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EXACT SOLUTIONS OF SYSTEM OF FOURTH-ORDER DIFFERENCE EQUATIONS

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ABSTRACT. In this paper, we derive the solutions of system difference equations

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(a + bx_{n-1}y_{n-3})}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(c + dy_{n-1}x_{n-3})}, \quad n \in \mathbb{N}_0,$$

where the parameters a, b, c, d are real numbers and the initial conditions x_{-i} and y_{-i} for $(i = 0, 1, 2, 3)$ are non zero real numbers.

1. INTRODUCTION

Nowadays, difference equations becomes a valuable tool in the modeling of many phenomena in various fields such as biology, epidemiology, physics, probability theory, etc. See for example [7, 27–29]. Accordingly, difference equations have attracted the attention of many researchers, see for example [1,5,6,8,12–18,21–24,26,30,31]. The following difference equations

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(1 + x_{n-1}y_{n-3})}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(\pm 1 \pm y_{n-1}x_{n-3})}. \tag{1.1}$$

has been studied by Almatrafi et al. [4].
In addition, Halim et al. [20] obtained the solution of the following system difference equation

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_n(a + by_{n-1}y_{n-2})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_n(a + bx_{n-1}y_{n-2})}. \tag{1.2}$$

In this paper, we generalize the solutions of the systems of nonlinear rational difference equations presented in [2], [3] and [4], which were established through a mere application of the induction principle.

2. MAIN RESULTS

In this section, we derive the admissible solutions of the system of difference equations given by

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(a + bx_{n-1}y_{n-3})}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(c + dy_{n-1}x_{n-3})}, \tag{2.1}$$

where $n \in \mathbb{N}_0$ and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}$ and y_0 are arbitrary non zero real numbers.

The system (2.1) can be written as

$$u_{n+1} = \frac{u_{n-1}}{a + bu_{n-1}}, \quad v_{n+1} = \frac{v_{n-1}}{c + dv_{n-1}}, \tag{2.2}$$

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by using the following change of variables

$$u_n = x_n y_{n-2}, \quad v_n = y_n x_{n-2}. \tag{2.3}$$

2.1. **Solutions of** $x_{n+1} = \dfrac{x_{n-1}}{\alpha + \beta x_{n-1}}$**.** In this section, we derive the closed form of the solution of the equation

$$x_{n+1} = \frac{x_{n-1}}{\alpha + \beta x_{n-1}}. \tag{2.4}$$

Let us define

$$x_n^{(j)} = x_{2n-j}, \quad j = 0, 1, \quad n \in \mathbb{N}_0. \tag{2.5}$$

Using this notation, we can write (2.4) as

$$x_{n+1}^{(j)} = \frac{x_n^{(j)}}{\alpha + \beta x_n^{(j)}}. \tag{2.6}$$

Equation (2.6) can be reduced to

$$w_{n+1} = \frac{(\alpha+1)w_n - \alpha}{w_n}, \tag{2.7}$$

by using the change of variable

$$x_n^{(j)} = \frac{1}{\beta}\left(w_n - \alpha\right), \quad j = 0, 1. \tag{2.8}$$

Now, we consider the difference equation (2.7) where the initial value w_0 is a non zero real number. The equality

$$w_n = \frac{z_n}{z_{n-1}} \tag{2.9}$$

reduces the equation (2.7) to the following one,

$$z_{n+1} - (\alpha+1)z_n - \alpha z_{n-1} = 0, \quad n \in \mathbb{N}_0. \tag{2.10}$$

Case $\alpha \neq 1$**.**

Lemma 2.1. *Consider the linear difference equation*

$$z_{n+1} - (\alpha+1)z_n + \alpha z_{n-1} = 0, \quad n \in \mathbb{N}_0, \tag{2.11}$$

with initial conditions $z_{-1}, z_0 \in \mathbb{R}$*. All solutions of equation* (2.11) *can be written in the form*

$$z_n = \frac{1}{1-\alpha}\left(z_0\left(1 - \alpha^{(n+1)}\right) - \alpha z_{-1}\left(1 - \alpha^n\right)\right). \tag{2.12}$$

Proof. Equation (2.11) is an homogeneous linear second order difference equation with constant coefficients, where z_0 and $z_{-1} \in \mathbb{R}$. These type of equations are usually solved by using the characteristic roots $\lambda_1 = \alpha$ and $\lambda_2 = 1$ of the characteristic polynomial $P(\lambda) = \lambda^2 - (1+\alpha)\lambda + \alpha$, and the general solution is given by

$$z_n = c_1 + c_2 a^{2n}.$$

where c_1 and c_2 are expressed in terms of the initial conditions z_0 and z_{-1} as

$$c_1 = \frac{z_0 - z_{-1}\alpha}{1-\alpha}$$
$$c_2 = \frac{\alpha(z_{-1} - z_0)}{1-\alpha}.$$

Then, the general solution of equation (2.11) is

$$z_n = \frac{1}{1-\alpha}\left(z_0\left(1-\alpha^{(n+1)}\right) - \alpha z_{-1}\left(1-\alpha^n\right)\right).$$

□

Case $\alpha = 1$**.**

Lemma 2.2. *Consider the linear difference equation*

$$z_{n+1} - 2z_n + z_{-1} = 0, \quad n \in \mathbb{N}_0, \tag{2.13}$$

with initial conditions $z_{-1}, z_0 \in \mathbb{R}$*. All solutions of equation* (2.13) *can be written in the form*

$$z_n = z_0(n+1) - z_{-1}n. \tag{2.14}$$

Proof. Equation (2.13), where z_0 and $z_{-1} \in \mathbb{R}$, is usually solved by using the characteristic roots $\lambda_1 = \lambda_2 = 1$ of the characteristic polynomial $P(\lambda) = (\lambda - 1)^2$. Thus, its general solution can be written in the following form

$$z_n = c_1 + c_2 n.$$

Using the initial conditions z_0 and z_{-1} and after some calculations we get

$$\begin{aligned} c_1 &= z_0 \\ c_2 &= z_0 - z_{-1}. \end{aligned}$$

Then, the general solution of equation (2.13) is

$$z_n = z_0(n+1) - z_{-1}n.$$

□

Using the above arguments, we can state the following theorem:

