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EXPLORING TWO MODIFIED ALI-MIKHAIL-HAQ COPULAS AND NEW BIVARIATE LOGISTIC DISTRIBUTIONS

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ABSTRACT. The Ali-Mikhail-Haq copula is a bivariate ratio-type Archimedean copula known for its simplicity and flexibility in modeling moderate negative and positive dependence structures, making it widely used in various fields. However, it is limited to capturing asymmetric dependence features, significant negative correlations, or versatile tail dependence properties. This research paper proposes two modifications of the Ali-Mikhail-Haq copula that overcome these limitations, but at the price of the loss of the positive dependence nature. Contrary to the common approach that focuses on modifying the corresponding generator function, we apply direct functional changes to the Ali-Mikhail-Haq copula. We thus perturb its Archimedean identity. The first copula has the particularity of being non-exchangeable, capable of reaching an interesting level of negative dependence correlations, and possessing flexible tail dependence properties. The second copula offers another modeling option; it is exchangeable like the Ali-Mikhail-Haq copula, but it benefits from a broader range of negative dependence correlations and more adaptable tail dependence properties. For the two proposed copulas, we investigate their main characteristics, including quadrant dependence, copula density function, conditional copulas, couples of value generation, extended variants via standard copula schemes, comprehensive copula orders, and weighted harmonic mean copula transformations. An application on two new bivariate logistic distributions in a two-component system context is given. When possible, numerical and graphical studies are given to strengthen the theory.

1. INTRODUCTION

The context, contributions and organization of the paper are now presented.

1.1. **Context.** Since the Sklar seminal work in [33], copulas have gained importance thanks to their ability to capture complex dependence features between random variables. Unlike traditional correlation measures, they offer a more nuanced approach by separating marginal distributions from the joint distribution. Thus, they allow a flexible representation of dependence structures. The basics of copulas can be found in the books [27] and [12]. In order to have a first mathematical element, the definition of a copula in the bivariate absolutely continuous context is recalled below.

Definition 1.1. We consider the bivariate absolutely continuous context. Let $D(u, v)$, $(u, v) \in [0, 1]^2$, be a differentiable function on $(0, 1)^2$. Then $D(u, v)$ is called a copula if it meets the following two requirements:

- **(I):** $D(u, 1) = u$, $D(1, v) = v$, $D(u, 0) = D(0, u) = 0$.
- **(II):** $\dfrac{\partial^2}{\partial u \partial v} D(u, v) \geq 0$.

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We refer to [27] for more information on these technical requirements.

This copula definition, taken into the associated bivariate absolutely continuous context, will be central to this study.

For decades, copulas have found applications in various fields, including finance, risk management, insurance, environmental science, reliability engineering, hydrology, and actuarial science. In parallel with this highly applicable activity, researchers often modify existing copulas or propose new ones to adapt their properties to specific challenges. As interdisciplinary demand for accurate and robust modeling increases, copulas continue to play a central role in advancing statistical methodologies and improving our understanding of multivariate dependence patterns. Recent studies in this regard include those in [8], [10], [24], [31], [4], [5] and [34].

In this research paper, we make our contribution to the subject by modifying, in a certain sense, the capabilities of the famous Ali-Mikhail-Haq (AMH) copula. Thus, in order to explain the interest of these contributions, a retrospective on this copula is necessary.

1.2. **On the AMH copula.** The AMH copula finds its origin in [1]. It was an intermediary mathematical tool to construct a single-parameter version of the Gumbel bivariate logistic distribution established in [16]. To be more precise, the AMH copula is defined as

$$C_{\ddagger}(u,v;a) = uv\frac{1}{1+a(1-u)(1-v)}, \quad (u,v)\in[0,1]^2, \tag{1.1}$$

with $a\in[-1,1]$. Based on this copula, by considering the standard logistic distribution for the two univariate marginal distributions, i.e., with the cumulative distribution function (CDF) $F(x)=[1+\exp(-x)]^{-1}$, $x\in\mathbb{R}$, the proposed single-parameter extended version of the Gumbel bivariate logistic distribution in [1] is

$$\begin{aligned} F(x,y;a) &= C_{\ddagger}[F(x),F(y);a] \\ &= F(x)F(y)\frac{1}{1+a[1-F(x)][1-F(y)]} \\ &= \frac{\exp(x+y)}{a+[\exp(x)+1][\exp(y)+1]}, \quad (x,y)\in\mathbb{R}^2, \end{aligned}$$

or equivalently, with more adequacy to the expression in [1],

$$F(x,y;a) = [1+\exp(-x)+\exp(-y)+(1+a)\exp(-x-y)]^{-1}, \quad (x,y)\in\mathbb{R}^2.$$

The choice of $a=-1$ yields the Gumbel bivariate logistic distribution as studied in [16].

