S(\overline{U^w})$ is bounded,*
(4) *If $\{x_n; n \in \mathbb{N}\} \subset \overline{U^w} \rightharpoonup x$ and $S(x_n) \rightharpoonup y$, then $y = A(y) \cdot B(x) + C(y)$.*

Then, either

$(\mathcal{A}_1)$ *$A(x) \cdot B(x) + C(x) = x$ has a solution, or*
$(\mathcal{A}_2)$ *there exist $x \in \partial_\Omega(U)$ and $\lambda \in (0,1)$ such that $A\left(\frac{x}{\lambda}\right) \cdot B(x) + C\left(\frac{x}{\lambda}\right) = \frac{x}{\lambda}$.*

Proof. Proceeding as in Theorem 3.2, and suppose that $(\mathcal{A}_2)$ does not occur and S has no fixed point in $\partial_\Omega(U)$ (otherwise, we are finished since $(\mathcal{A}_1)$ occurs). Then for all $x \in \partial_\Omega(U)$ and for all $\lambda \in [0,1]$, we have

$$\lambda S(x) \neq x.$$

Let

$$M = \left\{x \in \overline{U^w}, \lambda S(x) = x \text{ for some } \lambda \in [0,1]\right\}.$$

It is clear that M is nonempty and $M \cap \partial_\Omega(U) = \emptyset$. Using Inclusion

$$M \subset \overline{co}(S(M) \cup \{0\}),$$

we get

$$\beta(M) \leq \beta\left(\overline{co}(S(M) \cup \{0\})\right) = \beta\left(S(M)\right).$$

Taking into account the fact that S is condensing, by a contradiction argument we can obtain that $\beta(M) = 0$, or equivalently M is relatively weakly compact. We next prove that M is weakly closed. Let $x \in \overline{M^w}$. Since $\overline{M^w}$ is weakly compact the Eberlein-Smulian Theorem ensures the existence of a sequence $(x_n)_n$ such that $x_n \rightharpoonup x$. Notice that for each integer $n \in \mathbb{N}$, there exists $\lambda_n \in [0,1]$ such that $x_n = \lambda_n S(x_n)$. By extracting a subsequence, if necessary, we assume that $(\lambda_n)_n$ converges to some $\lambda \in [0,1]$. Without loss of generality, we may suppose that $\lambda \neq 0$. Otherwise since the set $\{S(x_n), n \in \mathbb{N}\}$ is bounded and $x_n = \lambda_n S(x_n) \rightharpoonup x$, we infer that $x = 0$, which implies that $\lambda S(x) = x$, and so $x \in M$. Now, since $\lambda > 0$, we infer that

$$S(x_n) \rightharpoonup \frac{x}{\lambda}.$$

From assumption (4), we get

$$\frac{x}{\lambda} = A(\frac{x}{\lambda}) \cdot B(x) + C(\frac{x}{\lambda}),$$

or equivalently $\lambda S(x) = x$, and so $x \in M$.
Keeping in mind that $M \cap \partial_\Omega(U) = \emptyset$, M is weakly compact, and $\partial_\Omega(U)$ is weakly closed since X endowed with the weak topology is a Tychonoff space, the Urysohn theorem for the weak topology [22] guarantees the existence of a weakly continuous mapping $\varphi : X \to [0, 1]$ such that

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in M; \\ 0, & \text{if } x \in \partial_\Omega(U). \end{cases}$$

We define the mapping $S_1 : X \to X$ by

$$S_1(x) = \begin{cases} \varphi(x)S(x), & \text{if } x \in \overline{U^w}; \\ 0, & \text{if } x \in X \setminus \overline{U^w}. \end{cases}$$

Furthermore, we readily check that

$$S_1(X) \subset \overline{co}(S(\overline{U^w}) \cup \{0\}). \tag{3.2}$$

Let $C = \overline{co}(S(\overline{U^w}) \cup \{0\})$. Clearly, C is a convex closed subset of X, and $S_1(C) \subset C$. Since $S(\overline{U^w})$ is bounded, so is $S(C)$. We consider the set

$$L = \{V \subset C \text{ such that } \overline{co}(V) \subset V, 0 \in V, T_1(V) \subset V\}.$$

The set L is nonempty since $C \in L$. Set $O = \bigcap_{V \in L} V$. Clearly, O is a closed convex subset of C, and $S_1(O) \subset O$. This implies that

$$\overline{co}(S_1(O) \cup \{0\}) \subset O,$$

from which we get

$$S_1(\overline{co}(S_1(O) \cup \{0\})) \subset S_1(O) \subset \overline{co}(S_1(O) \cup \{0\}).$$