Theorem 2.3. *Let* $\{w_n\}_{n\geq 0}$ *be an admissible solution of* (2.7)*. Then,*

$$w_n = \begin{cases} \dfrac{\alpha(1-\alpha^n) - w_0(1-\alpha^{n+1})}{\alpha(1-\alpha^{n-1}) - w_0(1-\alpha^n)}, & if \alpha \neq 1. \\[2ex] \dfrac{n - w_0(n+1)}{(n-1) - w_0 n}, & if \alpha = 1. \end{cases} \tag{2.15}$$

Let

$$x_n^{(j)} = \frac{1}{\beta}\left(w_n - \alpha\right).$$

If $\alpha \neq 1$, then

$$x_n^{(j)} = \frac{(\alpha - 1)x_0^{(j)}}{\alpha^n(\alpha - 1) - \beta(1-\alpha^n)x_0^{(j)}}.$$

By using

$$\frac{\alpha^n - 1}{\alpha - 1} = \sum_{i=0}^{n-1}\alpha^i, \quad w_0 = \alpha + \beta x_0^{(j)},$$

we get that the solution of the difference equation (2.6) is given by

$$x_n^{(j)} = \frac{x_0^{(j)}}{\alpha^n - \beta\left(\displaystyle\sum_{i=0}^{n-1}\alpha^i\right)x_0^{(j)}}. \tag{2.16}$$

If $\alpha = 1$, then the solution of equation (2.6) is given by

$$x_n^{(j)} = \frac{x_0^{(j)}}{1 + \beta n x_0^{(j)}},$$

for each $j \in \{0, 1\}$.
We can state the following theorem by taking into account the results described above as well as (2.5).

Theorem 2.4. *Let $\{x_n\}_{n \geq -1}$ be an admissible solution of* (2.4). *Then, for $n \geq 0$ the following statement are true:*

- *If$\alpha \neq 1$, then*

$$x_{2n} = \frac{x_0}{\alpha^n + \beta \left(\sum_{i=0}^{n-1} \alpha^i\right) x_0},$$

$$x_{2n-1} = \frac{x_{-1}}{\alpha^n + \beta \left(\sum_{i=0}^{n-1} \alpha^i\right) x_{-1}}$$

- *If $\alpha = 1$, then*

$$x_{2n} = \frac{x_0}{1 + \beta n x_0},$$

$$x_{2n-1} = \frac{x_{-1}}{1 + \beta n x_{-1}},$$

Now let $\{u_n, v_n\}_{n \geq -1}$ be an admissible solution of (2.2). Then, for $n \geq 0$, we have the following:

- If $\alpha \neq 1$, then,

$$u_{2n} = \frac{u_0}{a^n + b \left(\sum_{i=0}^{n-1} a^i\right) u_0},$$

$$u_{2n-1} = \frac{u_{-1}}{a^n + b \left(\sum_{i=0}^{n-1} a^i\right) u_{-1}},$$

$$v_{2n} = \frac{v_0}{c^n + d \left(\sum_{i=0}^{n-1} c^i\right) v_0},$$

$$v_{2n-1} = \frac{v_{-1}}{c^n + d \left(\sum_{i=0}^{n-1} c^i\right) v_{-1}},$$

- If $\alpha = 1$, then

$$\begin{aligned} u_{2n} &= \frac{u_0}{1+bnu_0}, \\ u_{2n-1} &= \frac{u_{-1}}{1+bnu_{-1}}, \\ v_{2n} &= \frac{v_0}{1+dnv_0}, \\ v_{2n-1} &= \frac{v_{-1}}{1+dnv_{-1}}, \end{aligned}$$

Now let

$$x_n = \frac{u_n}{y_{n-2}}, \tag{2.17}$$

$$y_n = \frac{v_n}{z_{n-2}}. \tag{2.18}$$

Using formula (2.17) and (2.18), and after some calculations, we have

$$x_{4n} = \frac{u_{4n}}{v_{4n-2}} x_{4n-4} \tag{2.19}$$

and

$$y_{4n} = \frac{v_{4n}}{u_{4n-2}} y_{4n-4}. \tag{2.20}$$

Also, we can obtain

$$x_{4n-1} = \frac{u_{4n-1}}{v_{4n-3}} x_{4n-5} \tag{2.21}$$

and

$$y_{4n-1} = \frac{v_{4n-1}}{u_{4n-3}} y_{4n-5}, \tag{2.22}$$

for any $n \in \mathbb{N}$.
By multiplying the equalities (2.19), (2.20), (2.21) and (2.22) from 0 to $n-1$, respectively, it follows that

$$x_{4n} = x_0 \prod_{i=0}^{n-1} \left(\frac{u_{4i}}{v_{4i-2}} \right), \tag{2.23}$$

$$y_{4n} = y_0 \prod_{i=0}^{n-1} \left(\frac{v_{4i}}{u_{4i-2}} \right), \tag{2.24}$$

$$x_{4n-1} = x_{-1} \prod_{i=0}^{n-1} \left(\frac{u_{4i-1}}{v_{4i-3}} \right), \tag{2.25}$$

$$y_{4n-1} = y_{-1} \prod_{i=0}^{n-1} \left(\frac{v_{4i-1}}{u_{4i-3}} \right). \tag{2.26}$$

If we substitute equations (2.23), (2.24), (2.25) and (2.26) into (2.19) and (2.20), we get

$$x_{4n-2} = \frac{v_{4n}}{y_{4n}} = \frac{v_{4n}}{y_0} \prod_{i=0}^{n-1} \left(\frac{u_{4i-2}}{v_{4i}} \right) \tag{2.27}$$

and

$$y_{4n-2} = \frac{u_{4n}}{x_{4n}} = \frac{u_{4n}}{x_0} \prod_{i=0}^{n-1} \left(\frac{v_{4i-2}}{u_{4i}} \right). \tag{2.28}$$

