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SOME NEW RESULTS ON BE-ALGEBRAS

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ABSTRACT. BE-algebra was introduced in 2007 by H. S. Kim and Y. H. Kim. Then this class of logical algebras was the focus of many researchers. In this paper, the concept of atoms in BE-algebras is introduced and analyzed, and, in addition, it is directly connected to two-membered BE-filters. A criterion was found for determining the existence of atoms in these algebras. In addition to the previous one, the paper designs two types of BE-algebra extensions by adding one element so that the additional element is an atom in it. In addition to the previous one, two new types of filters in BE-algebras are designed.

1. INTRODUCTION

Y. Imai and K. Is´eki introduced two classes of logical algebras: BCK-algebras and BCI-algebras ([2,5]). The concept of BE-algebra, as a generalization of a BCK-algebra, was introduced in 2007 by H S. Kim and Y. H. Kim in [6]. They also defined the notion of filters in these algebras. Then this class of abstract algebras was the subject of study by several researchers (see, for example [1,7–11]). The filter theory in BE-algebras was established by B. L. Meng in [8].

In this article, we introduce the concept of atoms in BE-algebras and describe its basic properties. Two criteria for recognizing atoms in BE-algebras were found, one of which refers to the specific subclass aBE**-algebras of the class of BE-algebras. In addition to the previous one, the paper designs two extensions of the BE-algebra A by adding one element $w \notin A$ so that

(i) the element w is the only atom in $A \cup \{w\}$, and

(ii) the element w is another atom in $A \cup \{w\}$.

Also, such a BE-algebra was considered in which all elements, except 1, are atoms in it. In addition to the previous one, two new types of filters in BE-algebras are designed.

2. PRELIMINARIES

In this section, we present the necessary notions and statements about them necessary to understand the material in Section 3.

Definition 2.1. ([6], Definition 1.) An algebra $(A, \cdot, 1)$ of type $(2, 0)$ is called a BE-algebra if the following holds:

(BE1) $(\forall x \in A)(x \cdot x = 1)$,

(BE2) $(\forall x \in A)(x \cdot 1 = 1)$,

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(BE3) $(\forall x \in A)(1 \cdot x = x)$,
(BE4) $(\forall x, y, z \in A)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

We denote this axiomatic system by **BE** and the structure generated by it as BE-algebra.

Example 2.2. ([6], Example 3) Let $A = \{1, a, b, c, d, e\}$ be a set with the internal operation in A determined by the following table

·	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
e	1	1	1	1	1	1

Then $(A, \cdot, 1)$ is a BE-algebra. The order relation $\leqslant$ in this BE-algebra is given by $\leqslant = \{(1,1), (a,1), (a,a), (b,1), (b,a), (b,b), (c,1), (c,c), (d,1), (d,a), (d,c), (d,d), (e,1), (e,a), (e,b), (e,c), (e,d), (e,e)\}$. □

A BE-algebra A is an aBE-algebra if it additionally satisfies the following condition ([3,11])
(BE5) $(\forall x, y \in A)((x \cdot y = 1 \wedge y \cdot x = 1) \implies x = y)$.
We denote this axiomatic system by **aBE** and the structure generated by it as aBE-algebra.

A BE-algebra A is an BE**-algebra if it additionally satisfies the following condition ([3,11])
(BE6) $(\forall x, y, z \in A)(y \cdot z = 1 \implies (x \cdot y) \cdot (x \cdot z) = 1)$.
We denote this axiomatic system by **BE**** and the structure generated by it as BE**-algebra.

Example 2.3. Let $A = \{1, a, b, c\}$ be a set with the internal operation in A determined by the following table

·	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	1
c	1	1	1	1

Then $(A, \cdot, 1)$ is a BE-algebra which is neither an aBE-algebra nor a BE**-algebra ([11], Example 3.6). □

The term 'BE-subalgebra' is defined in the usual way: A non-empty subset S of a BE-algebra A is said to be a subalgebra of A if the following holds
(S) $(\forall x, y \in A)((x \in S \wedge y \in S \implies x \cdot y \in S)$.
Since the BE-subalgebra S of a BE-algebra A is not-empty, we have $1 \in S$.

