ISSN: 2832-4293

Pan-American Journal of Mathematics 1 (2022), 1
https://doi.org/10.28919/cpr-pajm/1-1
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ON SHARP ESTIMATES OF COEFFICIENTS FOR POLYNOMIALS IN NORM-ATTAINABLE CLASSES

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ABSTRACT. New sharp estimates for coefficients of pseudo-monoidal type polynomials are given for the class of norm-attainable operators. Moreover, we show the existence of an equivalence relation in the case of the class of norm-attainable operators with the Bergman determinant for the class of composition operators.

1. INTRODUCTION

Following the quantum theory principle, studies on the estimation of coefficients of polynomials have been carried out with interesting results shown [3]. However, such studies have been done in different classes with a variety of estimates of coefficients obtained with varying degrees of sharpness [2]. Regarding norm-attainable classes, the characterization of several properties has been carried out (see [5] - [11]). These include the norm, numerical radius, spectral properties, and norm-attainability conditions [1]. Other aspects have also been considered in general Banach settings [12]. Nonetheless, obtaining the coefficients of pseudo-monoidal type polynomials in norm-attainable class has not received much consideration and this forms the basis of this work. Various coefficient estimates for classes of analytic functions of quasi q-sigmoid nature related to quasi-subordinate multiplicity with various determinants have been investigated in detail [4]. For instance, some researchers have worked on the type of Sakaguchi class whereby the considered meromorphic and holomorphic functions that are having equivalence relations with pseudo-subordinate coefficients in the class of global sigmoid functions. In [3], the author determined the bounds for coefficients and their inequalities for some classes of Sakaguchi type functions which are sharp but did not consider the class that conforms to norm-attainability conditions [7]. Comparisons done regarding determinants like Hankel determinant have been considered for the sharpness of these coefficients, however, more work needs to be done to determine the equivalence relations between various classes, for example, normal and self-adjoint classes of operators on Hardy sets. This work is organized in various sections as follows: 1. Introduction; 2. Preliminaries; 3. Main results and; 4. Conclusion.

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Submitted on Jun. 27, 2022.

2020 *Mathematics Subject Classification.* Primary 47H06; Secondary 47J25.

Key words and phrases. Sharp estimate, Coefficient, Analytic Functions, Norm-attainable class.
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The first author was supported in part by NRF Grant 903601.

2. Preliminaries

We provide some definitions and some known preliminary results which are useful in the sequel.

Definition 2.1. ([11], Definition 5) Let H be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on H. An operator $T \in B(H)$ that satisfies the norm-attainability condition, that is, if there exists a unit vector $h \in H$ such that $\|Th\| = \|T\|$ is called a norm-attainable operator. We denote the class of all norm-attainable operators by $NA(H)$.

Definition 2.2. ([4], Definition 2.7) Let the class of analytic, univalent and normalized functions in $\mathfrak{U}$ be represented by S. An analytic function r is said to be subordinate to R, written $r(z) \prec R(z)$, if there exists a function ξ analytic in $\mathfrak{U}$, with $\xi(0) = 0$ and $|\xi(z)| < 1$ and such that $r(z) = R(\xi(z))$.

Remark 2.3. Consider $NA(H)$ of all analytic functions satisfying the norm-attainability condition. The class of functions $y(z)$ analytic in $\mathfrak{U}$ such that $y(0) = 1$ and $Re(y(z)) > 0$ is denoted by Y.

Definition 2.4. ([3], Definition 2.1) For a polynomial $p \in NA(H)$, its coefficients are said to be pseudo-monoidal if they are q-sigmoid and satisfies the norm-attainability criterion.

3. Main results

Proposition 3.1. *Let $f(\beta) \in NA(H)$ and consider pseudo-monoidal coefficients of $p \in NA(H)$. Then we have*