Similarly,

$$x_{4n-3}=\frac{v_{4n-1}}{y_{4n-1}}=\frac{v_{4n-1}}{y_{-1}}\prod_{i=0}^{n-1}\left(\frac{u_{4i-3}}{v_{4i-1}}\right) \tag{2.29}$$

and

$$y_{4n-3}=\frac{u_{4n-1}}{x_{4n-1}}=\frac{u_{4n-1}}{x_{-1}}\prod_{i=0}^{n-1}\left(\frac{v_{4i-3}}{u_{4i-1}}\right). \tag{2.30}$$

Now, using the above arguments and taking into account that

$$u_0=x_0y_{-2},\quad v_0=y_0x_{-2},\quad u_{-1}=x_{-1}y_{-3},\quad v_{-1}=y_{-1}x_{-3},$$

we have the following:

Theorem 2.5. *Let $\{x_n,y_n\}_{n\geq -1}$ be an admissible solution of system* (2.1)*. Then, for $n\in\mathbb{N}_0$:*

1. *If $a\neq 1$ and $c\neq 1$, then the solution of system* (2.1) *is given by*

$$x_{4n-3}=\frac{x_{-1}^{n}y_{-3}^{n}\prod_{i=0}^{n-1}\left(c^{2i}+dy_{-1}x_{-3}\sum_{r=0}^{2i-1}c^{r}\right)}{y_{-1}^{n}x_{-3}^{n-1}\left(c^{2n}+dy_{-1}x_{-3}\sum_{r=0}^{2n-1}c^{r}\right)\prod_{i=0}^{n-1}\left(a^{2i-1}+bx_{-1}y_{-3}\sum_{r=0}^{2i-2}a^{r}\right)},$$

$$x_{4n-2}=\frac{x_{0}^{n}y_{-2}^{n}\prod_{i=0}^{n-1}\left(c^{2i}+dy_{0}x_{-2}\sum_{r=0}^{2i-1}c^{r}\right)}{y_{0}^{n}x_{-2}^{n-1}\left(c^{2n}+dy_{0}x_{-2}\sum_{r=0}^{2n-1}c^{r}\right)\prod_{i=0}^{n-1}\left(a^{2i-1}+bx_{0}y_{-2}\sum_{r=0}^{2i-2}a^{r}\right)},$$

$$x_{4n-1}=\frac{x_{-1}^{n+1}y_{-3}^{n}\prod_{i=0}^{n-1}\left(c^{2i-1}+dy_{-1}x_{-3}\sum_{r=0}^{2i-2}c^{r}\right)}{y_{-1}^{n}x_{-3}^{n}\prod_{i=0}^{n-1}\left(a^{2i}+bx_{-1}y_{-3}\sum_{r=0}^{2i-1}a^{r}\right)},\ x_{4n}=\frac{x_{0}^{n+1}y_{-2}^{n}\prod_{i=0}^{n-1}\left(c^{2i-1}+dy_{0}x_{-2}\sum_{r=0}^{2i-2}c^{r}\right)}{y_{0}^{n}x_{-2}^{n}\prod_{i=0}^{n-1}\left(a^{2i}+bx_{0}y_{-2}\sum_{r=0}^{2i-1}a^{r}\right)},$$

and

$$y_{4n-3}=\frac{y_{-1}^{n}x_{-3}^{n}\prod_{i=0}^{n-1}\left(a^{2i}+bx_{-1}y_{-3}\sum_{r=0}^{2i-1}a^{r}\right)}{x_{-1}^{n}y_{-3}^{n-1}\left(a^{2n}+bx_{-1}y_{-3}\sum_{r=0}^{2n-1}a^{r}\right)\prod_{i=0}^{n-1}\left(c^{2i-1}+dy_{-1}x_{-3}\sum_{r=0}^{2i-2}c^{r}\right)},$$

$$y_{4n-2}=\frac{y_{0}^{n}x_{-2}^{n}\prod_{i=0}^{n-1}\left(a^{2i}+bx_{0}y_{-2}\sum_{r=0}^{2i-1}a^{r}\right)}{x_{0}^{n}y_{-2}^{n-1}\left(a^{2n}+bx_{0}y_{-2}\sum_{r=0}^{2n-1}a^{r}\right)\prod_{i=0}^{n-1}\left(c^{2i-1}+dy_{0}x_{-2}\sum_{r=0}^{2i-2}c^{r}\right)},$$

$$y_{4n-1}=\frac{y_{-1}^{n+1}x_{-3}^{n}\prod_{i=0}^{n-1}\left(a^{2i-1}+bx_{-1}y_{-3}\sum_{r=0}^{2i-2}a^{r}\right)}{x_{-1}^{n}y_{-3}^{n}\prod_{i=0}^{n-1}\left(c^{2i}+dy_{-1}x_{-3}\sum_{r=0}^{2i-1}c^{r}\right)},\ y_{4n}=\frac{y_{0}^{n+1}x_{-2}^{n}\prod_{i=0}^{n-1}\left(a^{2i-1}+bx_{0}y_{-2}\sum_{r=0}^{2i-2}a^{r}\right)}{x_{0}^{n}y_{-2}^{n}\prod_{i=0}^{n-1}\left(c^{2i}+dy_{0}x_{-2}\sum_{r=0}^{2i-1}c^{r}\right)}.$$

2. *If $a\neq 1$ and $c=1$, then the solution of system* (2.1) *is given by*

$$x_{4n-3}=\frac{x_{-1}^{n}y_{-3}^{n}\prod_{i=0}^{n-1}\left(1+2diy_{-1}x_{-3}\right)}{y_{-1}^{n}x_{-3}^{n-1}\left(1+2dny_{-1}x_{-3}\right)\prod_{i=0}^{n-1}\left(a^{2i-1}+bx_{-1}y_{-3}\sum_{r=0}^{2i-2}a^{r}\right)},$$

$$x_{4n-2}=\frac{x_0^n y_{-2}^n \prod_{i=0}^{n-1}(1+2diy_0x_{-2})}{y_0^n x_{-2}^{n-1}(1+2dny_0x_{-2})\prod_{i=0}^{n-1}\left(a^{2i-1}+bx_0y_{-2}\sum_{r=0}^{2i-2}a^r\right)},$$