A BE-algebra A is said to be self-distributive ([6], Definition 7) if the following holds:
(BE7) $(\forall x, y, z \in A)(x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z))$.

In any of these algebras A, the order relation $\leqslant$ is introduced in the following way:

$$(\forall x, y \in A)(x \leqslant y \iff x \cdot y = 1).$$

It immediately follows from axioms (BE1) and (BE2) of BE-algebras:
(1) $(\forall x \in A)(x \leqslant x)$ and
(2) $(\forall x \in A)(x \leqslant 1)$.

The following statement follows from (BE4) and (BE2) ([6], Proposition 2):
(3) $(\forall x, y \in A)(x \leqslant y \cdot x)$,
(4) $(\forall x, t \in A)(x \leqslant (x \cdot y) \cdot y)$.

If A is a self-distributive BE-algebra, then it holds

(5) $(\forall x, y, z \in A)((x \leqslant y \wedge y \leqslant z). \Longrightarrow x \leqslant z)$.

Indeed, let $x \leqslant y$ and $y \leqslant z$. This means that $x \cdot y = 1$ and $y \cdot z = 1$. Then (BE7) has the form $x \cdot 1 = 1 \cdot (x \cdot z)$ which gives $1 = x \cdot z$. Thus $x \leqslant z$. So, in the self-distributive BE-algebra $\leqslant$ is a quasi-order relation on the set A since it is a reflexive and transitive relation.

A BE-algebra A is said to be transitive ([1], Definition 2.2) if the following holds

(BE8) $(\forall x, y, z \in A)(y \cdot z \leqslant (x \cdot y) \cdot (x \cdot z))$.

It has been shown that:

(6) If A is a self-distributive BE-algebra, then A is transitive ([1], Proposition 3.10); and

(7) If A is a transitive BE-algebra, then A satisfies the following condition ([8], Proposition 2.4)

$(\forall x, y, z \in A)(x \leqslant y \Longrightarrow (z \cdot x \leqslant z \cdot y \wedge y \cdot z \leqslant x \cdot z))$.

The previous implication can be understood as the compatibility of the quasi-order relation $\leqslant$ on a self-distributive BE-algebra A with the internal binary operation in A.

(8) In addition to the previous one, if A is a self-distributive BE-algebra, then A is a BE**-algebra. Indeed, from $y \cdot z = 1$ follows $1 = y \cdot z \leqslant (x \cdot y) \cdot (x \cdot z)$, which gives $(x \cdot y) \cdot (x \cdot z) = 1$, according to (2) and (5).

The concept of BE-filters in this algebraic structure is introduced by the following definition.

Definition 2.4. ([6], Definition 4) Let $(A, \cdot, 1)$ be a BE-algebra. Non-empty subset F of A is a BE-filter in A if:

(F1) $1 \in F$,

(F2) $(\forall x, y \in A)((x \in F \wedge x \cdot y \in F) \Longrightarrow y \in F)$.

It can be immediately concluded that the subset $F_0 = \{1\}$ in any BE-algebra A is a BE-filter in A.

Of course, in addition to the previous ones, the substructure F in a BE-algebra A also satisfies the following conditions:

(F3) $(\forall x, y \in A)((x \in F \wedge x \leqslant y) \Longrightarrow y \in F)$,

(F4) $(\forall x, y \in F)((x \in F \wedge y \in F) \Longrightarrow x \cdot y \in F)$.

We understand the last formula in the sense that every BE-filter in a BE-algebra A is a BE-subalgebra in A (see, for example [7], Lemma 2.2). The validity of the last formula can be proved as follows: From $x \in F$ and $y \in F$, according to (3), we get $y \leqslant x \cdot y$. Hence we have $x \cdot y \in F$ according to (F3).