$|\alpha_2| \leq |\frac{\delta_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma))}|,$

$$\begin{aligned}
|\alpha_3| &\leq \frac{1}{3-\mu(t^2+t\sigma+\sigma^2)}|\frac{\zeta_1 q_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right) q_0 - \left(\frac{\mu(\mu+1)(t+\sigma)^2}{2!} - 2\mu(t+\sigma)\right) \\
&\left(\frac{q_0^2\zeta_1^2}{4([1]_q!)^2(2-\mu(t+\sigma))^2}\right)|,
\end{aligned}$$

$$\begin{aligned}
|\alpha_4| &\leq \frac{1}{4-\mu(t^3+t^2\sigma+t\sigma^2+\sigma^3)}|\frac{\zeta_1 q_2}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right) q_1 + \left(\frac{\zeta_3}{2[1]_p!} + 2B\zeta_1\zeta_2 + D\zeta_1^3\right) q_0 \\
&- \left(\mu(\mu+1)(t+\sigma)^2 - \frac{\mu(\mu+1)(\mu+2)(t+\sigma)^3}{3!}\right)\left(\frac{q_0^3\zeta_1^3}{8([1]_q!)^3(2-\mu(t+\sigma))^3}\right) \\
&- (\mu(\mu+1)(t^3+t^2\sigma+t\sigma^2+\sigma^3) - 2\mu(t^2+t\sigma+\sigma^2)) - 3\mu(t+\sigma) \\
&\left(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma))(3-\mu(t^2+t\sigma+\sigma^2))}\right)(\frac{\zeta_1 q_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right) q_0 \\
&- \left(\frac{\mu(\mu+1)(t+\sigma)^2}{2!} + 2\mu(t+\sigma)\right)\left(\frac{q_0^2\zeta_1^2}{4([1]_q!)^2(2-\mu(t+\sigma))^2})\right)|
\end{aligned}$$

in which

$$\mathfrak{V} = \left(\frac{[2]_q! - 2([1]_p!)^2}{4([1]_q!)^2[2]_p!}\right)$$

$$\mathfrak{W} = \left(\frac{4([1]_q!)^3[2]_q! - 4([1]_p!)^2[3]_q! + [2]_q![3]_q!}{8([1]_q!)^3[2]_p![3]_q!)}\right)$$

Proof. Since $f(\beta) \in NA(H)$, we consider the upper half plane of H and by definition of the unit disc in the complex plane we have,

$$\frac{f'(\beta)\left(\frac{(t-\sigma)\beta}{f(t\beta)-f(\sigma\beta)}\right)^\mu - 1}{\phi(\beta)} = \psi_{p,u,v}(\beta) - 1, (\beta \in Z)$$

$$f'(\beta)\left(\frac{(t-\sigma)\beta}{f(t\beta)-f(\sigma\beta)}\right)^{\mu}-1=\phi(\beta)(\psi_{p,u,v}(\beta)-1) \tag{3.1}$$

$$f(t\beta)-f(\sigma\beta)=(t-\sigma)\beta+(t^2-\sigma^2)\alpha_2\beta^2+(t^3-\sigma^3)\alpha_3\beta^3+(t^4-\sigma^4)\alpha_4\beta^4+...$$

$$f'(\beta)=1+2\alpha_2\beta+3\alpha_3{\beta}^{^2}+4\alpha_4\beta^5+...$$

$f'(\beta)\left(\frac{(t-\sigma)\beta}{f(t\beta)-f(\sigma\beta)}\right)^{\mu}=$
$f'(\beta)\left[((t-\sigma)\beta)((t-\sigma)\beta+(t^2-\sigma^2)\alpha_2\beta^2+(t^3-\sigma^3)\alpha_3\beta^3+(t^4-\sigma^4)\alpha_4\beta^4+...)^{-1}\right]^{\mu}$
$f'(\beta)\left[((t-\sigma)\beta)((t-\sigma)\beta)^{-1}\left(1+(t+\sigma)\alpha_2\beta+(t^2+t\sigma+\sigma^2)\alpha_3\beta^2++...\right)^{-1}\right]^{\mu}$
$f'(\beta)[\big(1+(t+\sigma)\alpha_2\beta+(t^2+t\sigma+\sigma^2)\alpha_3\beta^2+(t^3+t^2\sigma+t\sigma^2\sigma^3)\alpha_4\beta^3+...\big)]^{-\mu}.$

Since f satisfies the norm-attainability condition, then by using the generalized binomial expansion, we obtain,