$$x_{4n-1}=\frac{x_{-1}^{n+1} y_{-3}^n \prod_{i=0}^{n-1}(1+d(2i-1)y_{-1}x_{-3})}{y_{-1}^n x_{-3}^{n}\prod_{i=0}^{n-1}\left(a^{2i}+bx_{-1}y_{-3}\sum_{r=0}^{2i-1}a^r\right)},$$

$$x_{4n}=\frac{x_0^{n+1} y_{-2}^n \prod_{i=0}^{n-1}(1+d(2i-1)y_0x_{-2})}{y_0^n x_{-2}^{n}\prod_{i=0}^{n-1}\left(a^{2i}+bx_0y_{-2}\sum_{r=0}^{2i-1}a^r\right)},$$

and

$$y_{4n-3}=\frac{y_{-1}^n x_{-3}^n \prod_{i=0}^{n-1}\left(a^{2i}+bx_{-1}y_{-3}\sum_{r=0}^{2i-1}a^r\right)}{x_{-1}^n y_{-3}^{n-1}\left(a^{2n}+bx_{-1}y_{-3}\sum_{r=0}^{2n-1}a^r\right)\prod_{i=0}^{n-1}(1+d(2i-1)y_{-1}x_{-3})},$$

$$y_{4n-2}=\frac{y_0^n x_{-2}^n \prod_{i=0}^{n-1}\left(a^{2i}+bx_0y_{-2}\sum_{r=0}^{2i-1}a^r\right)}{x_0^n y_{-2}^{n-1}\left(a^{2n}+bx_0y_{-2}\sum_{r=0}^{2n-1}a^r\right)\prod_{i=0}^{n-1}(1+d(2i-1)y_0x_{-2})},$$

$$y_{4n-1}=\frac{y_{-1}^{n+1} x_{-3}^n \prod_{i=0}^{n-1}\left(a^{2i-1}+bx_{-1}y_{-3}\sum_{r=0}^{2i-2}a^r\right)}{x_{-1}^n y_{-3}^n\prod_{i=0}^{n-1}(1+2diy_{-1}x_{-3})},\ y_{4n}=\frac{y_0^{n+1} x_{-2}^n \prod_{i=0}^{n-1}\left(a^{2i-1}+bx_0y_{-2}\sum_{r=0}^{2i-2}a^r\right)}{x_0^n y_{-2}^n\prod_{i=0}^{n-1}(1+2diy_0x_{-2})}.$$

3. *If $a=1$ and $c\neq 1$, then the solution of system* (2.1) *is given by*

$$x_{4n-3}=\frac{x_{-1}^n y_{-3}^n \prod_{i=0}^{n-1}\left(c^{2i}+dy_{-1}x_{-3}\sum_{r=0}^{2i-1}c^r\right)}{y_{-1}^n x_{-3}^{n-1}\left(c^{2n}+dy_{-1}x_{-3}\sum_{r=0}^{2n-1}c^r\right)\prod_{i=0}^{n-1}(1+b(2i-1)x_{-1}y_{-3})},$$

$$x_{4n-2}=\frac{x_0^n y_{-2}^n \prod_{i=0}^{n-1}\left(c^{2i}+dy_0x_{-2}\sum_{r=0}^{2i-1}c^r\right)}{y_0^n x_{-2}^{n-1}\left(c^{2n}+dy_0x_{-2}\sum_{r=0}^{2n-1}c^r\right)\prod_{i=0}^{n-1}(1+b(2i-1)x_0y_{-2})},$$

$$x_{4n-1}=\frac{x_{-1}^{n+1} y_{-3}^n \prod_{i=0}^{n-1}\left(c^{2i-1}+dy_{-1}x_{-3}\sum_{r=0}^{2i-2}c^r\right)}{y_{-1}^n x_{-3}^n\prod_{i=0}^{n-1}(1+2bix_{-1}y_{-3})},\ x_{4n}=\frac{x_0^{n+1} y_{-2}^n \prod_{i=0}^{n-1}\left(c^{2i-1}+dy_0x_{-2}\sum_{r=0}^{2i-2}c^r\right)}{y_0^n x_{-2}^n\prod_{i=0}^{n-1}(1+2bix_0y_{-2})},$$

and

$$y_{4n-3}=\frac{y_{-1}^n x_{-3}^n \prod_{i=0}^{n-1}(1+2bix_{-1}y_{-3})}{x_{-1}^n y_{-3}^{n-1}(1+2bnx_{-1}y_{-3})\prod_{i=0}^{n-1}\left(c^{2i-1}+dy_{-1}x_{-3}\sum_{r=0}^{2i-2}c^r\right)},$$

$$y_{4n-2}=\frac{y_0^n x_{-2}^n \prod_{i=0}^{n-1}(1+2bix_0y_{-2})}{x_0^n y_{-2}^{n-1}(1+2bnx_0y_{-2})\prod_{i=0}^{n-1}\left(c^{2i-1}+dy_0x_{-2}\sum_{r=0}^{2i-2}c^r\right)},$$

$$y_{4n-1}=\frac{y_{-1}^{n+1}x_{-3}^{n}\prod_{i=0}^{n-1}\left(1+b(2i-1)x_{-1}y_{-3}\right)}{x_{-1}^{n}y_{-3}^{n}\prod_{i=0}^{n-1}\left(c^{2i}+dy_{-1}x_{-3}\sum_{r=0}^{2i-1}c^{r}\right)},\ y_{4n}=\frac{y_{0}^{n+1}x_{-2}^{n}\prod_{i=0}^{n-1}\left(1+b(2i-1)x_{0}y_{-2}\right)}{x_{0}^{n}y_{-2}^{n}\prod_{i=0}^{n-1}\left(c^{2i}+dy_{0}x_{-2}\sum_{r=0}^{2i-1}c^{r}\right)}.$$