Some known facts and results about the AMH copula are recalled below. First, it is a member of the Archimedean family of copulas and is characterised by the following strict generator function:

$$\phi(t) = \log\left[\frac{1+a(1-t)}{t}\right], \quad t>0,$$

with $\lim_{t\to 0}\phi(t) = +\infty$. In fact, the AMH copula is one of the famous twenty-two single-parameter Archimedean copulas presented in [27, Table 4.1]. It is clearly exchangeable, i.e., $C_{\ddagger}(u,v;a)=C_{\ddagger}(v,u;a)$ for any $(u,v)\in[0,1]^2$, and associative, i.e., $C_{\ddagger}[C_{\ddagger}(u,v;a),w;a]=C_{\ddagger}[u,C_{\ddagger}(v,w;a);a]$ for any $(u,v,w)\in[0,1]^3$. Furthermore, it does not have tail dependence except for $a=-1$, where a rigid left tail of 0.5 is obtained (see [22, Theorem 3]). It is positively quadrant dependent for $a\in[-1,0]$, i.e., $C_{\ddagger}(u,v;a)\geq uv$ for any $(u,v)\in[0,1]^2$, and negatively quadrant dependent for $a\in(0,1]$, i.e., $C_{\ddagger}(u,v;a)\leq uv$ for any $(u,v)\in[0,1]^2$. It is one of the rare copulas to enjoy a simple and comprehensive harmonic mean property (see [35, Proposition 1]). On the correlation aspects, the range for the associated medial correlation is $[-0.2, 0.3333]$, and that of the associated Spearman rho is $[33-48\log(2), 4\pi^2-39]\approx[-0.2711, 0.4784]$ (see [22, Page 661]). Thus, the AMH copula has both positive and negative moderate dependence correlations. However, these results imply that it is not appropriate for asymmetric or moderate-to-strong negative dependence. Despite these limitations, the AMH copula has attracted attention for decades, revealing itself to be a suitable dependence model in various statistical scenarios. See, for instance, [14], [25], [27], [22], [35], and [9].

1.3. **Contributions.** With the idea of overcoming some limitations of the AMH copula and keeping a similar ratio-type form, we propose two modifications of it. They are based on direct functional changes of $C_{\ddagger}(u, v; a)$; we do not alter directly the associated strict generator function $\phi(t)$ and go beyond the standard Archimedean scheme. More precisely, the modified copulas are of the following form:

$$C_{\bullet}(u, v; a) = uv\frac{\psi(u)\varphi(v)}{\psi(u)\varphi(v) + a(1-u)(1-v)}, \quad (u, v) \in [0, 1]^2,$$

with $a \in \mathbb{R}$ (to be refined), $\psi(u) \in \{1, u\}$ and $\varphi(v) \in \{1, v\}$, or, equivalently,

$$C_{\bullet}(u, v : a) = uv\frac{1}{1 + a\Psi(u)\Phi(v)}, \quad (u, v) \in [0, 1]^2,$$

with $\Psi(u) = (1-u)/\psi(u) \in \{1-u, u^{-1}-1\}$ and $\Phi(v) = (1-v)/\varphi(v) \in \{1-v, v^{-1}-1\}$. Such a similar single-parameter ratio-type copula form was investigated in [6] but with original functions for $\Psi(u)$ and $\Phi(v)$, completely out of the polynomial denominator definition of the AMH copula. Of course, the admissible values for a making $C_{\bullet}(u, v; a)$ a valid copula remain a challenge that this research work addresses. However, applying our modification strategy results in some gains and losses. Indeed, after a thorough investigation, we lost sight of the positive dependence nature of the AMH copula and its Archimedean identity. For the gains, let us distinguish the two proposed copulas. The first one has the advantage of being non-exchangeable, capable of reaching an interesting level of negative dependence correlations, and possessing flexible tail dependence properties. The second proposed copula gains from other modeling angles. Indeed, it is exchangeable like the AMH copula, but it benefits from a broader range of negative dependence correlations and more adaptable tail dependence properties. These facts are emphasized via standard measures and benchmarks. On the other hand, for each of them, we investigate some important mathematical characteristics, such as the nature of the quadrant dependence, the copula density function and its shapes, the conditional copulas along with their uses in couples of value generation, several extended variants via standard copula schemes, comprehensive copula orders involving old and modern copulas, and the weighted harmonic mean copula transformatons. As a probability application, two new single-parameter variants of the Gumbel bivariate logistic distribution are established. Their uses in a two-component system context, with the consideration of minimum and maximum random variables, are also examined. The theory is reinforced using numerical and graphical investigation.

1.4. **Organization.** The rest of this research paper contains the following sections: Section 2 describes the first proposed copula, along with its main characteristics. Section 3 does the same for the second one. Two new bivariate logistic distributions are described in Section 4, along with a discussion in their implication in two-component random systems. A conclusion is formulated in Section 5.

2. First proposed ratio-type copula

2.1. **Description.** The first proposed ratio-type copula is explored in the subsequent proposition.

Proposition 2.1. *The following bivariate function is a copula for $a \in [0, 1]$:*

$$C_1(u, v; a) = uv\frac{u}{u + a(1-u)(1-v)}, \quad (u, v) \in [0, 1]^2. \tag{2.1}$$

Proof. The validation of **(I)** and **(II)** in Definition 1.1 forms the foundation of the proof. Let us start by examining **(I)**. For any $u \in [0,1]$, we obtain

$$C_1(u,1;a) = u \times 1 \times \frac{u}{u + a(1-u)(1-1)} = u \times \frac{u}{u} = u.$$

Also, for any $v \in [0,1]$, we have

$$C_1(1,v;a) = 1 \times v \times \frac{1}{1 + a(1-1)(1-v)} = v \times 1 = v.$$

On the other hand, for any $u \in [0,1]$, we get

$$C_1(u,0;a) = u \times 0 \times \frac{u}{u + a(1-u)(1-0)} = 0.$$

Similarly, for any $v \in [0,1]$, we have

$$C_1(0,v;a) = 0 \times v \times \frac{0}{0 + a(1-0)(1-v)} = 0.$$

The results above validate **(I)**.