Then, we get

$$\overline{co}(S_1(O) \cup \{0\}) \in L,$$

and consequently

$$O \subset \overline{co}(S_1(O) \cup \{0\}).$$

This discussion enables to obtain

$$O = \overline{co}(S_1(O) \cup \{0\}) \subset \overline{co}(S(O) \cup \{0\}).$$

Arguing as above, we can infer that O is weakly compact.
Now, we claim that S_1 is sequentially weakly continuous on O. It is enough to prove the sequential weak continuity of S. Indeed, let $\{x_n, n \in \mathbb{N}\}$ be a sequence of O such that $x_n \rightharpoonup x$. Since B satisfies the condition $(\mathcal{H}_2)$, then $\{B(x_n), n \in \mathbb{N}\}$ has a weakly convergent subsequence, and consequently it is relatively weakly compact, in view of Eberlian-Sumilian's Theorem. Using the inclusion

$$\{S(x_n), n \in \mathbb{N}\} \subset A(\{S(x_n), n \in \mathbb{N}\}) \cdot \{B(x_n), n \in \mathbb{N}\} + C(\{S(x_n), n \in \mathbb{N}\})$$

and the fact that X is a RWC-Banach algebra, we get

$$\begin{aligned}\beta\left(\{S(x_n), n \in \mathbb{N}\}\right) &\leq \left\|B(\overline{U^w})\right\| \beta\left(A\left(\{S(x_n), n \in \mathbb{N}\}\right)\right) + \beta\left(C\left(\{S(x_n), n \in \mathbb{N}\}\right)\right) \\ &\leq \left\|B(\overline{U^w})\right\| \Phi_A\left(\beta\left(\{S(x_n), n \in \mathbb{N}\}\right)\right) + \Phi_C\left(\beta\left(\{S(x_n), n \in \mathbb{N}\}\right)\right).\end{aligned}$$

Using a contradiction argument, we obtain that $\{S(x_n), n \in \mathbb{N}\}$ is relatively weakly compact, and so it has a subsequence, say $\left(S\left(x_{n_k}\right)\right)_k$, converges weakly to some $y \in X$. By hypothesis, we have $y = A(y) \cdot B(x) + C(y)$, or equivalently $y = S(x)$. Now, we will prove that the whole sequence $\{S(x_n), n \in \mathbb{N}\}$ converges weakly to $S(x)$. Otherwise, there exists a subsequence $\left(x_{n_j}\right)_j$ of $\{x_n, n \in \mathbb{N}\}$ and a weak neighborhood V of $S(x)$ such that

$$S\left(x_{n_j}\right) \notin V, \text{ for all } j \in \mathbb{N}.$$

Proceeding as above, we conclude the existence of a subsequence $\left(x_{n_{j_k}}\right)_k$ such that

$$S\left(x_{n_{j_k}}\right) \rightharpoonup S(x),$$

which is a contradiction and consequently the claim is proved.
The Arino-Gautier-Penot fixed point theorem [5], implies that there exists $u \in O$ such that $S_1(u) = u$. Now $u \in \overline{U^w}$ since $0 \in \overline{U^w}$. Consequently, $\varphi(u)S(u) = u$, and so $S(u) = u$. □

A combination of Theorem 3.9 with Lemma 3.3 yields to the following result.

Corollary 3.10. *Let X be a RWC-Banach algebra, $\Omega \subset X$ be a closed convex subset, $U \subset \Omega$ be a weakly open set, and such that $0 \in U$. Let $A, C : X \longrightarrow X$ be nonlinear operators satisfying Condition $(\mathcal{H}_2)$ and $B : \overline{U^w} \longrightarrow X$ be a strongly continuous operator such that:*

1. *A and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions Φ_A and Φ_C respectively,*
2. *$B(\overline{U^w})$ is bounded and $\left\|B(\overline{U^w})\right\| \ \Phi_A(r) + \Phi_C(r) < r, r > 0$.*
3. *$S : \overline{U^w} \to X$ is condensing with respect to the measure β, and $S(\overline{U^w})$ is bounded.*

Then, either

- $(\mathcal{A}_1)$ *$A(x) \cdot B(x) + C(x) = x$ has a solution, or*
- $(\mathcal{A}_2)$ *there exist $x \in \partial_\Omega(U)$ and $\lambda \in (0,1)$ such that $A\left(\frac{x}{\lambda}\right) \cdot B(x) + C\left(\frac{x}{\lambda}\right) = \frac{x}{\lambda}$.*

Remark 3.11.

