ISSN: 2832-4293

Pan-American Journal of Mathematics 3 (2024), 9
https://doi.org/10.28919/cpr-pajm/3-9

LYAPUNOV EXPONENTIAL STABILITY OF THE SHALLOW WATER EQUATIONS IN TRAPEZOIDAL CHANNEL

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ABSTRACT. This paper solves the problem of the exponential stability in L^2-norm of Saint-Venant equations linear hyperbolic system for a non-prismatic and non-rectangular channel. We consider the general case of systems containing not only both arbitrary friction and spatially varying slopes but also spatially varying channel dimensions (width and lateral slope), leading to non-uniform stationary states. An explicit quadratic Lyapunov function is constructed as a weighting function for steady-state small perturbations. We then show that local exponential stability of Saint-Venant equations linear system for a trapezoidal channel can be guaranteed in the L^2-norm by an appropriate choice of boundary feedback control. Finally, we give explicitly that control.

1. INTRODUCTION

The one-dimensional form of Saint-Venant's equation is one of the most widely used models in engineering for simulating shallow water flow. They are originally derived from Barré de Saint-Venant [1], [2], [3] in 1871. Despite their apparent simplicity, they capture a wealth of physical behavior that makes them fundamental tools for practical applications, notably regulating canals for agricultural management and regulating navigable rivers. Among others, the problem of developing control instruments for regulating water level and flow in open hydraulic systems has long been studied [26], [16], [20], [26].

The Saint-Venant equations constitute a nonlinear 2×2 1-D hyperbolic system. The stabilization of such systems by proportional or output boundary control have been studied for decades, one can cite for instance the pioneering work of Li and Greenberg [21] for a system of two homogeneous equations considered in the framework of the C^1 norm . This was later generalized by [16], [28], [22] to general nonhomogeneous systems and [6] in the framework of the H^2 norm. To our knowledge, the first result concerning the boundary stabilization of the Saint-Venant equations in themselves goes back to 1999 with [10] for the stability of the homogeneous linearizedsystem and its extension to the nonlinear homogeneous system in [11] , using proportional boundary conditions. Later, in 2008, using a semigroup approach and the method of the characteristics, the stabilization of the nonlinear homogeneous equations was achieved for sufficiently small friction and slope [20], [29] . The same type of result was shown in [9] using a Lyapunov approach while [8]

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Submitted on February 06, 2024.

2020 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 20C20.

Key words and phrases. Inhomogeneous equations of Saint-Venant, Trapezoidal channel, Lyapunov approach, Stabilization, Feedback control.

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dealt with the inhomogeneous Saint-Venant equations in the particular case where the steady-states are uniform. It is essential to note that Lyapunov approach, the quadratic Lyapunov function, is another method introduced in 1999 by Coron et al. [11] that guarantee the exponential stability of linear and nonlinear homogeneous hyperbolic systems when the boundary conditions satisfy an appropriate sufficient dissipativity property. Such boundary conditions are the so-called static boundary feedback control and lead to feedbacks that only depend on the measures at the boundaries. Other results exists using different boundary conditions for instance PI controls ([6], Chapter 8), [19], [5], [32], [31], [23] , full-state feedbacks resulting of a backstepping approach [15] (see [18], [17] for its application on variant systems based on the Saint-Venant equations) or stabilization by internal control [14] . In 2017, the Lyaponov method succeeded in stabilizing the inhomogeneous Saint-Venant equations for arbitrarily large friction but without slope [7], and only recently for every cross-sectional profile and every slope or friction [25], [24]. However, the stability of the Saint-Venant equations for trapezoidal channels has rarely been studied and described in detail, although it is mathematically very interesting and necessary for a realistic description of flow behavior. This corresponds to a variety of physical situations, such as in the case of water flow, where changes in channel dimensions, such as bottom width and side inclination. In this case of non-rectangular and non-prismatic channels, channel dimensions such as bottom width andside slope are assumedto vary linearly along length, making the approach more complicated. It is important to note that in engineering, the trapezoidal channel is preferred because it offers less resistance to flow and is easier to construct than the circular channel which the perfect channel.