Let us now focus on **(II)**. Using multiple differentiation procedures and suitable factorizations, for any $(u,v) \in (0,1)^2$, we find that

$$\begin{aligned}
\frac{\partial^2}{\partial u \partial v} C_1(u,v;a) &= -\frac{au^2 v}{[u + a(1-u)(1-v)]^2} - \frac{2a(1-u)u^2 v[1 - a(1-v)]}{[u + a(1-u)(1-v)]^3} \\
&- \frac{u^2[1-a(1-v)]}{[u + a(1-u)(1-v)]^2} + \frac{2a(1-u)uv}{[u + a(1-u)(1-v)]^2} + \frac{2u}{u + a(1-u)(1-v)} \\
&= u \frac{a^2(2-u)(1-u)(1-v) + K(u,v;a)}{[u + a(1-u)(1-v)]^3},
\end{aligned}$$

where

$$K(u,v;a) = au[u(v-2) - 3v + 3] + u^2.$$

Since $u + a(1-u)(1-v) \geq 0$, $2 - u \geq 0$, $1 - u \geq 0$ and $1 - v \geq 0$, it is clear that **(II)** is fulfilled if $K(u,v;a) \geq 0$. To demonstrate that, the key is a suitable factorization of $K(u,v;a)$; for $a \in [0,1]$, we have

$$\begin{aligned}
K(u,v;a) &= au[(1-u)(1-v) + 2(1-v)] + (1-a)u^2 \\
&\geq (1-a)u^2 \geq 0.
\end{aligned}$$

Thus, **(II)** is established. As a result, $C_1(u,v;a)$ is a valid copula, ending the proof. □

Let us name the copula $C_1(u,v;a)$ in Equation (2.1) as the first new ratio-type (FNRT) copula, and assume in the rest of this section that $a \in [0,1]$. To make a parallel with the expression of the AMH copula, we can rewrite it as

$$C_1(u,v;a) = uv \frac{1}{1 + a(u^{-1} - 1)(1-v)}, \quad (u,v) \in [0,1]^2.$$

Hence, the AMH and FNRT copulas have similar mathematical ingredients (ratio-type type, product uv, etc.), but the difference is that the term $1-u$ in the denominator of the AMH copula is replaced by $u^{-1} - 1$ and that the parameter a is restricted to the interval $[0,1]$. These changes, which can be considered minor at first glance, have notable consequences. Some of these consequences are discussed in the next subsection.

2.2. **Properties.** Some of the important properties of the FNRT copula are studied in the proposition below.

Proposition 2.2. *The FNRT copula possesses the following properties:*

- *It is not exchangeable for $a \in (0,1]$ (and exchangeable for $a = 0$).*
- *It is a decreasing function with respect to a.*
- *It is negative quadrant dependent.*
- *Its medial correlation has a range of values of $[-0.3333, 0]$.*
- *Its Spearman rho has a range of values of $[-0.4784, 0]$.*
- *It is upper left tail dependent only.*

Proof. The listed items are proved in turns below.

- We can always find (u_ι, v_ι) such that

$$\begin{aligned} C_1(u_\iota, v_\iota; a) &= u_\iota v_\iota \frac{u_\iota}{u_\iota + a(1-u_\iota)(1-v_\iota)} \\ &\neq u_\iota v_\iota \frac{v_\iota}{v_\iota + a(1-u_\iota)(1-v_\iota)} = C_1(v_\iota, u_\iota; a), \end{aligned}$$

implying that the FNRT copula is not exchangeable. For $a = 0$, the independence copula is obtained; we have $C_1(u, v; 0) = uv$, which is obviously exchangeable.

- Clearly, for $a_1 \leq a_2$, we have $a_1(1-u)(1-v) \leq a_2(1-u)(1-v)$, which implies that

$$C_1(u, v; a_2) = uv \frac{u}{u + a_2(1-u)(1-v)} \leq uv \frac{u}{u + a_1(1-u)(1-v)} = C_1(u, v; a_1).$$

So the FNRT copula is decreasing with respect to a.

- It follows from the above result that, for any $a \in [0,1]$, we have $C_1(u, v; a) \leq C_1(u, v; 0) = uv$, which corresponds to the negative quadrant property.
- Based on a formula in [27], the medial correlation of the FNRT copula is defined by

$$M(a) = 4C_1\left(\frac{1}{2}, \frac{1}{2}; a\right) - 1 = -\frac{a}{2+a}.$$

It is a decreasing function with respect to a, so, for $a \in [0,1]$, the associated range of values is $[M(1), M(0)]$. We have $M(0) = 0$ and $M(1) = -1/3 \approx -0.3333$. Thus the range of values for $M(a)$ is $[-0.3333, 0]$.

