ISSN: 2832-4293

Pan-American Journal of Mathematics 3 (2024), 7
https://doi.org/10.28919/cpr-pajm/3-7

EXISTENCE AND MULTIPLICITY RESULTS FOR AN ELLIPTIC PROBLEM INVOLVING MIXED LOCAL AND NONLOCAL OPERATOR

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ABSTRACT. In this work, we consider the following mixed local-nonlocal quasilinear ellipic problem

$$(P_\lambda)\begin{cases} -\Delta_p u + (-\Delta)^s_p u &= \lambda f(u) \quad \text{in } \Omega,\\ u &> 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain in $\mathbb{R}^N$ with $0 < s < 1 < p < N$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, that have a finite number of zeroes, changing sign between them. The main goal of this paper is to prove the existence and multiplicity of positive solutions for such problems by using variational methods.

1. INTRODUCTION

The aim of this paper is to discuss the existence and multiplicity of positive solutions to the following mixed local-nonlocal quasilinear ellipic problem

$$\begin{cases} -\Delta_p u + (-\Delta)^s_p u &= \lambda f(u) \quad \text{in } \Omega,\\ u &> 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ be a bounded regular domain with $0 < s < 1 < p < N$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous functions which changes sign and fulfill some suitable hypotheses that will be presice later.
Here $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is standard $p-$ Laplace operator and $(-\Delta)^s_p$ denotes the so-called *fractional $p-$Laplacian* operator, is defined as,

$$(-\Delta)^s_p u(x) := \text{P.V} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\, dy,$$

where P.V. denotes the principal value. For more details and properties of the fractional Laplacian, the interested reader is referred to [10, 32].

Problems driven by following operator

$$\mathcal{L}_{p,s}(.) := -\Delta_p(.) + (-\Delta)^s_p(.)$$

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Submitted on December 27, 2023.
2020 *Mathematics Subject Classification.* 35R11, 35B09, 35J35.
Key words and phrases. Mixed local and nonlocal $p-$ Laplace operators, exitence and mutiplicity, positive solutions, variational methods.
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have raised a certain interest in the last few years, for example in connection with the study of optimal animal foraging strategies (see for example [33]). For other applications in the differente fields generated by mixed operators, we refer readers [14,35,45,47] and references therein.

Recently, several investigations have concentrated to the mixed local and non-local operator from differents view, like the regularity theory, maximum principle, boundary Harnack principle, existence and non-existence results, calculus of variations, shape of optimization and eigenvalue problems see for instance, [6–9,11–13,16,17,25,26,29,30,34,39,41,44,48,50] and the references therein.

Throughout this paper, we assume the following assumptions:

$(\mathcal{H}_1)$ f is a continuous function such that $f(0) \geq 0$, and there are

$$0 < a_1 < b_1 < a_2 < .... < b_{m-1} < a_m$$

the zeroes of f, such that

$$\begin{cases} f \leq 0 & \text{in } (a_k, b_k), \\ f \geq 0 & \text{in } (b_k, a_{k+1}); \end{cases}$$

$(\mathcal{H}_2)$ $\int_{a_k}^{a_{k+1}} f(t)dt > 0, \forall k \in \Big\{1, 2, ..., m-1\Big\}$.

Before starting our main results, we begin by recalling some well known results related to our problem.

Notice that when operator $\mathcal{L}_{p,s}$ is replaced by $p-$ Laplacian operator in problem $(P)_\lambda$, we have that

$$\begin{cases} -\Delta_p u &=& \lambda f(u) & \text{in } \Omega, \\ u &>& 0 & \text{in } \Omega, \\ u &=& 0 & \text{in } \partial\Omega, \end{cases} \tag{1.2}$$

where $p > 1$ and λ is real positive parameter.

In the case of semilinear problem corresponding to $p = 2$, it's was considered in [46] where the authors have showed the existence and multilicity of positive solutions to problem (1.2) under the hypothesis $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, by using variational and topological methods and arguments with lower and upper solutions (see also [15,24,28]). The quasilinear case that is for $p \in (1, \infty)$, authors in [43] extented the results obtained in [46].

In [19] the authors generalize the results obtained [43,46] to delicate case of the $p, q-$ Laplacian operator that is

$$\begin{cases} -\Delta_p u - \Delta_q u &=& \lambda f(u) & \text{in } \Omega, \\ u &>& 0 & \text{in } \Omega, \\ u &=& 0 & \text{in } \partial\Omega, \end{cases} \tag{1.3}$$

where the existence and multiplicity of positive solutions have been showed by using variational methods.