1. Arguing as in the proof of Theorem 3.6, if A and C are Lipschizian and $B(\overline{U^w})$ is bounded. Then $S(\overline{U^w})$ is so. Therefore, taking into account the fact that every Lipschitzian with constant $\alpha \geq 0$ is a $\mathcal{D}$-Lipschizian mapping with $\mathcal{D}$-function $\varphi(\cdot)$ where $\varphi(t) = \alpha t, t \geq 0$, then Corollary 3.12 extend Theorem 2.2 in [27] to RWC-Banach algebras and relaxing the sequential weak continuity on A and C, by assuming only that A and C satisfy only Condition $(\mathcal{H}_2)$.
2. Corollary 3.12 extend Theorem 3.1 in [2] to RWC-Banach algebras, and relaxing the sequential weak continuity of A and C, by assuming only that A and C satisfy only Condition $(\mathcal{H}_2)$. In contrast to Theorem 3.1 in [2], the operator A in our results needs not be regular.
3. When $C \equiv x_0$ for some $x_0 \in X$, Corollary 3.12 extend Theorem 3.2 in [2] to RWC-Banach algebras, and relaxing the sequential weak continuity of A and C, by assuming only that A and C satisfy only Condition $(\mathcal{H}_2)$. In contrast to Theorem 3.2 in [2], the operator A in our results needs not be regular.

Corollary 3.12. *Let X be a RWC-Banach algebra, $\Omega \subset X$ be a closed and convex subset, $U \subset \Omega$ be a weakly open set, and such that $0 \in U$. Let $A, C : X \longrightarrow X$ be nonlinear operators satisfying Condition $(\mathcal{H}_2)$ and $B : \overline{U^w} \longrightarrow X$ be a strongly continuous operator such that:*

1. *A and C are $\mathcal{D}$-Lipschitzian with functions Φ_A and Φ_C respectively,*
2. *B is condensing and $S(\overline{U^w})$ is bounded,*
3. *$S(\overline{U^w})$ is bounded, and $\left\|B(\overline{U^w})\right\| \ \Phi_A(r) + \Phi_C(r) < (1-\delta)\, r, r > 0$, for some $\delta \geq \|A(S(\overline{U^w}))\|$.*

Then, either

- $(\mathcal{A}_1)$ *$A(x) \cdot B(x) + C(x) = x$ has a solution, or*
- $(\mathcal{A}_2)$ *there exist $x \in \partial_\Omega(U)$ and $\lambda \in (0,1)$ such that $A\left(\frac{x}{\lambda}\right) \cdot B(x) + C\left(\frac{x}{\lambda}\right) = \frac{x}{\lambda}$.*

Proof. According to Theorem 3.9, it is enough to claim that S define a condensing mapping on $\overline{U^w}$. Indeed, let M be a bounded subset of $\overline{U^w}$ with $\beta(M) > 0$. Taking into account that X is a RWC-Banach algebra, in view of Lemma 2.12, it follows from Inclusion

$$S(M) \subset A(S(M)) \cdot B(M) + C(S(M))$$

that

$$\begin{aligned}\beta\left(S(M)\right) &\leq \|B(M)\|\ \beta\left(A\left(S(M)\right)\right) + \|A(S(\overline{U^w})\|\beta(B(M)) + \beta\left(C\left(S(M)\right)\right)\\ &\leq \left\|B(\overline{U^w})\right\|\ \Phi_A\left(\beta\left(S(M)\right)\right) + \|A(S(\overline{U^w})\|\beta(M) + \Phi_C\left(\beta\left(S(M)\right)\right).\end{aligned}$$

In view of our assumptions, we get

$$\delta\beta(S(M)) < \|A(S(\overline{U^w})\|\beta(M) \leq \delta\beta(M),$$

which achieves our claim and completes the proof. □

Remark 3.13. Corollary 3.12 extends Theorem 3.5 in [2] to RWC-Banach algebras. In contrast to Theorem 3.5 in [2], the operator A in our results needs not be regular nor weakly compact and B in our results needs not be weakly compact.