Our contribution in this paper is that we successfully construct an explicit Lyapunov control function to control the local exponential stability in the L^2-norm of the linear Saint-Venant equations with static boundary feedbacks, to analyze the situation not only in the case where the friction and the slope is arbitrary, but also the dimensions of the channel. This enables us to design robust static feedback controllers to ensure exponential stability of the nonlinear system steady states. In particular, we deal with the case where the slope, width of the bottom and the lateral slope may vary with respect to the space variable . This is all the more important as not only is the slope likely to vary in a river but also the width of its bottom and the lateral slope, even sometimes over short distances. In the case where the width of the bottom is taken constant and the lateral slope zero we recover the case of the rectangular channel. This particular case of rectangular channel was solved in [12].

The organization of the paper is as follows. In Section 2, we give a description of the nonlinear Saint-Venant equations. In Section 3, the steady-state and the linearisation of the nonlinear system is firstly studied for a trapezoidal section. We then stady the exponential stability of the linearized system by constructing a quadratic Lyapunov function that enables us to get the exponential stability of the system by properly choosing the boundary feedback controls.

2. The Saint-Venant model

The most general version of $1D$ Saint-Venant equations with arbitrary varying slope, section profile and friction model is given by the following system [1] :

$$\begin{cases} \partial_t A + \partial_x(AV) = 0, \\ \\ \partial_t(AV) + \partial_x(V^2 A) + gA\partial_x H(A,x) = gA\big(S_0 - S_f(A,V,x)\big), \end{cases} \tag{2.1}$$

where $\mathbf{A}$ is the cross-sectional area of the water in the channel, $\mathbf{V}$ is the velocity, $\mathbf{H}(A,x)$ is the water depth, g is the gravity acceleration, $\mathbf{S}_0$ is the channel slope, $\mathbf{S}_f$ is the friction slope, it is usually defined by semi-empirical formulae proposed by hydraulic engineers in the late nineteenth or early twentieth centuries. In general, it only depends on the fluid quantities. One of the most popular is $S_f = k\dfrac{V^2}{A}$, where k is a constant friction coefficient.

For a trapezoidal section, the area is given by

$$A(A, x) = bH + mH^2, \tag{2.2}$$

where $b = b(x)$ is the channel width at the bottom and $m = m(x)$ is the inverse slope of the channel walls. Given that $H > 0$ and $m \pm 0$, we deduce the following relations from (2.2)

$$H(A, x) = \frac{\sqrt{b^2 + 4mA} - b}{2m} \tag{2.3}$$

Using (2.3), the system (2.1) can now be written as

$$\begin{cases} \partial_t A + \partial_x AV = 0, \\ \\ \partial_t V + V\partial_x V + \dfrac{g}{\sqrt{b^2 + 4mA}}\partial_x A = g(S_0 - S_f), \end{cases} \tag{2.4}$$

3. Exponential stability for the L^2-norm in a trapezoidal channel

3.1. **Steady-state and linearisation.** A steady-state is a constant state A^*, V^* which satisfies the relation

$$\begin{cases} A^* V^* = Q^*, \\ \\ V^*\partial_x V^* + \dfrac{g}{\sqrt{b^{*2} + 4m^*A^*}}\partial_x A^* = g(S_0 - k\dfrac{V^{*2}}{A^*}), \end{cases} \tag{3.1}$$

where $Q^* \geq 0$ is any given constant set point and corresponds to the flow rate.