Example 2.5. Let A be as in Example 2.2. Consider the subsets $F_0 = \{1\}$, $F_1 = \{1, a\}$, $F_2 = \{1, c\}$, $F_3 = \{1, a, b\}$, $F_4 = \{1, c, d\}$, $F_5 = \{1, a, b, e\}$ and $F_6 = \{1, c, d, e\}$ as filter candidates in A. F_0 is a BE-filter according to what was said above. The subset F_1 is not a BE-filter in A because, for example, we have $a \in F_1$ and $a \cdot b = a \in F_1$ but $b \notin F_1$. The subsets F_5 and F_6 are also not BE-filters in A. The subsets F_2, F_3 and F_4 are BE-filters in A. □

Example 2.6. Let $A = \{1, a, b, c, d\}$ be a set and with the internal operation in A determined by the following table

·	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	a	b	1	b
d	1	1	1	1	1

Then $(A, \cdot, 1)$ is a self-distributive BE-algebra ([6], Example 8). Here, the order relation $\leqslant$ is given by $\{(1,1),(a,1),(a,a),(b,1),(b,b),(c,1),(c,a),(c,c),(d,1),(d,a), (d,b),(d,c),(d,d)\}$. Subsets $F_0 = \{1\}$, $F_1 = \{1,a\}$, $F_2 = \{1,b\}$, $F_3 = \{1,a,c\}$ are BE-filters in A but subsets $F_4 = \{1,b,d\}$ and $F_5 = \{1,a,c,d\}$ are not BE-filters in A. □

3. The main results

This section is the central part of this paper. All statements in it, unless it is specifically highlighted, are implemented for the basic version of BE-algebra.

3.1. **Concept of atoms in BE-algebras.** First, we will determine the concept of atoms in a BE-algebra.

Definition 3.1. Let A be a BE-algebra. An element $(1 \neq)a \in A$ is an *atom* in A if

(A) $(\forall x \in A)(a \leqslant x \implies (x = a \vee x = 1))$

holds. The set of all atoms in A is denoted by $L(A)$.

It can immediately be concluded that:

Theorem 3.2. *$L(A)$ is an anti-chain.*

Proof. Let $a, b \in L(A)$ be such that $a \neq b$. If we assume that $a \leqslant b$, we would have $b = a$ or $b = 1$ because a is an atom in A. Since none of the obtained options is possible, we conclude that the elements a and b are not comparable. □

The following theorem gives a criterion for recognizing atoms in a quasi-ordered residuated system.

Theorem 3.3. *Let A be a BE-algebra and $a \in A$ such that $1 \neq a$. Then a is an atom in A if the set $\{1, a\}$ is a filter in A.*

Proof. Let the subset $\{1, a\}$ be filter in A. Then holds

$$(\forall x \in A)((a \in \{1,a\} \wedge a \leqslant x) \implies x \in \{1,a\})$$

according (F3). This means $x = 1$ or $x = a$. □

The converse of the previous theorem is not valid in BE-algebras, as the following example shows.

Example 3.4. Let $A = \{1, a, , c, d, e\}$ as in examples 2.2 and 2.5. The subset $\{1, c\}$ is a BE-filter in A so, according to Theorem 3.3, the element c is an atom in A. On the other hand, the element a is an atom in A even though the subset $\{1, a\}$ is not a BE-filter in A (compare with Example 2.5). □

The following example illustrates the application of the previous theorem.

Example 3.5. Let $A = \{1, a, b, c, d\}$ and operation $'\cdot'$ defined on A as follows:

·	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	a	1	1	1
c	1	a	d	1	d
d	1	a	c	c	1

Then $A = (A, \cdot, 1)$ is a BE-algebra the relation $'\leqslant'$ is defined as follows $\leqslant := \{(1,1),(a,1),(a,b),(a,c),(a,d),(b,b),(b,c),(b,d),(b,1),(c,c),(c,1),(d,d),(d,1)\}$.