The case of $\phi-$ Laplacian, it's was treated in [49], where the considered problem

$$\begin{cases} -div(\phi(|\nabla u|)\nabla u) &=& \lambda f(u) & \text{in } \Omega, \\ u &=& 0 & \text{in } \partial\Omega, \end{cases} \tag{1.4}$$

ϕ is C^1 function fulfilling some suitable conditions. The authors have obtained the same results as in [46] even if ϕ is not reflexive, (see also [20]).

Recently, Problem (1.1) has been treated by another type of operator, notably for a doubly anisotropic operator; see [40]. We refer the readers also, [18,21,23,27,31,36,42,51,52] for more general context and the references therein.

Needless to say, the references mentioned above do not exhaust the rich literature on the subject.

The main goal of this work is to establish the existence and multiplicity results for the mixed problem (1.1) under the hypothesis $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ by using variational methods. Although it is worth to mention that the presence of the nonlocal operator in the mixed equation cannot be neglected and such nonlocal affect is one of the main obstacle, see [7]. To overcome this difficulty, we simultaneously employ the theory developed for the $p-$ Laplacian and fractional $p-$ Laplacian operators to study the mixed problem (1.1). As far as we aware, our main results are new even in the semilinear case $p = 2$.

The paper is organized as follows. In the next section we give some preliminaries dealing with functional setting associated to our problem, like the concepts of solutions, some functional inequalities and useful lemmas are included that will be needed along of the paper. Section 3 is devoted to showing some preliminary results that will be useful for the proofs of our main results. In Section 4, we prove the existence and multiplicity result of solutions for Problem (1.1) by using Ekeland variational principle. The paper is ended with some concluding remarks.

2. The functional setting and tools

In this section we collect some well-known results on fractional Sobolev spaces and give some tools that will be used in the proofs of our main results.

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open-bounded set with $N \geq 2$ and $0 < s < 1 < p < \infty$ be the real numbers. The fractional Sobolev space is defined by,

$$W^{s,p}(\Omega) = \Big\{u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy < +\infty\Big\}.$$

$W^{s,p}(\Omega)$ is Banach space equipped with the norm,

$$\|u\|_{W^{s,p}(\Omega)} = \left(||u||_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy\right)^{\frac{1}{p}}.$$

Notice that, the space $W^{s,p}(\mathbb{R}^N)$ is defined analogously.

If we assume that, Ω is bounded regular domain of $\mathbb{R}^N$, we can define $W_0^{s,p}(\Omega)$ is the set of functions defined as,

$$W_0^{s,p}(\Omega) = \Big\{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega\Big\}$$

and

$$\|u\|_{W_0^{s,p}(\Omega)} = \left(\int\int_{D_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy\right)^{\frac{1}{p}},$$

where

$$D_\Omega := (\mathbb{R}^N \times \mathbb{R}^N) \setminus \big((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)\big).$$

Both $W^{s,p}(\Omega)$ and $W_0^{s,p}(\Omega)$ are reflexive Banach spaces, see [4,32] for more details.

The Sobolev space $W^{1,p}(\Omega)$ with $1 < p < \infty$, is defined as the Banach space of locally integrable weakly differentiable functions $u : \Omega \to \mathbb{R}^N$ endowed with norm

$$||u||_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)}.$$

The space $W^{1,p}(\mathbb{R}^N)$ is defined analogously as $W^{1,p}(\Omega)$.

To study mixed local and nonlocal problems, we use the following space

$$W_0^{1,p}(\Omega) = \left\{u \in W^{1,p}(\mathbb{R}^N) : u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega\right\}$$

equipped with norm

$$||u||_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy\right)^{\frac{1}{p}} \tag{2.1}$$

Notice that, $W_0^{1,p}(\Omega), ||.||_{W_0^{1,p}(\Omega)})$ is reflexive Banach space for $1 < p < \infty$, see for example [5].

Next, we have the following result where the proof can be find in [16].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be bounded Lipschitz domain and $0 < s < 1 < p < \infty$, then, there exists a constant $C = C(N, p, s)$ such that,*

$$||u||_{W^{s,p}(\Omega)} \le C||u||_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \tag{2.2}$$

We need also the following result, see [16] for more details.