Remark 3.1. We are interested in physical stationary states therefore we suppose that $A^* > 0$ and $V^* > 0$. Furthermore, We are interested on the steep slope regime as it is the most challenging situation to stabilize. In this regime, A^* tends to increase while V^* decreases and consequently the system moves away from the limit of the fluvial regime defined by the critical point where $\dfrac{gA^*}{\sqrt{b^{*2} + 4m^*A^*}} = V^{*2}$, in this case, we have

$$S_0 > k\frac{V^{*2}}{A^*} \tag{3.2}$$

Using (3.1), we get that V^* satisfies

$$\partial_x V^* = \frac{gV^*(S_0 - k\dfrac{V^{*2}}{A^*})}{V^{*2} - \dfrac{gA^*}{\sqrt{b^{*2} + 4m^*A^*}}} \tag{3.3}$$

Observe that the steady-states are therefore not necessarily uniform. We also suppose that the flow is in the fluvial regime. In this case the Froude number is strictly less than 1 (see [27] for instance) and the system needs to have a prescribed boundary condition at $x = 0$ and a boundary condition at $x = L$ to be well-posed. Since for each initial condition $\big(A^*(0), V^*(0)\big)$ satisfying $\dfrac{gA^*(0)}{\sqrt{b^{*2}(0) + 4m^*(0)A^*(0)}} > V^{*2}(0)$, there exists a unique maximal solution to (3.1), and this maximal solution exists as soon as the following condition is satisfied

$$\frac{gA^*}{\sqrt{b^{*2} + 4m^*A^*}} > V^{*2}. \tag{3.4}$$

For the linearization of the system, we define the perturbation functions a and v as

$$a(t,x) = A(t,x) - A^*(x), \quad \text{and} \quad v(t,x) = V(t,x) - V^*(x). \tag{3.5}$$

Using (3.5) in (2.4) and taking into account the relations (3.1) and (2.3) we obtain the linearization of the system (2.4) around the steady state as follows:

$$\begin{pmatrix} a \\ v \end{pmatrix}_t + \begin{pmatrix} V^* & A^* \\ \dfrac{g}{\sqrt{b^{*2}+4m^*A^*}} & V^* \end{pmatrix} \begin{pmatrix} a \\ v \end{pmatrix}_x + \begin{pmatrix} V^*_x & A^*_x \\ f^*_w & V^*_x + 2gk\dfrac{V^*}{A^*} \end{pmatrix} \begin{pmatrix} a \\ v \end{pmatrix} = 0, \tag{3.6}$$

where f^*_w is defined by

$$f^*_w = gk\frac{V^{*2}}{A^{*2}} + \frac{2m^*gA^*V^{*2}(S_0 - k\frac{V^{*2}}{A^*})}{\left(V^{*2}\sqrt{b^{*2}+4m^*A^*} - gA^*\right)\left(b^{*2}+4m^*A^*\right)}, \tag{3.7}$$

3.2. **Exponential stability of the linearized system.** In this section, we study the exponential stability of the linearized system (3.6) about a steady-state $(A^*, V^*)^T$ for the L^2-norm.

We assume that both ends of the channel are equipped with hydraulic controls (gates, pumps, mobile spillways, etc.) that allow to assign the values of the flow-rate. On-line measurements of the water levels at both ends $h(t,0)$ and $h(t,L)$ are assumed to be available for feedback control since the cross-sectional area of the water in the channel depends on the water depth. Obviously, instead of the flow-rates, we may as well consider the velocities $v(t,0)$ and $v(t,L)$ as being the control actions. Therefore we introduce the following boundary conditions: the following boundary conditions:

$$v(t,0) = k_0 a(t,0), \quad v(t,L) = k_1 a(t,L), \tag{3.8}$$

Conditions (3.8) are linear feedback static control laws with the tuning parameters k_0 and k_1.

The initial condition is given as follows

$$a(0,x) = a_0(x), \quad v(0,x) = v_0(x), \tag{3.9}$$

where $(a_0, v_0)^T \in L^2((0,L);\mathbb{R}^2)$. The Cauchy problem (3.6), (3.8) and (3.9) is well-posed (see [6], Appendix A). Note that the exponential stability of the linearized system is now a problem of null-stabilization for a and v.

The main result we establish is the following.