4. *If $a=1$ and $c=1$, then the solution of system* (2.1) *is given by*

$$x_{4n-3}=\frac{x_{-1}^{n}y_{-3}^{n}\prod_{i=0}^{n-1}\left(1+2diy_{-1}x_{-3}\right)}{y_{-1}^{n}x_{-3}^{n-1}\left(1+2dny_{-1}x_{-3}\right)\prod_{i=0}^{n-1}\left(1+b(2i-1)x_{-1}y_{-3}\right)},$$

$$x_{4n-2}=\frac{x_{0}^{n}y_{-2}^{n}\prod_{i=0}^{n-1}\left(1+2diy_{0}x_{-2}\right)}{y_{0}^{n}x_{-2}^{n-1}\left(1+2dny_{0}x_{-2}\right)\prod_{i=0}^{n-1}\left(1+b(2i-1)x_{0}y_{-2}\right)},$$

$$x_{4n-1}=\frac{x_{-1}^{n+1}y_{-3}^{n}\prod_{i=0}^{n-1}\left(1+d(2i-1)y_{-1}x_{-3}\right)}{y_{-1}^{n}x_{-3}^{n}\prod_{i=0}^{n-1}\left(1+2bix_{-1}y_{-3}\right)},\ x_{4n}=\frac{x_{0}^{n+1}y_{-2}^{n}\prod_{i=0}^{n-1}\left(1+d(2i-1)y_{0}x_{-2}\right)}{y_{0}^{n}x_{-2}^{n}\prod_{i=0}^{n-1}\left(1+2bix_{0}y_{-2}\right)},$$

and

$$y_{4n-3}=\frac{y_{-1}^{n}x_{-3}^{n}\prod_{i=0}^{n-1}\left(1+2bix_{-1}y_{-3}\right)}{x_{-1}^{n}y_{-3}^{n-1}\left(1+2bnx_{-1}y_{-3}\right)\prod_{i=0}^{n-1}\left(1+d(2i-1)y_{-1}x_{-3}\right)},$$

$$y_{4n-2}=\frac{y_{0}^{n}x_{-2}^{n}\prod_{i=0}^{n-1}\left(1+2bix_{0}y_{-2}\right)}{x_{0}^{n}y_{-2}^{n-1}\left(1+2bnx_{0}y_{-2}\right)\prod_{i=0}^{n-1}\left(1+d(2i-1)y_{0}x_{-2}\right)},$$

$$y_{4n-1}=\frac{y_{-1}^{n+1}x_{-3}^{n}\prod_{i=0}^{n-1}\left(1+b(2i-1)x_{-1}y_{-3}\right)}{x_{-1}^{n}y_{-3}^{n}\prod_{i=0}^{n-1}\left(1+2diy_{-1}x_{-3}\right)},\ y_{4n}=\frac{y_{0}^{n+1}x_{-2}^{n}\prod_{i=0}^{n-1}\left(1+b(2i-1)x_{0}y_{-2}\right)}{x_{0}^{n}y_{-2}^{n}\prod_{i=0}^{n-1}\left(1+2diy_{0}x_{-2}\right)}.$$

3. Some applications

In this section, we apply the previous results in order to show how some closed-form formulas for the solutions to the systems in (2.1), which were presented in [2] and [4] are obtained:

- When $a=b=c=d=1$ in system (2.1), we have that

$$x_{4n-3}=\frac{x_{-1}^{n}y_{-3}^{n}\prod_{i=0}^{n-1}\left(1+(2i-1)y_{-1}x_{-3}\right)}{y_{-1}^{n}x_{-3}^{n-1}\left(1+(2n-1)y_{-1}x_{-3}\right)\prod_{i=0}^{n-1}\left(1+(2i-2)x_{-1}y_{-3}\right)},$$

$$x_{4n-2}=\frac{x_{0}^{n}y_{-2}^{n}\prod_{i=0}^{n-1}\left(1+2iy_{0}x_{-2}\right)}{y_{0}^{n}x_{-2}^{n-1}\left(1+2ny_{0}x_{-2}\right)\prod_{i=0}^{n-1}\left(1+(2i-1)x_{0}y_{-2}\right)},$$

$$x_{4n-1}=\frac{x_{-1}^{n+1}y_{-3}^{n}\prod_{i=0}^{n-1}(1+(2i-2)y_{-1}x_{-3})}{y_{-1}^{n}x_{-3}^{n}\prod_{i=0}^{n-1}(1+(2i-1)x_{-1}y_{-3})},\ x_{4n}=\frac{x_{0}^{n+1}y_{-2}^{n}\prod_{i=0}^{n-1}(1+(2i-1)y_{0}x_{-2})}{y_{0}^{n}x_{-2}^{n}\prod_{i=0}^{n-1}(1+2ix_{0}y_{-2})},$$

and

$$y_{4n-3}=\frac{y_{-1}^{n}x_{-3}^{n}\prod_{i=0}^{n-1}(1+b(2i-1)x_{-1}y_{-3})}{x_{-1}^{n}y_{-3}^{n-1}(1+b(2n-1)x_{-1}y_{-3})\prod_{i=0}^{n-1}(1+(2i-2)y_{-1}x_{-3})},$$

$$y_{4n-2}=\frac{y_{0}^{n}x_{-2}^{n}\prod_{i=0}^{n-1}(1+2ix_{0}y_{-2})}{x_{0}^{n}y_{-2}^{n-1}(1+2nx_{0}y_{-2})\prod_{i=0}^{n-1}(1+(2i-1)y_{0}x_{-2})},$$

$$y_{4n-1}=\frac{y_{-1}^{n+1}x_{-3}^{n}\prod_{i=0}^{n-1}(1+(2i-2)x_{-1}y_{-3})}{x_{-1}^{n}y_{-3}^{n}\prod_{i=0}^{n-1}(1+(2i-1)y_{-1}x_{-3})},\ y_{4n}=\frac{y_{0}^{n+1}x_{-2}^{n}\prod_{i=0}^{n-1}(1+(2i-1)x_{0}y_{-2})}{x_{0}^{n}y_{-2}^{n}\prod_{i=0}^{n-1}(1+2iy_{0}x_{-2})}.$$

this agree with what was obtained is theorem 1 in [2].