4. Application to Chandrasekhar Integral Equations

Let $(\mathcal{E}, \|\cdot\|)$ be a Banach algebra satisfying the condition $(\overline{\mathcal{P}})$ and $\mathcal{X} := \mathcal{C}(J, \mathcal{E})$ be the Banach algebra of all continuous functions from J into $\mathcal{E}$ endowed with the sup-norm $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = \sup_{t \in J} \|f(t)\|, f \in \mathcal{X}.$$

Consider the following Chandrasekhar integral equation in $\mathcal{X}$:

$$x(t) = (\Xi_1 x)(t)\left[\left(\gamma(t) + \int_0^{\sigma(t)} \frac{t}{t+s}\kappa(s, x(\mu(s)))ds\right) \cdot u\right] + (\Xi_2 x)(t), \tag{4.1}$$

for $t \in J$, where $J := [0, 1]$, u is a fixed vector of $\mathcal{E}$ such that $u \neq 0$ and $x = x(t)$ is an unknown function. The problem (4.1) will be studied under the following conditions:

- (A_1) $\Xi_1, \Xi_2 : \mathcal{X} \longrightarrow \mathcal{X}$ are Lipschitzian with Lipschitz constants α_1 and α_2, resp., and satisfy Condition $(\mathcal{H}_2)$,
- (A_2) the functions $\sigma, \mu : J \longrightarrow J$ and $\gamma : J \longrightarrow \mathbb{R}$ are continuous,
- (A_3) the mapping $\kappa : J \times \mathcal{E} \longrightarrow \mathbb{R}$ is weakly sequentially continuous with respect to the second variable,
- (A_4) there exists $r > 0$ such that $|\kappa(s, x)| \leq m(s) \in L^1(J)$ for all $s, x \in J \times \mathcal{E}$ such that $\|x\|_\infty \leq r$, and $c = \int_0^1 \frac{m(s)}{t+s} ds.$

Define the operators $A, C : \mathcal{E} \to \mathcal{E}$ and $B : B_r \to \mathcal{E}$ by:

$$\begin{cases} (Ax)(t) &= (\Xi_1 x)(t)\\ (Bx)(t) &= \left(\gamma(t) + \displaystyle\int_0^{\sigma(t)} \frac{t}{t+s}\kappa(s, x(\mu(s))\, ds\right) \cdot u\\ (Cx)(t) &= (\Xi_2 x)(t). \end{cases}$$

Theorem 4.1. *Assume that $(A_1) - (A_4)$ hold. Let $\Omega = B_r$ and U be a weakly open subset of Ω such that $0 \in U$. In addition, suppose that*

$$\lambda A\left(\frac{x}{\lambda}\right) \cdot B(x) + \lambda C\left(\frac{x}{\lambda}\right) \neq x$$

for all $\lambda \in (0, 1)$ and $x \in \partial_\Omega(U)$. Then the Chandrasekhar Integral Equation (4.1) has a solution in Ω.

Proof. In view of Theorem 2.2 in [13], $\mathcal{X}$ is a RWC-Banach algebra. We shall prove that A, B and C satisfy all conditions of Corollary 3.4. For this purpose, we need three steps:
(i) From assumption (A_1), it follows that A and C are Lipschitzian with Lipschitz constants α_1 and α_2, respectively.
(ii) The operator B is strongly continuous. Indeed, let $\{x_n\}_{n=1}^{\infty} \subset \overline{U^w}$ such that $x_n \rightharpoonup x \in \overline{U^w}$. By Dobrakov's theorem we infer that

$$x_n(t) \rightharpoonup x(t), \text{ for each } t \in J.$$

Let $t \in J$, we have

$$\|Bx_n(t) - Bx(t)\| \leq \|u\| \int_0^1 \frac{t}{t+s} |\kappa(s, x_n(s)) - \kappa(s, x(s))| \, ds. \tag{4.2}$$

Since $x \mapsto \kappa(t, x)$ is sequentially weakly continuous, then

$$\kappa(s, x_n(s)) \to \kappa(s, x(s)), \text{ for all } s \in J.$$

Using the dominated convergence theorem and taking the supremum over t, we get

$$B(x_n) \to B(x).$$

(iii) Now we prove that $B(\overline{U^w})$ is relatively weakly compact. By definition,

$$B(\overline{U^w}) := \{B(x), x \in \overline{U^w}\}.$$

For all $t \in J$, we have

$$B(\overline{U^w})(t) = \{(B(x))(t), x \in \overline{U^w}\}.$$

We claim that $B(\overline{U^w})(t)$ is sequentially relatively weakly compact in $\mathcal{E}$. To see this, let $\{x_n, \ n \in \mathbb{N}\}$ be any sequence in $\overline{U^w}$, we have $(B(x_n))(t) = r_n(t) \cdot u$, where

$$r_n(t) = \left(\gamma(t) + \int_0^{\sigma(t)} \frac{t}{t+s} \kappa(s, x_n(\mu(s)) \, ds \right).$$