Theorem 3.2. *The linear Saint-Venant system in trapezoidal channel (3.6), (3.8) and (3.9) is exponentially stable for the L^2-norm provided that the boundary conditions satisfy*

$$k_0 \in \left(\frac{-g}{V^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}, -\frac{V^*(0)}{A^*(0)}\right), \tag{3.10}$$

and

$$k_1 \in \mathbb{R}\backslash\left[\frac{-g}{V^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}, -\frac{V^*(L)}{A^*(L)}\right]. \tag{3.11}$$

In order to prove Theorem 1, we use a direct Lyapunov approach where the time derivative of the Lyapunov function can be madevstrictly negative definite by an appropriate choice of thevboundary conditions, thus we introduce the following lemma.

Lemma 3.3. *For the linearized Saint-Venant system (3.6), (3.8) and (3.9), if the boundary conditions satisfy*

$$k_0 \in \left(\frac{-g}{V^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}, -\frac{V^*(0)}{A^*(0)} \right),$$

and

$$k_1 \in \mathbb{R} \backslash \left[\frac{-g}{V^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}, -\frac{V^*(L)}{A^*(L)} \right]$$

Then there are constants $\alpha_1 > 0$, $\alpha > 0$ *and* $g_1 \in C^1\big([0, L] \to (0, +\infty)\big)$, $g_2 \in C^1\big([0, L] \to (0, +\infty)\big)$ *such that the following control Lyapunov function candidate*

$$V(a, v) = \int_0^L \Big(g_1 + g_2\Big)\left(\frac{g}{A^*\sqrt{b^{*2}+4m^*A^*}} h^2 + 2\frac{g_1 - g_2}{g_1 + g_2}\sqrt{\frac{g}{A^*\sqrt{b^{*2}+4m^*A^*}}} vh + v^2 \right) dx \tag{3.12}$$

verifies:

$$V(a, v) \geq \alpha_1 \big(\|a\|^2_{L^2(0,L)} + \|v\|^2_{L^2(0,L)}\big) \tag{3.13}$$

for any $(a, v) \in L^2((0, L); \mathbb{R}^2)$*, where* $L^2(0, L)$ *denotes* $L^2((0, L); \mathbb{R})$*. If in addition,* $(a, v)^T$ *is a solution of the system (3.6), (3.8) and (3.9), we have*

$$\frac{d}{dt}(V(a(t, \cdot), v(t, \cdot))) \leq -\alpha V(a(t, \cdot), v(t, \cdot)). \tag{3.14}$$

Proof. Let us denote

$$M_1(w^*) = \begin{pmatrix} V^* & A^* \\ \dfrac{g}{\sqrt{b^{*2}+4m^*A^*}} & V^* \end{pmatrix}.$$

Under the subcritical condition (3.4), the matrix $M_1(w^*)$ has two real distinct eigenvalues λ_1 and $-\lambda_2$ with

$$\lambda_1(x) = \sqrt{\frac{gA^*}{\sqrt{b^{*2}+4m^*A^*}}} + V^*(x) > 0, \quad \lambda_2(x) = \sqrt{\frac{gA^*}{\sqrt{b^{*2}+4m^*A^*}}} - V^*(x) > 0. \tag{3.15}$$

We define the characteristic coordinates as follows

$$\xi_1 = v + h\sqrt{\frac{g}{A^*\sqrt{b^{*2}+4m^*A^*}}} \quad \text{and} \quad \xi_2 = v - h\sqrt{\frac{g}{A^*\sqrt{b^{*2}+4m^*A^*}}}, \tag{3.16}$$

with the inverse coordinate transformation

$$h = \frac{\xi_1 - \xi_2}{2}\sqrt{\frac{A^*\sqrt{b^{*2}+4m^*A^*}}{g}} \quad \text{and} \quad v = \frac{\xi_1 + \xi_2}{2}.$$