- Owing to a formula in [27], the Spearman rho of the FNRT copula is defined by

$$\rho(a) = 12 \int_0^1 \int_0^1 C_1(u, v; a) du dv - 3.$$

Since $C_1(u, v; a)$ is a decreasing function with respect to a, the same holds for $\rho(a)$. Thus, since $a \in [0,1]$, the associated range of values is $[\rho(1), \rho(0)]$. We have

$$\rho(0) = 12 \int_0^1 \int_0^1 C_1(u, v; 0) du dv - 3 = 12 \int_0^1 \int_0^1 uv du dv - 3 = 12 \times \left(\frac{1}{2}\right)^2 - 3 = 0$$

and, after tedious integral developments, we find that

$$\begin{aligned}\rho(1) &= 12\int_0^1\int_0^1 C_1(u,v;1)dudv-3\\ &= 12\int_0^1\left[\int_0^1 uv\frac{u}{u+(1-u)(1-v)}du\right]dv-3\\ &= 12\int_0^1\frac{v(3v-2)-2(v-1)^2\log(1-v)}{2v^2}dv-3\\ &= 12\left(\frac{7}{2}-\frac{\pi^2}{3}\right)-3\approx -0.4784.\end{aligned}$$

Therefore, the Spearman rho has a range of values of $[-0.4784, 0]$.

- The diverse kinds of tail dependence of the FNRT copula can be revealed by the calculus of the related parameters. In particular, owing to the formulas in [27] and [18], the lower left tail dependence parameter is given by

$$\lambda_{LL}=\lim_{u\to 0}\frac{C_1(u,u;a)}{u}=\lim_{u\to 0}\frac{u^2}{u+a(1-u)^2}=0,$$

the lower right tail dependence parameter is obtained as

$$\lambda_{LR}=\lim_{u\to 0}\frac{u-C_1(1-u,u;a)}{u}=\lim_{u\to 0}\frac{u-(1-u)^2u/[1-u+au(1-u)]}{u}=0,$$

the upper left tail dependence parameter is given by

$$\lambda_{UL}=\lim_{u\to 0}\frac{u-C_1(u,1-u;a)}{u}=\lim_{u\to 0}\frac{u-u^2(1-u)/[u+au(1-u)]}{u}=\frac{a}{a+1},$$

and the upper right tail dependence parameter is computed as

$$\lambda_{UR}=\lim_{u\to 1}\frac{1-2u+C_1(u,u;a)}{1-u}=\lim_{u\to 1}\frac{1-2u+u^3/[u+a(1-u)^2]}{1-u}=0.$$

As a result, since only $\lambda_{UL}=a/(a+1)\neq 0$ for $a\in(0,1]$, the FNRT copula is only upper left tail dependent.

The listed results are proved. □

Therefore, contrary to the AMH copula, the FNRT copula is non-exchangeable (for $a\in(0,1]$) so it can model asymmetric dependences, and it reaches a greater range of negative dependence than the AMH copula, i.e., for the medial correlation, we have $[-0.3333, 0]$ versus $[-0.2, 0]$ and for the Spearman rho, we have $[-0.4784, 0]$ versus $[-0.2711, 0]$. However, the FNRT copula cannot reach positive dependence.

2.3. **Functional analysis.** The FNRT copula is now analyzed from a functional perspective, with the help of graphics. Some related results are also discussed.

To begin, let us depict the FNRT copula for two different values of a with $a\in[0,1]$ in Figure 1. We mention that all the graphics and numerical study of this research paper are performed with the software R (see [28]).

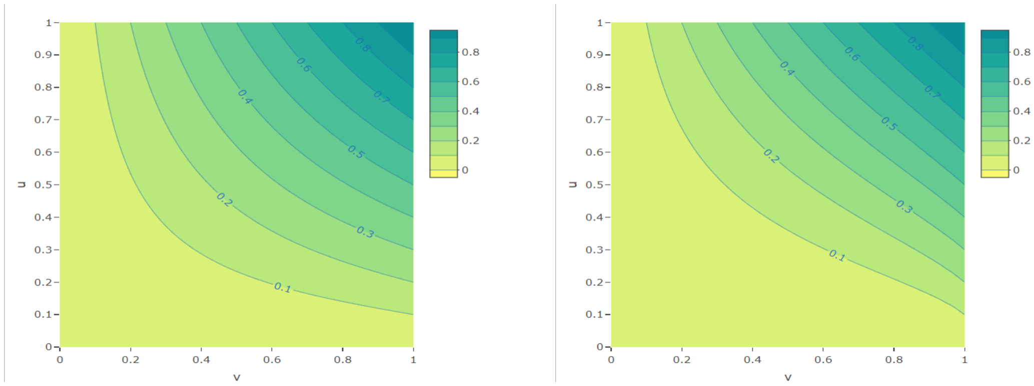

FIGURE 1. Plots of the contours of the FNRT copula for $a = 0.1$ (left) and $a = 0.9$ (right).

The analysis of these figures confirms the validity of the FNRT copula. The asymmetric contours are clearly visible for $a = 0.9$.

On the other hand, the copula density derived from the FNRT copula is obtained as

$$\begin{aligned} c_1(u, v; a) &= \frac{\partial^2}{\partial u \partial v} C_1(u, v; a) \\ &= u \frac{a^2(2-u)(1-u)(1-v) + au[(1-u)(1-v) + 2(1-v)] + (1-a)u^2}{[u + a(1-u)(1-v)]^3}, \\ &\quad (u, v) \in [0, 1]^2. \end{aligned}$$

The shapes of this function are important to understand the modeling ability of the FNRT copula. For a direct examination of them, we perform a graphical study. Figure 2 represents $c_1(u, v; a)$ for two different values of a with $a \in [0, 1]$.