It is clear that $\{r_n(t), n \in \mathbb{N}\}$ is a bounded, real sequence, so, by the Bolzano Weirstrass Theorem, there is a renamed subsequence such that

$$r_n(t) \longrightarrow r(t) \text{ in } \mathbb{R},$$

which implies

$$r_n(t) \cdot u \longrightarrow r(t) \cdot u \text{ in } \mathbb{R},$$

and, consequently,

$$(Bx_n)(t) \longrightarrow r(t) \cdot u \text{ in } \mathcal{E}.$$

Hence, we conclude that $B(\overline{U^w})(t)$ is sequentially relatively compact in $\mathcal{E}$, then $B(\overline{U^w})(t)$ is sequentially relatively weakly compact. Now, we have to prove that $B(\overline{U^w})$ is weakly equicontinuous on J. Let $\varepsilon > 0$, $x \in \overline{U^w}$, $x^* \in \mathcal{E}^*, t, t' \in J$ such that $|t - t'| \leq \varepsilon$.

$$\begin{aligned}
&\|x^*(B(x)(t) - B(x)(t'))\| \leq |x^*(u)|(|\gamma(t) - \gamma(t')|) \\
&+ |x^*(u)| \left| \int_0^{\sigma(t)} \frac{t}{t+s} \kappa(s, x(s)) ds - \int_0^{\sigma(t)} \frac{t}{t+s} \kappa(s, x(s)) ds \right| \\
&\leq |x^*(u)|(|\gamma(t) - \gamma(t')|) \\
&+ |x^*(u)| \left| \int_{\sigma(t')}^{\sigma(t)} \frac{t}{t+s} \kappa(s, x(s)) ds + \int_0^{\sigma(t')} \frac{t}{t+s} - \frac{t'}{t'+s} \kappa(s, x(s)) ds \right| \\
&\leq |x^*(u)|(|\gamma(t) - \gamma(t')|)
\end{aligned}$$

$$
\begin{aligned}
&+|x^*(u)|\left(\int_{\sigma(t')}^{\sigma(t)}\frac{t}{t+s}|\kappa(s,x(s))|ds+\int_0^{\sigma(t')}\left|\frac{t}{t+s}-\frac{t'}{t'+s}\right||m(s)|ds\right)\\
&\leq |x^*(u)|(|\gamma(t)-\gamma(t')|)\\
&+|x^*(u)|\left(\int_{\sigma(t')}^{\sigma(t)}\frac{t}{t+s}|\kappa(s,x(s))|ds+|t-t'|\int_0^1\frac{|m(s)|}{t+s}ds\right)\\
&\leq (l_1(\gamma,\varepsilon)+l_2(\sigma,\varepsilon)+l_3(m,\varepsilon))\,|x^*(u)|,
\end{aligned}
$$

where

$$l_1(\gamma,\varepsilon):=\sup\left\{|\gamma(t)-\gamma(t')|,t,t'\in J,|t-t'|\leq\varepsilon\right\},$$

$$l_2(\sigma,\varepsilon):=\sup\left\{\int_{\sigma(t')}^{\sigma(t)}\frac{t}{t+s}|\kappa(s,x(s))|ds,t,t'\in J,|t-t'|\leq\varepsilon,x\in\overline{U^w}\right\},$$

and

$$l_3(m,\varepsilon):=\sup\left\{|t-t'|\int_0^1\frac{|m(s)|}{t+s}ds,t,t'\in J,|t-t'|\leq\varepsilon\right\}.$$

Taking into account our assumptions and in view of the uniform continuity of the functions γ and σ, it follows that $l_1(\gamma,\varepsilon)\longrightarrow 0, l_2(\sigma,\varepsilon)\longrightarrow 0$ and $l_3(m,\varepsilon)\longrightarrow 0$ when $\varepsilon\longrightarrow 0$. Hence, the application of the Arzelà-Ascoli Theorem leads to have that $B(\overline{U^w})$ is sequentially relatively weakly compact in $\mathcal{X}$. Now, the Eberlein-Smulian Theorem yields that $B(\overline{U^w})$ is relatively weakly compact. On the other hand, in view of assertion (1) in Remark 3.11, we deduce that $S(\overline{U^w})$ is bounded.

The desired conclusion follows from a direct application of Corollary 3.4. The proof is complete. □

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