With these definitions and notations, the linearized Saint-Venant equations (3.6) are written in characteristic form:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_x - \begin{pmatrix} \gamma_1 & \delta_1 \\ \gamma_2 & \delta_2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0, \tag{3.17}$$

where

$$\gamma_1(x) = -\frac{\left(S_0 - k\frac{V^{*2}}{A^*}\right)\sqrt{gA^*\big(b^{*2}+4m^*A^*\big)}}{2\left(V^*\sqrt{\sqrt{b^{*2}+4m^*A^*}}+\sqrt{gA^*}\right)},$$

$$\delta_1(x) = \frac{\left(S_0 - k\frac{V^{*2}}{A^*}\right)\sqrt{gA^*\big(b^{*2}+4m^*A^*\big)}}{2\left(V^*\sqrt{\sqrt{b^{*2}+4m^*A^*}}-\sqrt{gA^*}\right)},$$

$$\gamma_2(x) = -\frac{1}{2}\Bigg[\left(\frac{2gA^*}{V^*} - \sqrt{gA^*\sqrt{b^{*2}+4m^*A^*}}\right)k\frac{V^{*2}}{A^{*2}} + \frac{gV^*\left(b^{*2}+4m^*A^*\right)^{3/4} - 2m^*A^*V^{*2}\sqrt{gA^*}}{\left(V^{*2}\sqrt{b^{*2}+4m^*A^*}-gA^*\right)\left(b^{*2}+4m^*A^*\right)^{3/4}}\left(S_0 - k\frac{V^{*2}}{A^*}\right)\Bigg],$$

$$\delta_2(x) = -\frac{1}{2}\Bigg[\left(\frac{2gA^*}{V^*} + \sqrt{gA^*\sqrt{b^{*2}+4m^*A^*}}\right)k\frac{V^{*2}}{A^{*2}} + \frac{gV^*\left(b^{*2}+4m^*A^*\right)^{3/4} + 2m^*A^*V^{*2}\sqrt{gA^*}}{\left(V^{*2}\sqrt{b^{*2}+4m^*A^*}-gA^*\right)\left(b^{*2}+4m^*A^*\right)^{3/4}}\left(S_0 - k\frac{V^{*2}}{A^*}\right)\Bigg].$$

For the flow in the steep slope fluvial regime we notice that γ_1 and δ_2 are negative.

From (3.16) and (3.8), we obtain the following boundary conditions for system (3.17)

$$\begin{pmatrix}\xi_1(t,0)\\ \xi_2(t,L)\end{pmatrix} = \underbrace{\begin{pmatrix}0 & m_0\\ m_1 & 0\end{pmatrix}}_{K}\begin{pmatrix}\xi_1(t,L)\\ \xi_2(t,0)\end{pmatrix}, \tag{3.18}$$

where

$$m_0 = \frac{k_0\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}+\sqrt{g}}{k_0\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}-\sqrt{g}}, \quad m_1 = \frac{k_1\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}+\sqrt{g}}{k_1\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}-\sqrt{g}} \tag{3.19}$$

Let us denote

$$\xi = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \quad \Lambda = \begin{pmatrix}\lambda_1 & 0\\ 0 & -\lambda_2\end{pmatrix}, \quad M = \begin{pmatrix}\gamma_1 & \delta_1\\ \gamma_2 & \delta_2\end{pmatrix} \quad \text{and} \quad |\Lambda| = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix},$$

and introduce the following candidate Lyapunov function :

$$V = \int_0^L \xi^T P(x)\xi dx. \tag{3.20}$$

The weighting matrix $P(x)$ is defined as follows:
$P(x) \triangleq \mathrm{diag}\{P_1e^{-\tau x}, P_2e^{\tau x}\}$, with $\tau > 0$ and $P_1, P_2 \in C^1\big([0,L] \to (0,+\infty)\big)$ two real positive functions.