Lemma 2.2. *Under the same hypothesis of the previous lemma, then, there exists a constant $C = C(N, p, s, \Omega)$ such that,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \le C \int_\Omega |\nabla u|^p dx, \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.3}$$

Remark 2.3. It clear that from the previous inequality and the norm of space $W_0^{1,p}(\Omega)$ defined in (2.1) is equivalent to

$$||u||_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}. \tag{2.4}$$

For the following Sobolev embedding, see, for example, [37].

Lemma 2.4. *The embedding operators*

$$W_0^{1,p}(\Omega) \hookrightarrow \begin{cases} L^q(\Omega) & \text{for } q \in [1, p^*], \text{ if } p \in (1, N) \\ L^q(\Omega) & \text{for } q \in [1, \infty), \text{ if } p = N, \\ L^\infty(\Omega) & \text{if } p > N \end{cases},$$

are continuous. Also, the above embeddings are compact, except for $q = p^* = \frac{pN}{N-p}$, if $1 < p < N$.

We define the notion of zero of Dirichlet boundary condition as follows,

Definition 2.5. We say that $u \le 0$ on $\partial\Omega$, if $u = 0$ in $\mathbb{R}^N \setminus \Omega$ and for every $\epsilon > 0$, we have that,

$$(u - \epsilon)_+ \in W_0^{1,p}(\Omega).$$

We say that, $u = 0$ on $\partial\Omega$, if u in nonnegative and $u \le 0$ on $\partial\Omega$.

Now, we need to precise the sense of the weak solution for the problem (1.1).

Definition 2.6. We say that $u \in W_0^{1,p}(\Omega)$ is a energy solution to (1.1), if

$$u > 0 \quad \text{in } \Omega, u = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega, \quad \text{and for every} \quad \phi \in W_0^{1,p}(\Omega), \text{ we have that}$$

$$\begin{aligned} &\int\int_{D_\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}} dxdy \\ &+ \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi dx = \lambda \int_\Omega f(u)\phi dx. \end{aligned} \tag{2.5}$$

Next, we state also the following algebraic inequality where the proof can be found in [22],

Lemma 2.7. *Let $1 < p < \infty$. Then for any $\xi_1, \xi_2 \in \mathbb{R}^N$, there exists a constant positive $C := C(p)$ such that*

$$\langle |\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2, \xi_1 - \xi_2 \rangle \ge C \frac{|\xi_1 - \xi_2|^2}{(|\xi_2| + |\xi_1|)^{2-p}}. \tag{2.6}$$

3. Some useful lemmas

In this section we show two preliminary results that will be used in the proof of the existence theorem.

Let us start by following result

Lemma 3.1. *Let $s \in (0,1)$ and $1 < p < N$. Assume that $g \in C(\mathbb{R})$ be a continuous function and $\beta_0 > 0$ be such that*

$$\begin{cases} g(\alpha) \geq 0 & \text{if } \alpha \in (-\infty, \beta_0), \\ g(\alpha) \leq 0 & \text{if } \alpha \in [\beta_0, +\infty). \end{cases}$$

If $u \in W_0^{1,p}(\Omega)$ is a weak solution of

$$\begin{cases} -\Delta_p u + (-\Delta)_p^s u &= g(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.1}$$

Then $u \geq 0$ a.e in Ω, $u \in L^\infty(\Omega)$ and $||u||_{L^\infty(\Omega)} \leq \beta_0$.

Proof. Let $u \in W_0^{1,p}(\Omega)$ be the weak solution to problem (3.1). Choosing u^- as test function to problem (3.1), we get

$$\int_\Omega |\nabla u^-|^p dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+ps}} dxdy = \int_\Omega g(u)u^- dx. \tag{3.2}$$

So, from Lemma 3.1 in [38], we get

$$|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^-(x)-u^-(y)) \geq 0, \quad \text{for all } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{3.3}$$