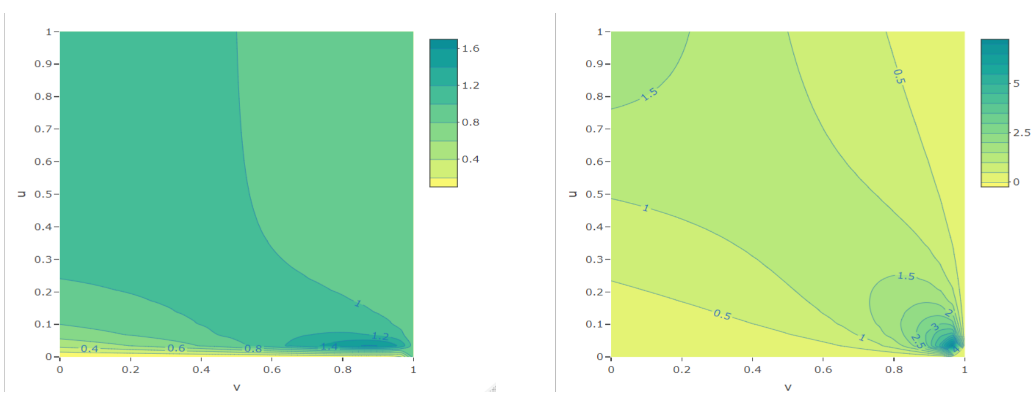

FIGURE 2. Plots of the contours of the FNRT copula density for $a = 0.1$ (left) and $a = 0.9$ (right).

From this figure, nuanced contours are observed for the FNRT copula density. Furthermore, the upper left tail dependence at the corner point $(0, 1)$ is flagrant; a certain concentration of high values is present.

On the other hand, the first conditional FNRT copulas is given by

$$C_{\sqcap}(u, v; a) = \frac{\partial}{\partial u} C_1(u, v; a) = uv \frac{a(2-u)(1-v) + u}{[u + a(1-u)(1-v)]^2}$$

and the second one is specified as

$$C_{\sqcup}(u,v;a) = \frac{\partial}{\partial v}C_1(u,v;a) = u^2\frac{u+a(1-u)}{[u+a(1-u)(1-v)]^2}, \quad (u,v)\in[0,1]^2.$$

In particular, the conditional copula $C_{\sqcap}(u,v;a)$ can be useful for generating a couple of values (u_ι, v_ι) from a random vector (U,V) having the FNRT copula as a cumulative distribution function (CDF). The standard generation scheme is as follows:

- Generate a couple of independent values (u_ι, w) based on the uniform distribution over $[0,1]$.
- Find v_ι satisfying $C_{\sqcap}(u_\iota, v_\iota; a) = w$.
- Take into account (u_ι, v_ι).

Figure 3 shows the scatter plots of 500 such generated couples of values for two different values of a with $a\in[0,1]$.

FIGURE 3. Scatter plots of 500 generated couples of values derived from the FNRT copula for $a = 0.1$ (left) and $a = 0.9$ (right).

With some distinct point clusters and a rather well-defined structure, this figure illustrates some types of dependent patterns. The most evident cluster is at the corner $(1,0)$, which is coherent with the upper left tail dependence nature of the FNRT copula.

2.4. **Related copulas.** Since the FNRT copula is not exchangeable (for $a\in(0,1]$), a natural copula comes by exchanging the variables u and v; the following bivariate function is a copula:

$$C_\star(u,v;a) = C_1(v,u;a) = uv\frac{v}{v+a(1-u)(1-v)}, \quad (u,v)\in[0,1]^2,$$

still with $a\in[0,1]$.

Also, the standard flipping and survival techniques described in [27] can be applied to the FNRT copula to generate new ratio-type copulas. In particular, the u-flipping FNRT copula is obtained as

$$\begin{aligned} C_\wedge(u,v;a) &= v - C_1(1-u,v;a) \\ &= v-(1-u)v\frac{1-u}{1-u+au(1-v)}, \quad (u,v)\in[0,1]^2, \end{aligned}$$

the v-flipping FNRT copula is given by

$$\begin{aligned} C_\vee(u,v;a) &= u - C_1(u,1-v;a) \\ &= u-u(1-v)\frac{u}{u+a(1-u)v}, \quad (u,v)\in[0,1]^2, \end{aligned}$$

and the survival FNRT copula is indicated as

$$\begin{aligned}C_{\curlywedge}(u,v;a) &= u+v-1+C_1(1-u,1-v;a)\\ &= u+v-1+(1-u)(1-v)\frac{1-u}{1-u+auv}, \quad (u,v)\in[0,1]^2.\end{aligned}$$

The power function technique elaborated in [11] can also be applied. Let us consider $b \in [0,1]$ and $c \in [0,1]$. Then a valid copula based on the FNRT copula is given by

$$\begin{aligned}C_{\diamond}(u,v;a,b,c) &= u^b v^c C_1(u^{1-b}, v^{1-c};a)\\ &= uv\frac{u^{1-b}}{u^{1-b}+a(1-u^{1-b})(1-v^{1-c})}, \quad (u,v)\in[0,1]^2.\end{aligned}$$

While adding greater flexibility to the FNRT copula, the parameters b and c complicate the mathematical scheme. From a practical point of view, there is a risk of the over-parameterization phenomenon.