$f'(\beta)\left(\frac{(t-\sigma)\beta}{f(t\beta)-f(\sigma\beta)}\right)^{\mu}-1=$
$(2-\mu(t+\sigma))\alpha_2\beta+\left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2\alpha_2^2-\mu(t^2+t\sigma+\sigma^2)\alpha_3-2\mu(t-\sigma)\alpha_2^2+3\alpha_3\right)\beta^2$
$+(\mu(\mu+1)(t^3+t^2\sigma+t\sigma^2+\sigma^3)\alpha_2\alpha_3-\mu(t^3+t^2\sigma+t\sigma^2+\sigma^3)\alpha_4-\frac{(\mu)(\mu+1)(\mu+2)}{3!}(t+\sigma)^3\alpha_2^3$
$\mu(\mu+1)(t+\sigma)^2\alpha_2^3-2\mu(t^2+t\sigma+\sigma^2)\alpha_2\alpha_3+4\alpha_4)\beta^3+...$

Moreover, since H is complex and the existence of the unit vector for the norm-attainability criterion, we have

$$\psi_{p,u,v}(\varpi(\beta))=1+\frac{\zeta_1}{2[1]_p!}\beta+\left(\frac{\zeta_2}{2[1]_p!}+B\zeta_1^2\right)\beta^2+\left(\frac{\zeta_3}{2[1]_p!}+2B\zeta_1\zeta_2+D\zeta_1^3\right)\beta^3+... \tag{3.2}$$

Now, using unitized involution property of $NA(H)$ under normalized conditions we get

$$\phi(\beta)=q_0+q_1\beta+q_2\beta^2+... \tag{3.3}$$

$\phi(\beta)(\psi_{p,u,v}(\varpi(\beta))-1)=$
$(q_0+q_1\beta+q_2\beta^2+...)\left(\frac{\zeta_1}{2[1]_p!}\beta+\left(\frac{\zeta_2}{2[1]_p!}+B\zeta_1^2\right)\beta^2+\left(\frac{\zeta_3}{2[1]_p!}+2B\zeta_1\zeta_2+D\zeta_1^3\right)+...\right)$
and then,
$q_0\left(\frac{\zeta_1}{2[1]_p!}\beta+(\frac{\zeta_2}{2[1]_p!}+B\zeta_1^2)\beta^2+(\frac{\zeta_3}{2[1]_p!}+2B\zeta_1\zeta_2+D\zeta_1^3)\beta^3+...\right)+q_1\beta\left(\frac{\zeta_1}{2[1]_p!}\beta+(\frac{\zeta_2}{2[1]_p!}+B\zeta_1^2)\beta^2+...\right)$
$+q_2\beta^2(\frac{\zeta_1}{2[1]_p!}\beta+(\frac{\zeta_2}{2[1]_p!}+...)\frac{q_0\zeta_1}{2[1]_p!}\beta+(\frac{\zeta_2}{2[1]_p!}+B\zeta_1^2)q_0\beta^2+(\frac{\zeta_3}{2[1]_p!}+2B\zeta_1\zeta_2+D\zeta_1^3)q_0\beta^3$
$+\frac{q_1\zeta_1}{2[1]_p!}\beta^2+(\frac{\zeta_2}{2[1]_p!}+B\zeta_1^2)+\frac{q_2\zeta_1}{2[1]_p!}\beta^3...$
$\phi(\beta)(\psi_{p,u,v}(\varpi(\beta))-1)=\frac{q_0\zeta_1}{2[1]_p!}\beta+\left(\frac{q_1\zeta_1}{2[1]_p!}+(\frac{\zeta_2}{2[1]_p!}+B\zeta_1^2)q_0\right)\beta^2+(\frac{q_2\zeta_1}{2[1]_p!}\beta^3+(\frac{\zeta_3}{2[1]_p!}+B\zeta_1^2)q_1+(\frac{\zeta_3}{2[1]_p!}+2B\zeta_1\zeta_2+D\zeta_1^3)q_0)\beta^3+...$
From Equation 3.2 Equation 3.3 we obtain,
$(2-\mu(t+\sigma))\alpha_2\beta+\left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2\alpha_2^2-\mu(t^2+t\sigma+\sigma^2)\alpha_3-2\mu(t-\sigma)\alpha_2^2+3\alpha_3\right)\beta^2$
$+(\mu(\mu+1)(t^3+t^2\sigma+t\sigma^2+\sigma^3)\alpha_2\alpha_3-\mu(t^3+t^2\sigma+t\sigma^2+\sigma^3)\alpha_4-\frac{(\mu)(\mu+1)(\mu+2)}{3!}(t+\sigma)^3\alpha_2^3$
$+(\mu(\mu+1)(t+\sigma)^2\alpha_2^3-2\mu(t^2+t\sigma+\sigma^2)\alpha_2\alpha_3-3\mu(t-\sigma)\sigma_2\sigma_3+4\alpha_4)\beta^3+...$
$=\frac{q_0\zeta_1}{2[1]_p!}\beta+\left(\frac{q_1\zeta_1}{2[1]_p!}+(\frac{\zeta_2}{2[1]_p!}+B\zeta_1^2)q_0\right)\beta^2+\left(\frac{q_2\zeta_1}{2[1]_p!}+(\frac{\zeta_2}{2[1]_p!}+B\zeta_1^2)q_1+(\frac{\zeta_3}{2[1]_p!}+2B\zeta_1\zeta_2+D\zeta_1^3)q_0\right)\beta^3+...$
But the coefficients of β,β^2,β^3 are pseudo-monoidal and so we do comparison to obtain $\alpha_2,\alpha_3,\alpha_4$, and we get