- When $a=b=1$ and $c=d=-1$ in system (2.1), we have that

$$x_{4n-3}=\frac{x_{-1}^{n}y_{-3}^{n}}{y_{-1}^{n}x_{-3}^{n-1}\prod_{i=0}^{n-1}(1+(2i-1)x_{-1}y_{-3})},\quad x_{4n-1}=\frac{(-1)^{n}x_{-1}^{n+1}y_{-3}^{n}(1+y_{-1}x_{-3})^{n}}{y_{-1}^{n}x_{-3}^{n}\prod_{i=0}^{n-1}(1+2ix_{-1}y_{-3})},$$

$$x_{4n-2}=\frac{x_{0}^{n}y_{-2}^{n}}{y_{0}^{n}x_{-2}^{n-1}\prod_{i=0}^{n-1}(1+(2i-1)x_{0}y_{-2})},\quad x_{4n}=\frac{(-1)^{n}x_{0}^{n+1}y_{-2}^{n}(1+y_{0}x_{-2})^{n}}{y_{0}^{n}x_{-2}^{n}\prod_{i=0}^{n-1}(1+2ix_{0}y_{-2})},$$

and

$$y_{4n-3}=\frac{(-1)^{n}y_{-1}^{n}x_{-3}^{n}\prod_{i=0}^{n-1}(1+2ix_{-1}y_{-3})}{x_{-1}^{n}y_{-3}^{n-1}(1+2nx_{-1}y_{-3})(1+y_{-1}x_{-3})^{n}},\quad y_{4n-1}=\frac{y_{-1}^{n+1}x_{-3}^{n}\prod_{i=0}^{n-1}(1+(2i-1)x_{-1}y_{-3})}{x_{-1}^{n}y_{-3}^{n}},$$

$$y_{4n-2}=\frac{(-1)^{n}y_{0}^{n}x_{-2}^{n}\prod_{i=0}^{n-1}(1+2ix_{0}y_{-2})}{x_{0}^{n}y_{-2}^{n-1}(1+2nx_{0}y_{-2})(1+dy_{0}x_{-2})^{n}},\quad y_{4n}=\frac{y_{0}^{n+1}x_{-2}^{n}\prod_{i=0}^{n-1}(1+(2i-1)x_{0}y_{-2})}{x_{0}^{n}y_{-2}^{n}}.$$

this agree with what was obtained is Theorem 2 in [4].

- When $a=b=d=1$ and $c=-1$ in system (2.1), we have that

$$x_{4n-3}=\frac{x_{-1}^{n}y_{-3}^{n}}{y_{-1}^{n}x_{-3}^{n-1}\prod_{i=0}^{n-1}(1+(2i-1)x_{-1}y_{-3})},\quad x_{4n-1}=\frac{x_{-1}^{n+1}y_{-3}^{n}(-1+y_{-1}x_{-3})^{n}}{y_{-1}^{n}x_{-3}^{n}\prod_{i=0}^{n-1}(1+2ix_{-1}y_{-3})},$$

$$x_{4n-2}=\frac{x_{0}^{n}y_{-2}^{n}}{y_{0}^{n}x_{-2}^{n-1}\prod_{i=0}^{n-1}(1+(2i-1)x_{0}y_{-2})},\quad x_{4n}=\frac{x_{0}^{n+1}y_{-2}^{n}(-1+y_{0}x_{-2})^{n}}{y_{0}^{n}x_{-2}^{n}\prod_{i=0}^{n-1}(1+2ix_{0}y_{-2})},$$

and

$$y_{4n-3} = \frac{y_{-1}^{n} x_{-3}^{n} \prod_{i=0}^{n-1} (1+2ix_{-1}y_{-3})}{x_{-1}^{n} y_{-3}^{n-1} (1+2nx_{-1}y_{-3})(-1+y_{-1}x_{-3})^{n}}, \quad y_{4n-1} = \frac{y_{-1}^{n+1} x_{-3}^{n} \prod_{i=0}^{n-1} (1+(2i-1)x_{-1}y_{-3})}{x_{-1}^{n} y_{-3}^{n}},$$

$$y_{4n-2} = \frac{(-1)^{n} y_{0}^{n} x_{-2}^{n} \prod_{i=0}^{n-1} (1+2ix_{0}y_{-2})}{x_{0}^{n} y_{-2}^{n-1} (1+2nx_{0}y_{-2})(1+dy_{0}x_{-2})^{n}}, \quad y_{4n} = \frac{y_{0}^{n+1} x_{-2}^{n} \prod_{i=0}^{n-1} (1+(2i-1)x_{0}y_{-2})}{x_{0}^{n} y_{-2}^{n}}.$$

this agree with what was obtained is theorem 3 in [4].

- When $a = b = c = 1$ and $d = -1$ in system (2.1), we have that

$$x_{4n-3} = \frac{x_{-1}^{n} y_{-3}^{n} \prod_{i=0}^{n-1} (1-2iy_{-1}x_{-3})}{y_{-1}^{n} x_{-3}^{n-1} (1-2ny_{-1}x_{-3}) \prod_{i=0}^{n-1} (1+(2i-1)x_{-1}y_{-3})}, \quad x_{4n-1} = \frac{x_{-1}^{n+1} y_{-3}^{n} \prod_{i=0}^{n-1} (1-(2i-1)y_{-1}x_{-3})}{y_{-1}^{n} x_{-3}^{n} \prod_{i=0}^{n-1} (1+2ix_{-1}y_{-3})},$$

$$x_{4n-2} = \frac{x_{0}^{n} y_{-2}^{n} \prod_{i=0}^{n-1} (1-2iy_{0}x_{-2})}{y_{0}^{n} x_{-2}^{n-1} (1-2ny_{0}x_{-2}) \prod_{i=0}^{n-1} (1+(2i-1)x_{0}y_{-2})}, \quad x_{4n} = \frac{x_{0}^{n+1} y_{-2}^{n} \prod_{i=0}^{n-1} (1-(2i-1)y_{0}x_{-2})}{y_{0}^{n} x_{-2}^{n} \prod_{i=0}^{n-1} (1+2ix_{0}y_{-2})},$$

and

$$y_{4n-3} = \frac{y_{-1}^{n} x_{-3}^{n} \prod_{i=0}^{n-1} (1+2ix_{-1}y_{-3})}{x_{-1}^{n} y_{-3}^{n-1} (1+2nx_{-1}y_{-3}) \prod_{i=0}^{n-1} (1-(2i-1)y_{-1}x_{-3})}, \quad y_{4n-1} = \frac{y_{-1}^{n+1} x_{-3}^{n} \prod_{i=0}^{n-1} (1+(2i-1)x_{-1}y_{-3})}{x_{-1}^{n} y_{-3}^{n} \prod_{i=0}^{n-1} (1-2iy_{-1}x_{-3})},$$