Another option is the mixture technique, as described in [27]. It ensures that the following integral bivariate function is a copula:

$$\begin{aligned}C_{\triangle}(u,v) &= \int_0^1 C_1(u,v;a)da = \int_0^1 uv\frac{u}{u+a(1-u)(1-v)}da\\ &= uv\frac{\log\left[1+(u^{-1}-1)(1-v)\right]}{(u^{-1}-1)(1-v)}.\end{aligned}$$

It represents a new non-exchangeable ratio-type logarithmic copula of the literature. Since $a \in [0,1]$, we can eventually extend it in a parametric way by the duplication-parameter technique developed in [7] as

$$\begin{aligned}C_{\triangledown}(u,v;a) &= \int_0^1 C_1(u,v;ab)db = \int_0^1 uv\frac{u}{u+ab(1-u)(1-v)}db\\ &= uv\frac{\log\left[1+a(u^{-1}-1)(1-v)\right]}{a(u^{-1}-1)(1-v)}.\end{aligned}$$

When $a \to 0$, the independence copula is obtained. Thus this single-parametric non-exchangeable copula has the advantage to be general in form and cover the indepence case. It may be valuable to consider a dependence problem demonstrating such characteristics.

2.5. **Additional properties.** Since the FNRT copula is not exchangeable, it is obviously not Archimedean, contrary to the AMH copula.

In addition, the FNRT and AMH copulas are related by the following nonlinear equation:

$$\frac{uv}{1-u}\left(\frac{1}{C_{\ddagger}(u,v;a)}-\frac{u}{C_1(u,v;a)}\right)=1.$$

As for any valid copula, the FNRT copula is 1-Lipschitz, i.e., for any $(u_1,u_2,v_1,v_2)\in[0,1]^4$, we have

$$|C_1(u_1,v_1;a)-C_1(u_2,v_2;a)| \le |u_2-u_1| - |v_2-v_1|,$$

so

$$\left|u_1v_1\frac{u_1}{u_1+a(1-u_1)(1-v_1)}-u_2v_2\frac{u_2}{u_2+a(1-u_2)(1-v_2)}\right| \le |u_2-u_1| - |v_2-v_1|.$$

This bivariate inequality can be of independent interest in the field of analysis.

The FNRT copula benefits from tail increasing characteristics, as demonstrated below. For any $(u,v)\in(0,1)^2$, we have

$$\frac{\partial}{\partial u}\left(\frac{C_1(u,v;a)}{u}\right)=\frac{a(1-v)v}{[u+a(1-u)(1-v)]^2}\ge 0$$

and

$$\frac{\partial}{\partial v}\left(\frac{C_1(u,v;a)}{v}\right)=\frac{au^2(1-u)}{[u+a(1-u)(1-v)]^2}\geq 0,$$

implying that the FNRT copula is left tail increasing (for both variables u and v). Furthermore, we have

$$\frac{\partial}{\partial u}\left(\frac{v-C_1(u,v;a)}{1-u}\right)=\frac{a^2v(1-v)^2}{[u+a(1-u)(1-v)]^2}\geq 0$$

and

$$\frac{\partial}{\partial v}\left(\frac{u-C_1(u,v;a)}{1-v}\right)=\frac{au(1-u)[u-a(1-u)]}{[u+a(1-u)(1-v)]^2}\geq 0,$$

implying that the FNRT copula is right tail increasing (for both variables u and v). More details on the notion of tail increase can be found in [1].

After tedious differentiation steps and factorizations, we establish that

$$\frac{\partial^2}{\partial u^2}C_1(u,v;a)+\frac{\partial^2}{\partial v^2}C_1(u,v;a)=2a\frac{a[(1-u)^2u^2+(1-v)^2v]+(1-u)u^3}{[uv+a(1-u)(1-v)]^3}\geq 0.$$

By following the notion of harmonicity in [27], we conclude that the FFNRT copula is subharmonic.

The FNRT copula satisfies interesting copula orders, as discussed below. The following exponential inequality is well-known: $\exp(t)\geq 1+t$ for $t\in\mathbb{R}$. Therefore, for any $(u,v)\in[0,1]^2$, we have $\exp[a(u^{-1}-1)(1-v)]\geq 1+a(u^{-1}-1)(1-v)$, which implies that

$$C_1(u,v;a)=uv\frac{u}{u+a(1-u)(1-v)}\geq uv\exp[-a(u^{-1}-1)(1-v)]=C_\dagger(u,v;a),$$

where $C_\dagger(u,v;a)$ is a modified version of the Celebioglu-Cuadras copula (see [4]).

For any $(u,v)\in[0,1]^2$, we have $1-u\geq u(1-u)$, which can be rewritten as $u^{-1}-1\geq 1-u$, and $1+a(u^{-1}-1)(1-v)\geq 1+a(1-u)(1-v)$. Therefore, an inequality involving both the FNRT and AMH copulas is

$$C_1(u,b;a)=uv\frac{u}{u+a(1-u)(1-v)}\leq uv\frac{1}{1+a(1-u)(1-v)}=C_\ddagger(u,v;a),$$

where $C_\ddagger(u,v;a)$ is the AMH copula as described in Equation (1.1).

The two above results thus give the following copula orders:

$$C_\dagger(u,v;a)\leq C_1(u,b;a)\leq C_\ddagger(u,v;a), \quad (2.2)$$

and so position the FNRT copula into two well-referenced copulas.