$$(2-\mu(t+\sigma))\alpha_2\beta=\frac{q_0\zeta_1}{2[1]_q!}\beta$$

$$\alpha_2 = \frac{1}{2-\mu(t+\sigma)}\left(\frac{q_0\zeta_1}{2[1]_q!}\right)$$
$$\alpha_2 = \frac{q_0\zeta_1}{2[1]_q!(2-\mu(t+\sigma))}$$
$$|\alpha_2| \leq |\frac{q_0\zeta_1}{2[1]_q!(2-\mu(t+\sigma))}| \tag{3.4}$$

Also,

$\left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2\alpha_2^2 - \mu(t^2+t\sigma+\sigma^2)\alpha_3 - 2\mu(t-\sigma)\alpha_2^2 + 3\alpha_3\right)\beta^2 = \left(\frac{q_1\zeta_1}{2[1]_p!} + (\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2)q_0\right)\beta^2$

$\alpha_3 = \frac{1}{(3-\mu(t^2+t\sigma+\sigma^2))}\left(\frac{q_1\zeta_1}{2[1]_p!} + (\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2)q_0 - \left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2 - 2\mu(t-\sigma)\right)\left(\frac{q_0\zeta_1}{2[1]_q!(2-\mu(t+\sigma))}\right)^2\right)$

$|\alpha_3| \leq \frac{1}{3-\mu(t^2+t\sigma+\sigma^2)}|\frac{q_1\zeta_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2)q_0\right) - \left(\frac{\mu(\mu+1)(t+\sigma)^2}{2!} - 2\mu(t-\sigma)\right)$
$\left(\frac{q_0^2\zeta_1^2}{4[1]_q!(2-\mu(t+\sigma))^2}\right)|.$

And,

$(\mu(\mu+1)(t^3+t^2\sigma+t\sigma^2+\sigma^3)\alpha_2\alpha_3 - \mu(t^3+t^2\sigma+t\sigma^2+\sigma^3)\alpha_4 - \frac{(\mu)(\mu+1)(\mu+2)}{3!}(t+\sigma)^3\alpha_2^3 + (\mu(\mu+1)(t+\sigma)^2\alpha_2^3 - 2\mu(t^2+t\sigma+\sigma^2)\alpha_2\alpha_3 - 3\mu(t-\sigma)\sigma_2\sigma_3 + 4\alpha_4)\beta^3 + ...$
$= \left(\frac{q_2\zeta_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)q_1 + \left(\frac{\zeta_3}{2[1]_p!} + 2B\zeta_1\zeta_2 + D\zeta_1^3\right)q_0\right)\beta^3 + ...$

$\alpha_4 = \frac{1}{4-\mu(t^3+t^2\sigma+t\sigma^2+\sigma^3)}(\frac{q_2\zeta_1}{2[1]_p!} + (\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2)q_1 + (\frac{\zeta_3}{2[1]_p!} + 2B\zeta_1\zeta_2 + D\zeta_1^3)q_0 - (\mu(\mu+1)(t+\sigma)^2$
$-\frac{\mu(\mu+1)(\mu+2)(t+\sigma)^3}{3!})\left(\frac{q_0^3\zeta_1^3}{8([1]_p!)^3(2-\mu(t+\sigma))^3}\right) - (\mu(\mu+1)(t^3+t^2\sigma+t\sigma^2+\sigma^3) - 2\mu(t^2+t\sigma+\sigma^2))$
$-3\mu(t+\sigma)\left(\frac{q_0\zeta_1}{2[1]_q!(2-\mu(t+\sigma))(3-\mu(t^2+t\sigma+\sigma^2))}\right)(\frac{q_1\zeta_1}{2[1]_p!} + (\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2)q_0) -$
$(\frac{\mu(\mu+1)(t+\sigma)^2}{2!} + 2\mu(t+\sigma))(\frac{q_0^2\zeta_1^2}{4[1]_q!)^2(2-\mu(t+\sigma))^2})))$