The time derivative of V along the solutions of (3.20) is

$$\dot{V} = \int_0^L \Big(\partial_t \xi^T P(x)\xi + \xi^T P(x)\partial_t \xi\Big)dx$$
$$= \int_0^L \Big(-\partial_x \xi^T \Lambda P(x)\xi - \xi^T P(x)\Lambda \partial_x \xi + \xi^T M^T P(x)\xi + \xi^T P(x)M\xi\Big)dx. \quad (3.21)$$

Then, integrating by parts, we obtain:

$$\dot{V} = -\big[\xi^T \Lambda P(x)\xi\big]_0^L + \int_0^L \xi^T\Big(-\tau|\Lambda|P(x) + M^T P(x) + P(x)M\Big)\xi dx$$

In order to complete the proof, we have to find a matrix $P = \text{diag}\{P_1, P_2\}$ such that $\xi^T\big(M^T(0)P(0) + P(0)M(0)\big)\xi$ is a negative semi definite quadratic form and to find the range of admissible values of the tuning k_0 and k_1 such that the boundary conditions are dissipative ($\rho(K) < 1$ where $\rho(K) = \|\Delta K \Delta^{-1}\|$ is the norm for the matrix K and $\Delta = \text{diag}\{\sqrt{P_1\lambda_1}, \sqrt{P_2\lambda_2}\}$) (see [13]).

For the matrix P, a straightforward choice is

$$P_1 = \lambda_1(x) > 0, \quad P_2 = \lambda_2(x) > 0,$$

where $\lambda_1(x)$ and $\lambda_2(x)$ are given by (3.15). Furthermore by using (3.2) and (3.4) we obtain

$$\gamma_1 < 0, \qquad \delta_2 < 0 \qquad \text{and} \qquad \delta_2\gamma_1\lambda_1\lambda_2 > \big(\delta_1\lambda_1 + \gamma_2\lambda_2\big)^2, \quad (3.22)$$

since then the quadratic form is

$$\xi^T\big(M^T P + PM\big)\xi = -2|\gamma_1|\lambda_1\Big[\xi_1 + \big(\frac{\delta_1\lambda_1 + \gamma_2\lambda_2}{2\gamma_1\lambda_1}\big)\xi_2\Big]^2 - \frac{4\delta_2\gamma_1\lambda_1\lambda_2 - \big(\delta_1\lambda_1 + \gamma_2\lambda_2\big)^2}{2|\gamma_1|\lambda_1}\xi_2^2 \quad \leq 0. \quad (3.23)$$

Since $\xi^T\big(M^T P + PM\big)\xi$ is a negative semi definite quadratic form and

$$\Delta K \Delta^{-1} = \begin{pmatrix} 0 & m_0\dfrac{\lambda_1(0)}{\lambda_2(0)} \\ m_1\dfrac{\lambda_2(L)}{\lambda_1(L)} & 0 \end{pmatrix}. \quad (3.24)$$

Then, the dissipativity condition $\rho(K) < 1$ is a matter to be selected m_0 and m_1 such that

$$m_0^2\Big(\frac{\lambda_1(0)}{\lambda_2(0)}\Big)^2 < 1 \qquad \text{and} \qquad m_1^2\Big(\frac{\lambda_2(L)}{\lambda_1(L)}\Big)^2 < 1$$

- $m_0^2\dfrac{\lambda_1^2(0)}{\lambda_2^2(0)} < 1 \Rightarrow \Big(m_0\dfrac{\lambda_1(0)}{\lambda_2(0)} - 1\Big)\Big(m_0\dfrac{\lambda_1(0)}{\lambda_2(0)} + 1\Big) < 0$

$$\Rightarrow \Big(\frac{k_0\sqrt{A^*(0)\sqrt{b^{*2}(0) + 4m^*(0)A^*(0)}} + \sqrt{g}}{k_0\sqrt{A^*(0)\sqrt{b^{*2}(0) + 4m^*(0)A^*(0)}} - \sqrt{g}} \times \frac{\sqrt{\dfrac{gA^*(0)}{\sqrt{b^{*2}(0) + 4m^*(0)A^*(0)}}} + V^*(0)}{\sqrt{\dfrac{gA^*(0)}{\sqrt{b^{*2}(0) + 4m^*(0)A^*(0)}}} - V^*(0)} - 1\Big)$$