The weighted harmonic mean transformation of two FNRT copulas with different parameters is still a FNRT copula with a special parameter. Indeed, let $a_1\in[0,1]$, $a_2\in[0,1]$ and $b\in[0,1]$, and

$$C_\smile(u,v;a_1,a_2,b)=\frac{1}{b/C_1(u,v;a_1)+(1-b)/C_1(u,v;a_2)},$$

be such a weighted harmonic mean of the copulas $C_1(u,v;a_1)$ and $C_1(u,v;a_2)$, with weight b. Then, by expliciting $C_1(u,v;a_1)$ and $C_1(u,v;a_2)$, we establish that

$$\begin{aligned}C_\smile(u,v;a_1,a_2,b)&=\frac{uv}{b[1+a_1(u^{-1}-1)(1-v)]+(1-b)[1+a_2(u^{-1}-1)(1-v)]}\\&=\frac{uv}{1+[ba_1+(1-b)a_2](u^{-1}-1)(1-v)}=C_1[u,v;ba_1+(1-b)a_2].\end{aligned}$$

Based on Proposition 2.1, we thus obtain the FNRT copula with parameter $ba_1+(1-b)a_2$, which is valid if $ba_1+(1-b)a_2\in[0,1]$, thus proving the statement.

This ends the study of the FNRT copula. The second proposed ratio-type copula is developed in the next section.

3. Second proposed ratio-type copula

3.1. **Description.** The second proposed ratio-type copula is examined in the subsequent proposition.

Proposition 3.1. *The following bivariate function is a copula for $a \in [0,1]$:*

$$C_2(u,v;a) = uv\frac{uv}{uv + a(1-u)(1-v)}, \quad (u,v) \in [0,1]^2. \tag{3.1}$$

Proof. We follow the structure of the proof of Proposition 2.1; the validation of **(I)** and **(II)** in Definition 1.1 is the challenge. Let us begin with the study of **(I)**. For any $u \in [0,1]$, we obtain

$$C_2(u,1;a) = u \times 1 \times \frac{u \times 1}{u \times 1 + a(1-u)(1-1)} = u \times \frac{u}{u} = u.$$

Also, for any $v \in [0,1]$, we have

$$C_2(1,v;a) = 1 \times v \times \frac{1 \times v}{1 \times v + a(1-1)(1-v)} = v \times \frac{v}{v} = v.$$

On the other hand, for any $u \in [0,1]$, we obtain

$$C_2(u,0;a) = u \times 0 \times \frac{u \times 0}{u \times 0 + a(1-u)(1-0)} = 0.$$

Similarly, for any $v \in [0,1]$, we get

$$C_2(0,v;a) = 0 \times v \times \frac{0 \times v}{0 \times v + a(1-0)(1-v)} = 0.$$

The equalities in **(I)** are proved.

Let us now demonstrate **(II)**. With several differentiation techniques and appropriate factorizations, for any $(u,v) \in (0,1)^2$, we obtain

$$\begin{aligned}
\frac{\partial^2}{\partial u \partial v} C_2(u,v;a) &= -\frac{(a+1)u^2v^2}{[uv + a(1-u)(1-v)]^2} \\
&+ \frac{2u^2v^2[u - a(1-u)][v - a(1-v)]}{[uv + a(1-u)(1-v)]^3} - \frac{2u^2v[v - a(1-v)]}{[uv + a(1-u)(1-v)]^2} \\
&- \frac{2uv^2[u - a(1-u)]}{[uv + a(1-u)(1-v)]^2} + \frac{4uv}{uv + a(1-u)(1-v)} \\
&= uv\frac{a^2(2-u)(1-u)(2-v)(1-v) + L(u,v;a)}{[uv + a(1-u)(1-v)]^3},
\end{aligned}$$

where

$$L(u,v;a) = auv[u(2v-3) - 3v + 3] + u^2v^2.$$

Since $uv + a(1-u)(1-v) \geq 0$, $2-u \geq 0$, $1-u \geq 0$, $2-v \geq 0$ and $1-v \geq 0$, the assumption **(II)** is fulfilled if $L(u,v;a) \geq 0$. Let us prove it via an appropriate factorization. For $a \in [0,1]$, we have

$$\begin{aligned}
L(u,v;a) &= 3auv(1-u)(1-v) + (1-a)u^2v^2 \\
&\geq (1-a)u^2v^2 \geq 0.
\end{aligned}$$

This establishes **(II)**. The proof concludes the fact that $C_2(u,v;a)$ is a valid copula. □

Let us name the copula $C_2(u,v;a)$ in Equation (3.1) as the second new ratio-type (SNRT) copula and and assume in the rest of this section that $a \in [0,1]$. Similarly to the AMH and FNRT copulas, we can rewrite it in another ratio-type form as follows:

$$C_2(u,v;a) = uv\frac{1}{1 + a(u^{-1}-1)(v^{-1}-1)}, \quad (u,v) \in [0,1]^2.$$

Hence, the AMH and SNRT copulas have similar mathematical ingredients (ratio-type type, product uv, etc.), but the difference is that the terms $1-u$ and $1-v$ in the denominator of the AMH copula are replaced by $u^{-1}-1$ and $v^{-1}-1$, respectively, and that the parameter a is restricted to the interval $[0,1]$. As the next subsection highlights, these modifications make the SNRT copula original in some aspects.

3.2. **Properties.** Some of the important properties of the SNRT copula are studied in the result below.