$|\alpha_4| \leq \frac{1}{4-\mu(t^3+t^2\sigma+t\sigma^2+\sigma^3)}|\frac{q_2\zeta_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)q_1 + \left(\frac{\zeta_3}{2[1]_p!} + 2B\zeta_1\zeta_2 + D\zeta_1^3\right)q_0 - (\mu(\mu+1)(t+\sigma)^2 -$
$\frac{\mu(\mu+1)(\mu+2)(t+\sigma)^3}{3!})\left(\frac{q_0^3\zeta_1^3}{8([1]_p!)^3(2-\mu(t+\sigma))^3}\right) - (\mu(\mu+1)(t^3+t^2\sigma+t\sigma^2+\sigma^3) - 2\mu(t^2+t\sigma+\sigma^2)) -$
$3\mu(t+\sigma)\left(\frac{q_0\zeta_1}{2[1]_q!(2-\mu(t+\sigma))(3-\mu(t^2+t\sigma+\sigma^2))}\right)(\frac{q_1\zeta_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)q_0) - \left(\frac{\mu(\mu+1)(t+\sigma)^2}{2!} + 2\mu(t+\sigma)\right)$
$\left(\frac{q_0^2\zeta_1^2}{4[1]_q!)^2(2-\mu(t+\sigma))^2}\right))|.$ □

It is worth noting that when we set $p=1$, $|\alpha_2|$ and $|\alpha_3|$, we obtain a specialized case in a Bergman space. We next proceed to give a sharp estimate of the pseudo-coefficients in the next lemma.

Lemma 3.2. *Let $f(\beta) \in NA(H)$ and consider pseudo-monoidal coefficients of $p \in NA(H)$. If $\delta \in \Re$ then* $|\alpha_3 - \delta\alpha_2^2| \leq \frac{1}{3-\mu(t^2+t\sigma+\sigma^2)}|\frac{q_1\zeta_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)q_0 - \left(\frac{\mu(\mu+1)(t+\sigma)^2}{2!} - 2\mu(t+\sigma)\right)$
$\left(\frac{q_0^2\zeta_1^2}{4[1]_q!)^2(2-\mu(t+\sigma))^2}\right) - \delta\left(\frac{q_0^2\zeta_1^2(3-\mu(t^2+t\sigma+\sigma^2))}{4[1]_q!)^2(2-\mu(t+\sigma))^2}\right)|.$

Proof. Suppose that the polynomials in $NA(H)$ are orthogonal and their coefficients are of pseudo-monidal type, then by Proposition 3.1

$\alpha_3 - \delta\alpha_2^2 = \frac{1}{3-\mu(t^2+t\sigma+\sigma^2)}(\frac{q_1\zeta_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)q_0 - \left(\frac{\mu(\mu+1)(t+\sigma)^2}{2!} - 2\mu(t+\sigma)\right)$
$\left(\frac{q_0^2\zeta_1^2}{4[1]_q!)^2(2-\mu(t+\sigma))^2}\right)) - \delta\left(\frac{q_0^2\zeta_1^2}{4[1]_q!)^2(2-\mu(t+\sigma))^2}\right)$

$\alpha_3 - \delta\alpha_2^2 = \frac{1}{3-\mu(t^2+t\sigma+\sigma^2)}(\frac{q_1\zeta_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)q_0 - \left(\frac{\mu(\mu+1)(t+\sigma)^2}{2!} - 2\mu(t+\sigma)\right)$
$\left(\frac{q_0^2\zeta_1^2}{4[1]_q!)^2(2-\mu(t+\sigma))^2}\right) - \delta\left(\frac{q_0^2\zeta_1^2(3-\mu(t^2+t\sigma+\sigma^2))}{4[1]_q!)^2(2-\mu(t+\sigma))^2}\right))$