Using (3.2) and (3.3), we obtain

$$\int_{\Omega\cap\{u<0\}} |\nabla u|^p dx \leq \int_{\Omega\cap\{u<0\}} g(u)u dx,$$

so, by using the definition of g, $g(u)u \leq 0$ when $u < 0$, we have $\int_{\Omega\cap\{u<0\}} |\nabla u|^p dx \leq 0$, and therefore necessarily the set $\Omega \cap \{u < 0\}$ is a null measure, hence, we have that $u \geq 0$ a.e in Ω.
On the other hand, to show that u is bounded, we use $\varphi^+ \equiv (u-\beta_0)^+$ as test function in problem (3.1), we have

$$\begin{aligned} &\int\int_{D_\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi^+(x)-\varphi^+(y))}{|x-y|^{N+ps}} dxdy \\ &+ \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi^+ dx = \int_\Omega g(u)\varphi^+ dx. \end{aligned}$$

So,

$$\int_\Omega |\nabla \varphi^+|^p dx \leq \int_\Omega g(u)\varphi^+ dx,$$

which yields

$$\int_\Omega |\nabla (u-\beta_0)^+|^p dx \leq \int_{\Omega\cap\{u\geq\beta_0\}} g(u)(u-\beta_0) dx \leq 0.$$

Hence $(u-\beta_0)^+ \equiv 0$ and we conclude that $u \leq \beta_0$ a.e in . As desired. □

We now consider, for each $k \in \Big\{2, ..., m\Big\}$, the truncation of f given by

$$f_k(\alpha) = \begin{cases} f(0) & \text{if } \alpha \leq 0, \\ f(\alpha) & \text{if } 0 \leq \alpha \leq a_k, \\ 0 & \text{if } \alpha > a_k; \end{cases}$$

For each > 0, let us consider the functional $J_{,k}$ defined as follows

$$J_{\lambda,k}(u) = \frac{1}{p}\int_{\Omega} |\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy - \lambda \int_{\Omega} F_k(u)dx,$$

where $F_k(\sigma) = \int_0^\sigma f_k(t)dt$. Let us denote by $\mathcal{C}_{\lambda,k}$ the set of critical points of $J_{\lambda,k}$.

Notice that $J_{\lambda,k}$ is the energy functional of the following problem

$$(P)_k \begin{cases} -\Delta_p u + (-\Delta)^s_p u &= \lambda f_k(u) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and its weak solutions are the critical points of the functional $J_{,k}$.

A direct consequence of the previous lemma is the following result

Lemma 3.2. *Let $s \in (0,1)$ and $1 < p < N$. $u \in \mathcal{C}_{\lambda,k}$, if, and only if, u is a nonnegative weak solution of the problem $(P)_k$ and $u \in L^\infty(\Omega)$ with $||u||_{L^\infty(\Omega)} \le a_k$. Consequently, u is a nonnegative weak solution of problem* (1.1).

4. Existence and multiplicity results

In this section we study the existence and multiplicity of positive solutions to Problem (1.1) under the hypothesis $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$.

Let us start by following result,

Theorem 4.1. *For each $k \in \Big\{1, 2, ..., m-1\Big\}$, problem $(P_\lambda)_k$, possesses a nonnegative solution $u = u_{\lambda,k}$ for every $\lambda > 0$ with $||u_{\lambda,k}||_{L^\infty(\Omega)} \le a_k$.*

Proof. A direct consequence of Lemma 3.1, we have that $||u_{\lambda,k}||_{L^\infty(\Omega)} \le a_k$.

Since f_k is bounded function we have that

$$m_k|t| \le \int_0^t m_k d\sigma \le F_k(t) \le \int_0^t M_k d\sigma \le M_k|t|,$$

therefore, by using Höder and Sobolev inequalities, it follows that

$$\begin{aligned} J_{\lambda,k}(u) &= \frac{1}{p}\int_{\Omega} |\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy - \int_{\Omega} F_k(u)dx \\ &\ge \frac{1}{p}\int_{\Omega} |\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dx - \lambda \int_{\Omega} M_k|u|dx, \\ &\ge \frac{1}{p}||u||^p_{W^{1,p}_0(\Omega)} - M_k\lambda C||u||_{W^{1,p}_0(\Omega)}. \end{aligned} \tag{4.1}$$

Now, by using the fact $||u||_{W^{1,p}_0(\Omega)} \to +\infty$, we get $J_{,k} \to +\infty$, and the coercivity of $J_{\lambda,k}$ follows. Moreover from (4.1), implies that $J_{\lambda,k}$ is bounded from below in $W^{1,p}_0(\Omega)$ and so we have

$$J^k_\infty := \inf_{W^{1,p}_0(\Omega)} J_{\lambda,k}.$$

Since $J_{\lambda,k}$ is continuous (indeed $J_{\lambda,k}$ is of C^1),which implies $J_{\lambda,k}$ is lower semi-continuous. So, by using Ekeland Variational Principle we get the existence $\{u_n\}$ such that

$$J_{\lambda,k}(u_n) \to J^k_\infty \text{ and } J'_{\lambda,k}(u_n) \to 0 \text{ in } (W^{1,p}_0(\Omega))'.$$