$$\Big(\frac{k_0\sqrt{A^*(0)\sqrt{b^{*2}(0) + 4m^*(0)A^*(0)}} + \sqrt{g}}{k_0\sqrt{A^*(0)\sqrt{b^{*2}(0) + 4m^*(0)A^*(0)}} - \sqrt{g}} \times \frac{\sqrt{\dfrac{gA^*(0)}{\sqrt{b^{*2}(0) + 4m^*(0)A^*(0)}}} + V^*(0)}{\sqrt{\dfrac{gA^*(0)}{\sqrt{b^{*2}(0) + 4m^*(0)A^*(0)}}} - V^*(0)} + 1\Big) < 0$$

$$\Rightarrow \Big(\frac{k_0\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}+\sqrt{g}}{k_0\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}-\sqrt{g}}\times\frac{\sqrt{g}A^*(0)+\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}V^*(0)}{\sqrt{g}A^*(0)-\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}V^*(0)}-1\Big)$$
$$\Big(\frac{k_0\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}+\sqrt{g}}{k_0\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}-\sqrt{g}}\times\frac{\sqrt{g}A^*(0)+\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}V^*(0)}{\sqrt{g}A^*(0)-\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}V^*(0)}+1\Big)<0$$

$$\Rightarrow \frac{2k_0A^*(0)V^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}+2gA^*(0)}{\left(k_0\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}-\sqrt{g}\right)\left(\sqrt{g}A^*(0)-\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}V^*(0)\right)}$$
$$\frac{2k_0\sqrt{g}A^*(0)\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}+2\sqrt{g}\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}V^*(0)}{\left(k_0\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}-\sqrt{g}\right)\left(\sqrt{g}A^*(0)-\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}V^*(0)\right)}<0$$

$$\Rightarrow 4A^*(0)\sqrt{g}A^*(0)\sqrt{A^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}}\left(k_0V^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}+g\right)$$
$$\left(k_0A^*(0)+V^*(0)\right)<0$$
$$\Rightarrow \left(k_0V^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}+g\right)\left(k_0A^*(0)+V^*(0)\right)<0$$

$$\Rightarrow k_0\in\left(-\frac{g}{V^*(0)\sqrt{b^{*2}(0)+4m^*(0)A^*(0)}},-\frac{V^*(0)}{A^*(0)}\right),$$

we have also

- $m_1^2\dfrac{\lambda_2^2(L)}{\lambda_1^2(L)}<1\Rightarrow\left(m_1\dfrac{\lambda_2(L)}{\lambda_1(L)}-1\right)\left(m_1\dfrac{\lambda_2(L)}{\lambda_1(L)}+1\right)<0$

$$\Rightarrow \Big(\frac{k_1\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}-\sqrt{g}}{k_1\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}+\sqrt{g}}\times\frac{\sqrt{\frac{gA^*(L)}{\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}}-V^*(L)}{\sqrt{\frac{gA^*(L)}{\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}}+V^*(L)}-1\Big)$$
$$\Big(\frac{k_1\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}-\sqrt{g}}{k_1\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}+\sqrt{g}}\times\frac{\sqrt{\frac{gA^*(L)}{\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}}-V^*(L)}{\sqrt{\frac{gA^*(L)}{\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}}+V^*(L)}+1\Big)<0$$

$$\Rightarrow \Big(\frac{k_1\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}-\sqrt{g}}{k_1\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}+\sqrt{g}}\times\frac{\sqrt{g}A^*(L)-\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}V^*(L)}{\sqrt{g}A^*(L)+\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}V^*(L)}-1\Big)$$
$$\Big(\frac{k_1\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}-\sqrt{g}}{k_1\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}+\sqrt{g}}\times\frac{\sqrt{g}A^*(L)-\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}V^*(L)}{\sqrt{g}A^*(L)+\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}V^*(L)}+1\Big)$$
$$< 0$$

$$\Rightarrow \frac{-2k_1A^*(L)V^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}-2gA^*(L)}{\left(k_1\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}+\sqrt{g}\right)\left(\sqrt{g}A^*(L)+\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}V^*(L)\right)}$$
$$\frac{2k_1\sqrt{g}A^*(L)\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}-2\sqrt{g}\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}V^*(L)}{\left(k_1\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}+\sqrt{g}\right)\left(\sqrt{g}A^*(L)+\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}V^*(L)\right)}<0$$