$|\alpha_3 - \delta\alpha_2^2| \leq \frac{1}{3-\mu(t^2+t\sigma+\sigma^2}|)\frac{q_1\zeta_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)q_0 - \left(\frac{\mu(\mu+1)(t+\sigma)^2}{2!} - 2\mu(t+\sigma)\right)$
$\left(\frac{q_0^2\zeta_1^2}{4[1]_q!)^2(2-\mu(t+\sigma))^2}\right) - \delta\left(\frac{q_0^2\zeta_1^2(3-\mu(t^2+t\sigma+\sigma^2))}{4[1]_q!)^2(2-\mu(t+\sigma))^2}\right)|.$

This is a sharp estimate of Cauchy-Bunyakowski-Schwarz inequality type. □

Next, we give our main theorem which shows that in $NA(H)$, the estimates for the coefficients are sharper compared to other classes.

Theorem 3.3. *Let $f(\beta) \in NA(H)$ and consider pseudo-monoidal coefficients of $p \in NA(H)$. If $\delta \in \Re$, then*
$|\alpha_2\alpha_4 - \delta\alpha_3^2| \leq |(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma)(4-\mu(t^3+t^2\sigma+t\sigma^2+\sigma^3))}(\frac{q_2\zeta_1}{2[1]_p!} + (\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2)q_1$
$+(\frac{\zeta_3}{2[1]_p!} + 2B\zeta_1\zeta_2 + D\zeta_1^3)q_0 - \left(\mu(\mu+1)(t+\sigma)^2 - \frac{\mu(\mu+1)(\mu+2)(t+\sigma)^3}{3!}\right)\left(\frac{q_0^3\zeta_1^3}{8([1]_p!)^3(2-\mu(t+\sigma))^3}\right) -$
$(\mu(\mu+1)(t^3+t^2\sigma+t\sigma^2+\sigma^3) - 2\mu(t^2+t\sigma+\sigma^2)) - 3\mu(t+\sigma)\left(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma))(3-\mu(t^2+t\sigma+\sigma^2))}\right)$
$(\frac{q_1\zeta_1}{2[1]_p!} + (\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2)q_0 - \left(\frac{\mu(\mu+1)(t+\sigma)^2}{2!} + 2\mu(t+\sigma)\right)\left(\frac{q_0^2\zeta_1^2}{4[1]_q!)^2(2-\mu(t+\sigma))^2}\right)))) - \frac{\delta}{3-\mu(t^2+t\sigma+\sigma^2)^2}$
$(\frac{q_1^2\zeta_1^2}{4([1]_p!)^2} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)^2 q_0^2 + \left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2 - 2\mu(t-\sigma)\right)^2\left(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma))}\right)^4 + (\frac{\zeta_1\zeta_2q_1}{2[1]_p!} +$
$\frac{B\zeta_1^3q_1}{[1]_p!})q_0 - \left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2 - 2\mu(t-\sigma)\right)\left(\frac{q_0^2\zeta_1^3q_1}{4([1]_p!)^3(2-\mu(t+\sigma))^2}\right) - \left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2 - 2\mu(t-\sigma)\right)$
$\left(\frac{q_0^2\zeta_1^2\zeta_2}{4([1]_p!)^3(2-\mu(t+\sigma))^2} + (\frac{Bq_0\zeta_1^4}{2([1]_p!)^2(2-\mu(t+\sigma))^2}\right)|$