Since $\{u_n\}$ is a $(PS)_{J^k_\infty}$ sequence of $J_{\lambda,k}$, so, by using the fact that $J_{\lambda,k}$ is coercive, the boundedness of $\{u_n\}$ in $W_0^{1,p}(\Omega)$ follows. Hence, by applying Lemma 2.4, we get subsequence $\{u_n\}$ such that

$$u_n \rightharpoonup u, \text{ weakly in } W_0^{1,p}(\Omega), \tag{4.2}$$

$$u_n \to u, \text{ strongly in } L^q(\Omega) \text{ for every } 1 \leq q < p^*, \tag{4.3}$$

and

$$u_n(x) \to u(x), \text{ a.e in } \Omega. \tag{4.4}$$

On the other hand, we have that

$$\langle J'_{\lambda,k}(u_n), \varphi\rangle = o(1), \tag{4.5}$$

for every $\varphi \in W_0^{1,p}(\Omega)$.
Thus, by choosing $\varphi := (u_n - u)$ in (4.5), we get

$$\begin{aligned}&\int\int_{D_\Omega} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy\\ &+\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi(x)dx - \lambda\int_\Omega f_k(u_n)\varphi(x)dx = o(1),\end{aligned}$$

which gives

$$\begin{aligned}&\int\int_{D_\Omega} \frac{\Big(|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y)) - |u(x)-u(y)|^{p-2}(u(x)-u(y))\Big)(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy\\ &+\int\int_{D_\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy + \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi(x)dx\\ &+\int_\Omega \Big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\Big)\nabla\varphi dx - \lambda\int_\Omega f_k(u_n)\varphi(x)dx = o(1).\end{aligned} \tag{4.6}$$

Consequently, by using (4.2), we get

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi(x)dx = o(1), \tag{4.7}$$

$$\int\int_{D_\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy = o(1), \tag{4.8}$$

and

$$\Big(|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y)) - |u(x)-u(y)|^{p-2}(u(x)-u(y))\Big)(\varphi(x)-\varphi(y)) \geq 0, \tag{4.9}$$

Combining with (4.6)-(4.9) and using Lemma 2.7 for $p \geq 2$, it holds that

$$c||\nabla u_n - \nabla u||^p_{W_0^{1,p}(\Omega)} \leq o(1) + \lambda M_k \int_\Omega (u_n - u)dx, \tag{4.10}$$

so, by invoking (4.3), we conclude that

$$u_n \to u, \text{ strongly in } W_0^{1,p}(\Omega).$$

The same result can be obtained for $1 < p < 2$, by using in a similar way Lemma 2.7 in the case $1 < p < 2$, which end the proof. □

We are now going to state and prove our second existence result.

Theorem 4.2. *Let $s \in (0,1)$ and $1 < p < N$. Assume that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. Then, there exists $\lambda^* > 0$, such that for every $\lambda \in (\lambda^*, +\infty)$ problem* (1.1) *possesses at least $(m-1)$ nonnegative weak solutions u_i with*

$$u_i \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \text{ and } a_i \leq ||u_i||_{L^\infty(\Omega)} \leq a_{i+1}, \quad \forall i \in \Big\{1, 2, ..., m-1\Big\}.$$

Proof. Let $u \in W_0^{1,p}(\Omega)$ be a solution to problem $(P_\lambda)_k$, then by using Lemma 3.1, we get $u \in L^\infty(\Omega)$ and $0 \leq u < a_{k-1}$ a.e in Ω, so, $f_{k-1}(u) = f(u)$ and hence u is also a solution to (1.1).