$$\Rightarrow -4A^*(L)\sqrt{g}A^*(L)\sqrt{A^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}}\left(k_1V^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}+g\right)$$
$$\left(k_1A^*(L)+V^*(L)\right)<0$$

$$\Rightarrow \left(k_1V^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}+g\right)\left(k_1A^*(L)+V^*(L)\right)\geq 0$$

$$\Rightarrow k_1\in\mathbb{R}\backslash\left[\frac{-g}{V^*(L)\sqrt{b^{*2}(L)+4m^*(L)A^*(L)}},-\frac{V^*(L)}{A^*(L)}\right].$$

Let us define

$$g_1 \triangleq \lambda_1 e^{-\tau x}, \qquad g_2 \triangleq \lambda_2 e^{\tau x}, \tag{3.25}$$

for any $(h,v)\in L^2((0,L);\mathbb{R}^2)$, consider the result of the change of variable as in (3.16) and (3.25), we have

$$\begin{aligned} V &= \int_0^L \xi^T P(x)\xi dx \\ &= \int_0^L \begin{pmatrix}\xi_1 & \xi_2\end{pmatrix}\begin{pmatrix}P_1e^{-\tau x} & 0\\ 0 & P_2e^{\tau x}\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}dx \\ &= \int_0^L P_1e^{-\tau x}\xi_1^2+P_2e^{\tau x}\xi_2^2dx \\ &= \int_0^L \lambda_1e^{-\tau x}\xi_1^2+\lambda_2e^{\tau x}\xi_2^2dx \end{aligned}$$

$$= \int_0^L \lambda_1 e^{-\tau x}\left(v + h\sqrt{\frac{g}{A^*\sqrt{b^{*2}+4m^*A^*}}}\right)^2 + \lambda_2 e^{\tau x}\left(v - h\sqrt{\frac{g}{A^*\sqrt{b^{*2}+4m^*A^*}}}\right)^2 dx$$

$$= \int_0^L \left(g_1 + g_2\right)\left(\frac{g}{A^*\sqrt{b^{*2}+4m^*A^*}}h^2 + 2\frac{g_1 - g_2}{g_1 + g_2}\sqrt{\frac{g}{A^*\sqrt{b^{*2}+4m^*A^*}}}vh + v^2\right)dx.$$

Thus, we get the expression of Lyapunov function candidate as in (3.12). Moreover, According to the result (3.23), there exists $\alpha > 0$ such that we get (3.14). The proof of Lemma 1 is completed. □

Using Lemma 1, we shall finally prove Theorem 3.1 that is now straightforward.

Proof. **of Theorem 1** If the boundary conditions (3.10)-(3.11) are satisfied, according to Lemma 1, there exist a constant $\alpha > 0$ and a candidate control Lyapunov function V defined by (3.12) such that if $(a, v)^T$ is a solution of the system (3.6), (3.8) and (3.9), we have

$$\frac{d}{dt}(V(a(t,\cdot), v(t,\cdot))) \leq -\alpha V(a(t,\cdot), v(t,\cdot)) \tag{3.26}$$

in the distribution sense which implies the exponential stability of the linearized system (3.6), (3.8) and (3.9) for the L^2-norm □

4. Conclusion

In this paper, we adopt the exponential stabilization technique of Ababacar Diagne et al. [13] to address the problem of the exponential stability of the Saint-Venant equations in a trapezoidal channel whose ends are equipped with physical devices where controls acting as feedback are implemented. The boundary conditions of the system are linked to these devices and an explicit boundary condition has been given which guarantees the exponential stability of the linear system in L^2-norm. To this end, we first linearized the nonlinear system to obtain a corresponding linearized system for the case of a trapezoidal channel. Then, we studied this linearized system and proved the exponential stability result in the L^2 norm by constructing a quadratic Lyapunov function. Finally, we obtained explicitly a requirement of the appropriate boundary conditions to obtain this stability.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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