Proof. Suppose that the polynomials in $NA(H)$ are orthogonal and their coefficients are of pseudo-monidal type, then by Lemma 3.2
$\alpha_2\alpha_4 - \delta\alpha_3^2 = \left(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma))}\right)(\frac{1}{4-\mu(t^3+t^2\sigma+t\sigma^2+\sigma^3)}(\frac{q_2\zeta_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)q_1$
$+\left(\frac{\zeta_3}{2[1]_p!} + 2B\zeta_1\zeta_2 + D\zeta_1^3\right)q_0 - \left(\mu(\mu+1)(t+\sigma)^2 - \frac{\mu(\mu+1)(\mu+2)(t+\sigma)^3}{3!}\right)\left(\frac{q_0^3\zeta_1^3}{8([1]_p!)^3(2-\mu(t+\sigma))^3}\right) -$
$(\mu(\mu+1)(t^3+t^2\sigma+t\sigma^2+\sigma^3) - 2\mu(t^2+t\sigma+\sigma^2)) - 3\mu(t+\sigma)\left(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma))(3-\mu(t^2+t\sigma+\sigma^2))}\right)$
$(\frac{q_1\zeta_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)q_0 - \left(\frac{\mu(\mu+1)(t+\sigma)^2}{2!} + 2\mu(t+\sigma)\right)\left(\frac{q_0^2\zeta_1^2}{4[1]_q!)^2(2-\mu(t+\sigma))^2}\right)))) - \delta\frac{1}{3-\mu(t^2+t\sigma+\sigma^2)^2}$
$(\frac{q_1\zeta_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)q_0 - \left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2 - 2\mu(t-\sigma)\right)\left(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma))}\right)^2))^2$
$\alpha_2\alpha_4 - \delta\alpha_3^2 = \left(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma))}\right)(\frac{1}{4-\mu(t^3+t^2\sigma+t\sigma^2+\sigma^3)}(\frac{q_2\zeta_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)q_1$
$+\left(\frac{\zeta_3}{2[1]_p!} + 2B\zeta_1\zeta_2 + D\zeta_1^3\right)q_0 - \left(\mu(\mu+1)(t+\sigma)^2 - \frac{\mu(\mu+1)(\mu+2)(t+\sigma)^3}{3!}\right)\left(\frac{q_0^3\zeta_1^3}{8([1]_p!)^3(2-\mu(t+\sigma))^3}\right) -$
$(\mu(\mu+1)(t^3+t^2\sigma+t\sigma^2+\sigma^3) - 2\mu(t^2+t\sigma+\sigma^2)) - 3\mu(t+\sigma)\left(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma)(3-\mu(t^2+t\sigma+\sigma^2))}\right)$
$(\frac{q_1\zeta_1}{2[1]_p!} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)q_0 - \left(\frac{\mu(\mu+1)(t+\sigma)^2}{2!} + 2\mu(t+\sigma)\right)\left(\frac{q_0^2\zeta_1^2}{4[1]_q!)^2(2-\mu(t+\sigma))^2}\right)))) - \frac{\delta}{3-\mu(t^2+t\sigma+\sigma^2)^2}$
$(\frac{q_1^2\zeta_1^2}{4([1]_p!)^2} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)^2 q_0^2 + \left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2 - 2\mu(t-\sigma)\right)^2\left(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma))}\right)^4 + (\frac{\zeta_1\zeta_2q_1}{2[1]_p!} +$
$\frac{B\zeta_1^3q_1}{[1]_p!})q_0 - \left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2 - 2\mu(t-\sigma)\right)\left(\frac{q_0^2\zeta_1^3q_1}{4([1]_p!)^3(2-\mu(t+\sigma))^2}\right) - \left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2 - 2\mu(t-\sigma)\right)$
$\left(\frac{q_0^2\zeta_1^2\zeta_2}{4([1]_p!)^3(2-\mu(t+\sigma))^2} + (\frac{Bq_0\zeta_1^4}{2([1]_p!)^2(2-\mu(t+\sigma))^2}\right)|$

$|\alpha_2\alpha_4 - \delta\alpha_3^2| \leq |(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma)(4-\mu(t^3+t^2\sigma+t\sigma^2+\sigma^3))}(\frac{q_2\zeta_1}{2[1]_p!} + (\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2)q_1$
$+(\frac{\zeta_3}{2[1]_p!} + 2B\zeta_1\zeta_2 + D\zeta_1^3)q_0 - \left(\mu(\mu+1)(t+\sigma)^2 - \frac{\mu(\mu+1)(\mu+2)(t+\sigma)^3}{3!}\right)\left(\frac{q_0^3\zeta_1^3}{8([1]_p!)^3(2-\mu(t+\sigma))^3}\right) -$
$(\mu(\mu+1)(t^3+t^2\sigma+t\sigma^2+\sigma^3) - 2\mu(t^2+t\sigma+\sigma^2)) - 3\mu(t+\sigma)\left(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma))(3-\mu(t^2+t\sigma+\sigma^2))}\right)$
$(\frac{q_1\zeta_1}{2[1]_p!} + (\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2)q_0 - \left(\frac{\mu(\mu+1)(t+\sigma)^2}{2!} + 2\mu(t+\sigma)\right)\left(\frac{q_0^2\zeta_1^2}{4[1]_q!)^2(2-\mu(t+\sigma))^2}\right)))) - \frac{\delta}{3-\mu(t^2+t\sigma+\sigma^2)^2}$
$(\frac{q_1^2\zeta_1^2}{4([1]_p!)^2} + \left(\frac{\zeta_2}{2[1]_p!} + B\zeta_1^2\right)^2 q_0^2 + \left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2 - 2\mu(t-\sigma)\right)^2\left(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma))}\right)^4 + (\frac{\zeta_1\zeta_2q_1}{2[1]_p!} +$
$\frac{B\zeta_1^3q_1}{[1]_p!})q_0 - \left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2 - 2\mu(t-\sigma)\right)\left(\frac{q_0^2\zeta_1^3q_1}{4([1]_p!)^3(2-\mu(t+\sigma))^2}\right) - \left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2 - 2\mu(t-\sigma)\right)$
$\left(\frac{q_0^2\zeta_1^2\zeta_2}{4([1]_p!)^3(2-\mu(t+\sigma))^2} + (\frac{Bq_0\zeta_1^4}{2([1]_p!)^2(2-\mu(t+\sigma))^2}\right)|.$