$$y_{4n-2} = \frac{y_{0}^{n} x_{-2}^{n} \prod_{i=0}^{n-1} (1+2ix_{0}y_{-2})}{x_{0}^{n} y_{-2}^{n-1} (1+2nx_{0}y_{-2}) \prod_{i=0}^{n-1} (1-(2i-1)y_{0}x_{-2})}, \quad y_{4n} = \frac{y_{0}^{n+1} x_{-2}^{n} \prod_{i=0}^{n-1} (1+(2i-1)x_{0}y_{-2})}{x_{0}^{n} y_{-2}^{n} \prod_{i=0}^{n-1} (1-2iy_{0}x_{-2})}.$$

this agree with what was obtained is theorem 4 in [4].

- When $a = b = -1$ and $c = d = 1$ in system (2.1), we have that

$$x_{4n-3} = \frac{(-1)^{n} x_{-1}^{n} y_{-3}^{n} \prod_{i=0}^{n-1} (1+2iy_{-1}x_{-3})}{y_{-1}^{n} x_{-3}^{n-1} (1+2ny_{-1}x_{-3})(1+x_{-1}y_{-3})^{n}}, \quad x_{4n-1} = \frac{x_{-1}^{n+1} y_{-3}^{n} \prod_{i=0}^{n-1} (1+(2i-1)y_{-1}x_{-3})}{y_{-1}^{n} x_{-3}^{n}},$$

$$x_{4n-2} = \frac{(-1)^{n} x_{0}^{n} y_{-2}^{n} \prod_{i=0}^{n-1} (1+2iy_{0}x_{-2})}{y_{0}^{n} x_{-2}^{n-1} (1+2ny_{0}x_{-2})(1+x_{0}y_{-2})^{n}}, \quad x_{4n} = \frac{x_{0}^{n+1} y_{-2}^{n} \prod_{i=0}^{n-1} (1+(2i-1)y_{0}x_{-2})}{y_{0}^{n} x_{-2}^{n}},$$

and

$$y_{4n-3} = \frac{y_{-1}^{n} x_{-3}^{n}}{x_{-1}^{n} y_{-3}^{n-1} \prod_{i=0}^{n-1} (1+(2i-1)y_{-1}x_{-3})}, \quad y_{4n-1} = \frac{(-1)^{n} y_{-1}^{n+1} x_{-3}^{n} (1+x_{-1}y_{-3})^{n}}{x_{-1}^{n} y_{-3}^{n} \prod_{i=0}^{n-1} (1+2iy_{-1}x_{-3})},$$

$$y_{4n-2} = \frac{y_{0}^{n} x_{-2}^{n}}{x_{0}^{n} y_{-2}^{n-1} \prod_{i=0}^{n-1} (1+(2i-1)y_{0}x_{-2})}, \quad y_{4n} = \frac{y_{0}^{n+1} x_{-2}^{n} (1+x_{0}y_{-2})^{n}}{x_{0}^{n} y_{-2}^{n} \prod_{i=0}^{n-1} (1+2iy_{0}x_{-2})}.$$

this agree with what was obtained is 1 in [2].

- When $a = b = -1$ and $c = d = -1$ in system (2.1), we have that

$$x_{4n-3} = \frac{(-1)^n x_{-1}^n y_{-3}^n}{y_{-1}^n x_{-3}^{n-1} (1 + x_{-1}y_{-3})^n}, \quad x_{4n-1} = \frac{(-1)^n x_{-1}^{n+1} y_{-3}^n \prod_{i=0}^{n-1} (1 + y_{-1}x_{-3})^n}{y_{-1}^n x_{-3}^n},$$
$$x_{4n-2} = \frac{(-1)^n x_0^n y_{-2}^n}{y_0^n x_{-2}^{n-1} (1 + x_0 y_{-2})^n}, \quad x_{4n} = \frac{(-1)^n x_0^{n+1} y_{-2}^n (1 + y_0 x_{-2})^n}{y_0^n x_{-2}^n},$$

and

$$y_{4n-3} = \frac{(-1)^n y_{-1}^n x_{-3}^n}{x_{-1}^n y_{-3}^{n-1} (1 + y_{-1}x_{-3})^n}, \quad y_{4n-1} = \frac{(-1)^n y_{-1}^{n+1} x_{-3}^n (1 + x_{-1}y_{-3})^n}{x_{-1}^n y_{-3}^n},$$
$$y_{4n-2} = \frac{(-1)^n y_0^n x_{-2}^n}{x_0^n y_{-2}^{n-1} (1 + y_0 x_{-2})^n}, \quad y_{4n} = \frac{y_0^{n+1} x_{-2}^n (1 + x_0 y_{-2})^n}{x_0^n y_{-2}^n}.$$

this agree with what was obtained is theorem 2 in [2].

4. Conclusions

In this study, we mainly obtained solutions of the system of rational difference equations system.

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(a + bx_{n-1}y_{n-3})}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(c + dy_{n-1}x_{n-3})}, \quad n \in \mathbb{N}_0,$$

where the parameters a, b, c, d are real numbers and the initial conditions x_{-i} and y_{-i} for $i = 0, 1, 2, 3$, are non zero real numbers. Our results generalized the results obtained in [2] and [4].

References

[1] M.M. Alzubaidi, E.M. Elsayed, analytical and solutions of fourth order difference equations, Commun. Adv. Math. Sci. 2 (2019), 9-21. https://doi.org/10.33434/cams.447757.

[2] M.B. Almatrafi, Solutions structures for some systems of fractional difference equations, Open J. Math. Anal. 3 (2019), 52-61. https://doi.org/10.30538/psrp-oma2019.0032.

[3] M.B. Almatrafi, Analysis of solutions of some discrete systems of rational difference equations, J. Comp. Anal. Appl. 29 (2020), 355-368.