Subsets $F_1 = \{1\}$, $F_2 = \{1, c\}$, $F_3 = \{1, d\}$ are BE-filters in A. From here, according to Theorem 3.3, it immediately follows that the elements c and d are atoms in A. □

The following theorem says something more about the set $L(A)$ of all atoms for a given BE-algebra A.

Theorem 3.6. *Let A be a BE-algebra and $a \in L(A)$.*
(a) $(\forall\, x \in A)((a \cdot x) \cdot x = a \,\vee\, (a \cdot x) \cdot x = 1)$.
(b) $(\forall\, x \in A)(x \cdot a = a \,\vee\, x \cdot a = 1)$.

Proof. (a) Let A be a BE-algebra, $a \in L(A)$ and $x \in A$ be an arbitrary element. We have $a \leqslant (a \cdot x) \cdot x$ according to (4). Since a is an atom in A, from here we get $a = (a \cdot x) \cdot x$ or $(a \cdot x) \cdot x = 1$.

(b) For the elements a and $x \in A$, we have $a \leqslant x \cdot a$ by (3). Thus $x \cdot a = a$ or $x \cdot a = 1$ since a is an atom in A. □

The converse of statement (a) in the Theorem 3.6 is also valid:

Theorem 3.7. *Let A be a BE-algebra and $a \in A$. If the element a satisfies the condition* (a), *then a is an atom in A.*

Proof. Let $x \in A$ be such that $a \leqslant x$. This means $a \cdot x = 1$. Then $(a \cdot x) \cdot x = 1 \cdot x = x$ according (BE3). If $(a \cdot x) \cdot x = a$, then $a = x$. If $(a \cdot x) \cdot x = 1$, we have $x = 1$. This proves that a is an atom in A. □

However, if we additionally assume that the BE-algebra A satisfies some additional conditions, we have:

Theorem 3.8. *Let A be a self-distributive aBE-algebra and $a \in A$. If the element a satisfies the condition* (b), *then a is an atom in A.*

Proof. Let $x \in A$ be such that $a \leqslant x$. Then $1 = a \cdot a \leqslant a \cdot x \leqslant 1$ according to (7) and (2). From here we get $xa = 1$ according to (BE5). This means that $x \leqslant a$. From here, according to (7) again, we get $x = a$, which means that a is an atom in A. □

Example 3.9. Let $A = \{1, a, b, c\}$ be a set with the internal operation in A determined by the following table

·	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1

Then $(A, \cdot, 1)$ is an aBE**-algebra-algebra ([11], Example 3.8). Here, the order relation is given by $\leqslant = \{(1,1), (a,1), (a,a), (b,1), (b,b), (c,1), (c,a), (c,b), (c,c)\}$. The subsets $\{1, a\}$ and $\{1, b\}$ are BE-filters in A. Therefore, the elements a and b are atoms in A. □

Also valid

Theorem 3.10. *The subset $L(A) \cup \{1\}$ of a BE-algebra A is a BE-subalgebra.*

Proof. Since $a \leqslant b \cdot a$ by (3), it follows $b \cdot a = a$ or $b \cdot a = 1$ because a is an atom in A. Since the second option is not possible according to Theorem 3.2, we have $b \cdot a = a$. The statement $a \cdot b = b$ can be proved analogously to the previous proof. □

The following example shows that there are cases of BE-algebras in which $A = L(A) \cup \{1\}$.

Example 3.11. Let $A = \{1, a, b, c, d\}$ and operation '·' defined on A as follows:

·	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	d
c	1	a	b	1	d
d	1	a	b	c	1

Then $A = (A, \cdot, 1)$ is a BE-algebra. The relation $'\leqslant'$ is defined as follows $\leqslant := \{(1,1),(a,a),(a,1),(b,b),(b,1),(c,c),(c,1),(d,d),(d,1)\}$. Subsets $F_1 = \{1\}$, $F_2 = \{1,a\}$, $F_3 = \{1,b\}$, $F_4 = \{1,c\}$, $F_5 = \{1,d\}$ are BE-filters in A. According to Theorem 3.3, all elements, except 1, are atoms in this BE-algebra. □

The following proposition shows one property of this BE-algebra.