This completes the proof which shows an equivalence relation with the Bergman determinant for composition operators. □

The following is an immediate consequence.

Corollary 3.4. *Let $f(\beta)\in B(H)$ and consider pseudo-monoidal coefficients of $p\in B(H)$. If $\delta\in\mathbb{C}$ then*
$|\alpha_2\alpha_4-\delta\alpha_3^2|\leq|(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma)(4-\mu(t^3+t^2\sigma+t\sigma^2+\sigma^3))}(\frac{q_2\zeta_1}{2[1]_p!}+(\frac{\zeta_2}{2[1]_p!}+B\zeta_1^2)q_1$
$+(\frac{\zeta_3}{2[1]_p!}+2B\zeta_1\zeta_2+D\zeta_1^3)q_0-\left(\mu(\mu+1)(t+\sigma)^2-\frac{\mu(\mu+1)(\mu+2)(t+\sigma)^3}{3!}\right)\left(\frac{q_0^3\zeta_1^3}{8([1]_p!)^3(2-\mu(t+\sigma))^3}\right)-$
$(\mu(\mu+1)(t^3+t^2\sigma+t\sigma^2+\sigma^3)-2\mu(t^2+t\sigma+\sigma^2))-3\mu(t+\sigma)\left(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma))(3-\mu(t^2+t\sigma+\sigma^2))}\right)$
$(\frac{q_1\zeta_1}{2[1]_p!}+(\frac{\zeta_2}{2[1]_p!}+B\zeta_1^2)q_0-\left(\frac{\mu(\mu+1)(t+\sigma)^2}{2!}+2\mu(t+\sigma)\right)\left(\frac{q_0^2\zeta_1^2}{4[1]_q!)^2(2-\mu(t+\sigma))^2}\right))))-\frac{\delta}{3-\mu(t^2+t\sigma+\sigma^2)^2}$
$(\frac{q_1^2\zeta_1^2}{4([1]_p!)^2}+\left(\frac{\zeta_2}{2[1]_p!}+B\zeta_1^2\right)^2q_0^2+\left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2-2\mu(t-\sigma)\right)^2\left(\frac{q_0\zeta_1}{2[1]_p!(2-\mu(t+\sigma))}\right)^4+(\frac{\zeta_1\zeta_2q_1}{2[1]_p!}+$
$\frac{B\zeta_1^3q_1}{[1]_p!})q_0-\left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2-2\mu(t-\sigma)\right)\left(\frac{q_0^2\zeta_1^3q_1}{4([1]_p!)^3(2-\mu(t+\sigma))^2}\right)-\left(\frac{(\mu)(\mu+1)}{2!}(t+\sigma)^2-2\mu(t-\sigma)\right)$
$\left(\frac{q_0^2\zeta_1^2\zeta_2}{4([1]_p!)^3(2-\mu(t+\sigma))^2}+(\frac{Bq_0\zeta_1^4}{2([1]_p!)^2(2-\mu(t+\sigma))^2}\right)|.$

Proof. Follows analogously from the proof of Theorem 3.3 in a general Banach space setting with consideration to the completeness property. □

4. Conclusion

In conclusion, we have obtained new sharp estimates for the coefficients of polynomials which are given for the norm-attainable class. Moreover, we have shown the existence of an equivalence relation on the Bergman determinant for the class of composition operators.

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