[4] M.B. Almatrafi, E.M. Elsayed, Solutions and formulae for some systems of difference equations, MathLAB J. 1 (2018), 356-369.

[5] M.B. Almatrafi, M.M. Alzubaidi, Analysis of the qualitative behaviour of an eighth-order fractional difference equation, Open J. Discr. Math. 2 (2019), 41-47. https://doi.org/10.30538/psrp-odam2019.0010.

[6] R. Abo-Zeid, H. Kamal, Global behavior of two rational third order difference equations, Univ. J. Math. Appl. 2 (2019), 212-217. https://doi.org/10.32323/ujma.626465.

[7] M. Berkal, J.F. Navarro, Qualitative behavior of a two-dimensional discrete-time prey-predator model, Comp. Math. Methods. 2021 (2021), e1193. https://doi.org/10.1002/cmm4.1193.

[8] M. Berkal, K. Berehal, N. Rezaiki, Representation of solutions of a system of five-order nonlinear difference equations, J. Appl. Math. Inf. 40 (2021), 409-431. https://doi.org/10.14317/jami.2022.409.

[9] M. Berkal, R. Abo-Zeid, Solvability of a second-order rational system of difference equations, Fund. J. Math. Appl. 6 (2023), 232-242. https://doi.org/10.33401/fujma.1383434.

[10] M. Berkal, R. Abo-zeıd, On a rational $(P + 1)$th order difference equation with quadratic term, Univ. J. Math. Appl. 5 (2022), 136-44. https://doi.org/10.32323/ujma.1198471.

[11] M. Berkal, J.F. Navarro, R. Abo-zeid, Global behavior of solutions to a higher-dimensional system of difference equations with lucas numbers coefficients, Math. Comp. Appl. 29 (2024), 28. https://doi.org/10.3390/mca29020028.

[12] E.M. Elsayed, Solution for systems of difference equations of rational form of order two , Comp. Appl. Math. 33, (2014), 751-765. https://doi.org/10.1007/s40314-013-0092-9.

[13] E.M. Elsayed, T.F. Ibrahim, Periodicity and solutions for some systems of nonlinear rational difference equations, Hacet. J. Math. Stat. 44 (2015), 1361-1390. https://doi.org/10.15672/HJMS.2015449653.

[14] E.M. Elsayed, M.M. El-Dessoky, Dynamics and global behavior for a fourth-order rational difference equation, Hacet. J. Math. Stat. 42, (2013), 479-494.

[15] M.M. El-Dessoky, *Solution for rational systems of difference equations of order three*, Mathematics, 2016 (2016), 53. https://doi.org/10.3390/math4030053.

[16] M.M. El-Dessoky, A. Khaliq, A. Asiri, On some rational systems of difference equations, J. Nonlinear Sci. Appl. 11 (2017), 49-72. https://doi.org/10.22436/jnsa.011.01.05.

[17] M. Gümüş, On a competitive system of rational difference equations, Univ. J. Math. Appl. 2 (2019), 224-228. https://doi.org/10.32323/ujma.649122.

[18] Y. Halim, M. Berkal, A. Khelifa, On a three-dimensional solvable system of difference equations, Turk. J. Math. 44 (2020), 1263-1288. https://doi.org/10.3906/mat-2001-40.

[19] A. Khelifa, Y. Halim, M. Berkal, On the solutions of a system of $(2p+1)$ difference equations of higher order. Miskolc Math. Notes, 22 (2021), 331-350. https://doi.org/10.18514/MMN.2021.3385.

[20] Y. Halim, A. Khelifa, M. Berkal, Representation of solutions of a two-dimensional system of difference equations, Miskolc Math. Notes, 21 (2020), 203-218. https://doi.org/10.18514/MMN.2020.3204.

[21] Y. Halim, A. Khelifa, M. Berkal, et al., On a solvable system of p difference equations of higher order, Period. Math. Hung. 85 (2022), 109-127. https://doi.org/10.1007/s10998-021-00421-x.

[22] T.F. Ibrahim, N. Touafek, Max-type system of difference equations with positive two periodic sequences, Math. Methods Appl. Sci. 37 (2014), 2562-2569. https://doi.org/10.1002/(ISSN)1099-1476.

[23] M. Kara, Y. Yazlik, Solvability of a system of nonlinear difference equations of higher order, Turk. J. Math., 43 (2019), 1533-1565. https://doi.org/10.3906/mat-1902-24.

[24] A. Khelifa, Y. Halim, M. Berkal, Solutions of a system of two higher-order difference equations in terms of Lucas sequence, Univ. J. Math. Appl. 2, (2019), 202-211. https://doi.org/10.32323/ujma.610399.

[25] A. Khelifa, Y. Halim, A. Bouchair, M. Berkal, On a system of three difference equations of higher order solved in terms of Lucas and Fibonacci numbers, Math. Slovaca, 70 (2020), 641-656. https://doi.org/10.1515/ms-2017-0378.

[26] A.Q. Khan, Q. Din, M.N. Qureshi, T.F. Ibrahim, Global behavior of an anticompetitive system of fourth-order rational difference equations, Comput. Ecol. Softw. 4 (2014), 35-46.

[27] H. Oliveira, S. Elaydi, R. Luís, An economical model with Allee effect, J. Differ. Equ. 15 (2009), 877-894. https://doi.org/10.1080/10236190802468920.

[28] X. Liu, A note on the existence of periodic solution in discrete predator-prey models, Appl. Math. Model. 34 (2010), 2477-2483. https://doi.org/10.1016/j.apm.2009.11.012.

[29] R. Luís, S. Elaydi, Open problems in some competition models, J. Diff. Equ. 17 (2011), 1873-1877. https://doi.org/10.1080/10236198.2011.559468.

[30] S. Stević, Representation of solutions of bilinear difference equations in terms of generalized Fibonacci sequences, Elec. J. Qual. Theory Diff. Equ. 67 (2014), 1-15. https://doi.org/10.14232/ejqtde.2014.1.67.

[31] S. Stević, On some solvable systems of difference equations, Appl Math Comp. 218 (2012), 5010-5018. https://doi.org/10.1016/j.amc.2011.10.068.