Theorem 3.12. *Let A be a BE-algebra in which all elements, except 1, are atoms in A. Then any BE-subalgebra in A is a BE-filter in A.*

Proof. Let S a BE-subalgebra in a BE-algebra A and let $x, y \in A$ be such that $x \in S$ and $x \cdot y \in S$. Since $y \leqslant x \cdot y$, according to (3), we get $x \cdot y = 1$ or $x \cdot y = y \in S$ since y is an atom in A. Assume that $x \cdot y = 1$ holds. This means that $x \leqslant y$ and, therefore, we have $y = 1 \in S$ or $y = x \in S$ because x is an atom in A. This shows that S is a BE-filter in A. □

3.2. **Two types of extensions of BE-algebras.**

Theorem 3.13. *Let $A = (A, \cdot, 1)$ be a BE-algebra and $a \notin A$. We can extend the system $A = (A, \cdot, 1)$ to the system $B = (A \cup \{a\}, *, 1)$ so that the element a is an atom in the system B.*

Proof. System B can be created in the following way:

$$x * y = \begin{cases} x \cdot y & \text{for } x \in A \ \wedge \ y \in A, \\ 1 & \text{for } x \in A \setminus \{1\} \ \wedge \ y = a, \\ a & \text{for } x = 1 \ \wedge \ y = a, \\ y & \text{for } x = a \ \wedge \ y \in A, \\ 1 & \text{for } x = a \ \wedge \ y = a. \end{cases}$$

It should be checked whether the structure created in this way satisfies the axioms (B1)-(B4).

(i) The formula (B1) is a valid formula in system B because $a \cdot a = 1$ is holds.

(ii) The formula (B2) is a valid formula in system B because $a \cdot 1 = 1$ is holds.

(iii) The formula (B3) is a valid formula in system B because $1 \cdot a = a$ is holds.

(iv) The validity of the formula (B4) is determined by replacing one, two or all three variables in (B4) with the letter a:

The substitution $x = a$ gives $a \cdot (y \cdot z) = y \cdot z = y \cdot (a \cdot z)$. The substitution $y = a$ gives $x \cdot (a \cdot z) = x \cdot z = a \cdot (y \cdot z)$. The substitution $z = a$ gives $x \cdot (y \cdot a) = x \cdot 1 = 1 = y \cdot 1 = y \cdot (y \cdot a)$.

The substitution $x = a$ and $y = a$ gives $a \cdot (a \cdot z) = a \cdot z = z$. The substitution $x = a$ and $z = a$ gives $a \cdot (y \cdot a) = a \cdot 1 = 1 = y \cdot 1 = y \cdot (a \cdot a)$. The substitution $y = a$ and $z = a$ gives $x \cdot (a \cdot a) = x \cdot 1 = 1 = a \cdot 1 = a \cdot (x \cdot a)$.

So, the structure $B = (A \cup \{a\}, *, 1)$ is a BE-algebra. It is immediately clear that a is an atom in B, because if $a \leqslant x$ holds, then $1 = a \cdot x = x$. □

Example 3.14. Let $A = \{1, a, b, c, d\}$ as in Example 3.5. Here is $L(A) = \{c, d\}$. Let us put $B = A \cup \{w\}$ and define the operations in B as follows

$*$	1	a	b	c	d	w
1	1	a	b	c	d	w
a	1	1	1	1	1	1
b	1	a	1	1	1	1
c	1	a	d	1	d	1
d	1	a	c	c	1	1
w	1	a	b	c	d	1

It is obvious that w is a single atom in the system B. Hence $L(B) = \{w\}$. □

However:

Theorem 3.15. *The extension of a BE-algebra $A = (A, \cdot, 1)$ to the BE-algebra $B = (A \cup \{a\}, *, 1)$ can also be realized so that the set $L(A)$ of all atoms of the system A is expanded by one element, that is $L(B) = L(A) \cup \{a\}$.*

Proof. Let us take $a \notin A$ and form the set $B = A \cup \{a\}$ and design the operation $*$ on B in the following way:

$$x * y = \begin{cases} x \cdot y & \text{for } x \in A \ \wedge \ y \in A, \\ 1 & \text{for } x \in A \setminus L(A) \cup \{1\} \ \wedge \ y = a, \\ a & \text{for } x = 1 \ \wedge \ y = a, \\ a & \text{for } x \in L(A) \ \wedge \ y = a \\ y & \text{for } x = a \ \wedge \ y \in A \setminus \{1\}, \\ 1 & \text{for } x = a \ \wedge \ y = 1, \\ 1 & \text{for } x = a \ \wedge \ y = a. \end{cases}$$

The validity of formulas (BE1)-(BE4) for system B should be checked. Since it is obvious that (BE1)-(BE3) are valid formulas for the system B, we will check the validity of formula (BE4). Since it is valid for $x, y, z \in A$, it remains to check its validity when we replace one, two or all three variables in (BE4) with the letter a. For $x = a$, we have $a \cdot (y \cdot z) = y \cdot z = y \cdot (a \cdot z)$. For $y = a$, we have $x \cdot (a \cdot z) = x \cdot z = a \cdot (x \cdot z)$. For $z = a$, we have $x \cdot (y \cdot a) = x \cdot 1 = 1 = y \cdot 1 = y \cdot (x \cdot a)$. The substitution $x = a$ and $y = a$ gives $a \cdot (a \cdot z) = a \cdot z = z$. The substitution $y = a$ and $z = a$ gives $x \cdot (a \cdot a) = x \cdot 1 = 1 = a \cdot 1 = a \cdot (x \cdot a)$. The substitution $x = a$ and $z = a$ gives $a \cdot (y \cdot a) = a \cdot 1 = 1 = y \cdot 1 = y \cdot (a \cdot z)$.

So, the structure $B = (A \cup \{a\}, *, 1)$ is a BE-algebra. Let us show that a is an atom in B. Let $x \in A \setminus L(A)$ be such that $a \leqslant x$. Then $a \cdot x = 1$ and from here $(a \cdot x) \cdot x = 1 \cdot x = x$. On the other hand, we have $a \cdot x = x$ and $(a \cdot x) \cdot x = x \cdot x = 1$. Hence, $x = 1$. This means that a is an atom in B. □

The following example illustrates the previous theorem.

Example 3.16. Let $A = \{1, a, b, c, d\}$ as in Example 3.5. Elements c and d are atoms in this quasi-ordered residuated system: $L(A) = \{c, d\}$. Let us put $B = A \cup \{w\}$ and define the operations in B as follows

$*$	1	a	b	c	d	w
1	1	a	b	c	d	w
a	1	1	1	1	1	1
b	1	a	1	1	1	1
c	1	a	d	1	d	w
d	1	a	c	c	1	w
w	1	a	b	c	d	1

From the table, which determined the $*$ operation, it can be seen that $a \leqslant c < 1 \wedge b \leqslant c < 1$, $a \leqslant d < 1 \wedge b \leqslant d < 1$, and $a \leqslant w < 1 \ \wedge \ b \leqslant w < 1$. So, $L(B) = \{c, d, w\}$. □

4. Final remarks

The concept of BE-algebra is a generalization of the concept of BCK-algebra. aBE-algebra and BE**-algebra are two mutually independent special subclasses of the class BE-algebras, and both of them are a special subclass of the class aBE**-algebras. This report discussed the concept of atoms in the previously mentioned BE-algebra classes. In addition, two different extensions of BE-algebras were analyzed, which enable the designed extension to have at least one atom in it. One addition to this report could be the analysis of atoms in implicative BE-algebras that were introduced in [11].

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