To show the last part of the theorem, we claim that, for each $k \in \{2, ..., m\}$, there exists $\lambda_k > 0$ such that for all $\lambda > \lambda_k$, we have $u_{k,\lambda} \notin \mathcal{C}_{\lambda,k-1}$ where

$$J_{\lambda,k}(u_{k,\lambda}) = \inf_{v \in W_0^{1,p}(\Omega)} J_{\lambda,k}(v).$$

In first, for $\delta > 0$, let us consider the open set

$$\Omega_\delta := \Big\{x \in \Omega, \quad dist(x, \partial\Omega) < \delta\Big\}$$

and

$$\alpha_k = F(a_k) - \max_{0<t<a_{k-1}} |F(t)| = F(a_k) - C_k,$$

so, by invoking hypothesis $(\mathcal{H}_2)$, it holds that $\alpha_k > 0$.

On the other hand, let $w_\delta \in \mathcal{C}_c^\infty(\Omega)$ such that $0 \leq w_\delta \leq a_k$ and $w_\delta = a_k$, so, we have that

$$\int_\Omega F(w_\delta) \geq \int_\Omega F(a_k) - 2C_k|\Omega_\delta|.$$

Therefore, for each $u \in W_0^{1,p}(\Omega)$ such that $0 \leq u \leq a_{k-1}$, we obtain

$$\int_\Omega F(w_\delta) - \int_\Omega F(u) \geq \alpha_k|\Omega| - 2C_k|\Omega_\delta|. \tag{4.11}$$

Hence, by choosing $\delta > 0$ such that

$$\gamma_k = \alpha_k|\Omega| - 2C_k|\Omega_\delta| > 0,$$

we have

$$\begin{aligned}
&J_{\lambda,k}(w_\delta) - J_{\lambda,k-1}(u_{\lambda,k-1}) = \frac{1}{p}\int_\Omega \left(|\nabla w_\delta|^p - |\nabla u_{,k-1}|^p\right) dx \\
&+\frac{1}{p}\int\int_{D_\Omega} \left(\frac{|w_\delta(x) - w_\delta(y)|^p}{|x-y|^{N+ps}} - \frac{|u_{\lambda,k-1}(x) - u_{\lambda,k-1}(y)|^p}{|x-y|^{N+ps}}\right) dxdy \\
&-\lambda\int_\Omega \left(F_k(w_\delta) - F_k(u_{\lambda,k-1})\right) dx \\
&\leq \frac{1}{p}\int_\Omega |\nabla w_\delta|^p dx + \frac{1}{p}\int\int_{D_\Omega} \frac{|w_\delta(x) - w_\delta(y)|^p}{|x-y|^{N+ps}} - \lambda\gamma_k = \frac{1}{p}||w_\delta||_{W_0^{1,p}(\Omega)} - \lambda\gamma_k.
\end{aligned} \tag{4.12}$$

where in the last inequality we have used (4.11). Therefore, by choosing large enough in (4.12), it holds that

$$J_{\lambda,k}(w_\delta) - J_{\lambda,k-1}(u_{\lambda,k-1}) < 0,$$

and for $w = w_\delta$, we derive that

$$J_{\lambda,k}(u_{\lambda,k}) \leq J_{\lambda,k}(w) \leq J_{\lambda,k}(u_{\lambda,k-1}). \tag{4.13}$$

Consequently by using Lemma 3.2 and the previous inequelity, we get $u_{\lambda,k}$ and $u_{\lambda,k-1}$ are two distinct solutions to Problem (1.1).

Now, we claim that

$$a_{k-1} \leq ||u_{\lambda,k}||_{L^\infty(\Omega)} \leq a_k.$$

To do this, we assume by contradiction

$$0 \leq u_{\lambda,k} < a_{k-1},$$

so, we necessarily would have that

$$J_{\lambda,k-1}(u_{\lambda,k-1}) \leq J_{\lambda,k}(u_{\lambda,k}) = J_{\lambda,k}(u_{\lambda,k}),$$

which is a contradiction with (4.13) and claim follows. Hence the result follows. □

5. Concluding remarks

In this section we give some remarks related to our problem.

Under the hypothesis $(\mathcal{H})_1$ and $(\mathcal{H})_2$:

• We can generalize all the results obtained here to the following to nonlocal quasilinear elliptic problem

$$\begin{cases} (-\Delta)_p^s u &= \lambda f(u) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s \in (0,1)$ and $1 < p < \frac{N}{s}$.

• By some minor modifcations, we can obtain similar results for nonlocal problem of the form

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_p^{s_2} u &= \lambda f(u) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s_1,\ s_2 \in (0,1)$ with $s_1 < s_2$ and $1 < p < \frac{N}{s_1}$.

Competing interests. The authors declare no competing interests.

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