Proposition 3.2. *The SNRT copula possesses the following properties:*

- *It is exchangeable.*
- *it is a decreasing function with respect to a.*
- *it is negative quadrant dependent.*
- *Its medial correlation has a range of values of $[-0.5, 0]$.*
- *Its Spearman rho has a range of values of $[-0.7011, 0]$.*
- *It is lower right and upper left tails dependent only.*

Proof. The listed items are proved in turns below.

- For any $(u,v) \in [0,1]^2$, we have
$$C_2(u,v;a) = uv\frac{uv}{uv + a(1-u)(1-v)} = vu\frac{vu}{vu + a(1-v)(1-u)} = C_2(v,u;a),$$
implying that the SNRT copula is exchangeable.
- Clearly, for $a_1 \leq a_2$, we have $a_1(1-u)(1-v) \leq a_2(1-u)(1-v)$, which implies that
$$C_2(u,v;a_2) = uv\frac{uv}{uv + a_2(1-u)(1-v)} \leq uv\frac{uv}{uv + a_1(1-u)(1-v)} = C_2(u,v;a_1).$$
Hence, the SNRT copula is decreasing with respect to a.
- It follows from the above result that, for any $a \in [0,1]$, we have $C_2(u,v;a) \leq C_2(u,v;0) = uv$. The SNRT copula has thus a negative quadrant property.
- The medial correlation of the SNRT copula is indicated by
$$M(a) = 4C_2\left(\frac{1}{2}, \frac{1}{2}; a\right) - 1 = -\frac{a}{1+a}.$$
Since it is a decreasing function with respect to a for $a \in [0,1]$, the associated range of values is $[M(1), M(0)]$. We have $M(0) = 0$ and $M(1) = -1/2 = -0.5$. Thus the range of values for $M(a)$ is $[-0.5, 0]$.
- The Spearman rho of the SNRT copula is defined by
$$\rho(a) = 12\int_0^1\int_0^1 C_2(u,v;a)dudv - 3.$$
Since $C_2(u,v;a)$ is a decreasing function with respect to a, this also applies to $\rho(a)$. Thus, by taking into account that $a \in [0,1]$, the associated range of values is $[\rho(1), \rho(0)]$. We have
$$\rho(0) = 12\int_0^1\int_0^1 C_2(u,v;0)dudv - 3 = 12\int_0^1\int_0^1 uvdudv - 3 = 0$$
and, by the use of numerical techniques, we find that
$$\begin{aligned}\rho(1) &= 12\int_0^1\int_0^1 C_2(u,v;1)dudv - 3\\ &= 12\int_0^1\left[\int_0^1 uv\frac{uv}{uv + (1-u)(1-v)}du\right]dv - 3 \approx 12 \times 0.191575 - 3\\ &= -0.7011.\end{aligned}$$

Therefore, the Spearman rho has a range of values of $[-0.7011, 0]$.

- Let us investigate the four tail dependence parameters. The lower left tail dependence parameter is given by

$$\lambda_{LL} = \lim_{u\to 0} \frac{C_2(u,u;a)}{u} = \lim_{u\to 0} \frac{u^3}{u^2 + a(1-u)^2} = 0,$$

the lower right tail dependence parameter is obtained as

$$\lambda_{LR} = \lim_{u\to 0} \frac{u - C_2(1-u,u;a)}{u} = \lim_{u\to 0} \frac{u - (1-u)^2u^2/[(1-u)u + au(1-u)]}{u} = \frac{a}{a+1},$$

the upper left tail dependence parameter is given by

$$\lambda_{UL} = \lim_{u\to 0} \frac{u - C_2(u,1-u;a)}{u} = \lim_{u\to 0} \frac{u - u^2(1-u)^2/[u(1-u) + au(1-u)]}{u} = \frac{a}{a+1},$$

and the upper right tail dependence parameter is computed as

$$\lambda_{UR} = \lim_{u\to 1} \frac{1 - 2u + C_2(u,u;a)}{1-u} = \lim_{u\to 1} \frac{1 - 2u + u^4/[u^2 + a(1-u)^2]}{1-u} = 0.$$

As a result, since $\lambda_{LR} = \lambda_{UL} = a/(a+1) \neq 0$ for $a \in (0,1]$, the SNRT copula is lower right and upper left tails dependent.

This ends the proof. □

In comparison to the FNRT copula, the SNRT copula is exchangeable exclusively. Furthermore, it has a wider range of negative values for the medial correlation and Spearman rho than the FNRT copula, i.e., for the medial correlation, we have $[-0.5, 0]$ versus $[-0.3333, 0]$ and for the Spearman rho, we have $[-0.7011, 0]$ versus $[-0.4784, 0]$. Thus, the FNRT and SNRT copulas, even if constructed from a similar ratio-type scheme, accomplish different modeling goals.

On the other hand, if we compare the SNRT and AMH copulas, we observe that it has greater negative dependence flexibility, i.e., for the medial correlation, $[-0.5, 0]$ versus $[-0.2, 0]$, and for the Spearman rho, $[-0.7011, 0]$ versus $[-0.2711, 0]$, which are more than twice for both. However, it has no positive dependence, contrary to the AMH copula. Also, the SNRT copula is dependent on the lower right and left tails, which is advantageous if these characteristics need to be modulated based on a data analysis; the AMH copula is more rigid on these aspects.

3.3. **Functional analysis.** We now perform a functional analysis of the SNRT copula. To begin, let us depict it for two different values of a with $a \in [0, 1]$ in Figure 4.

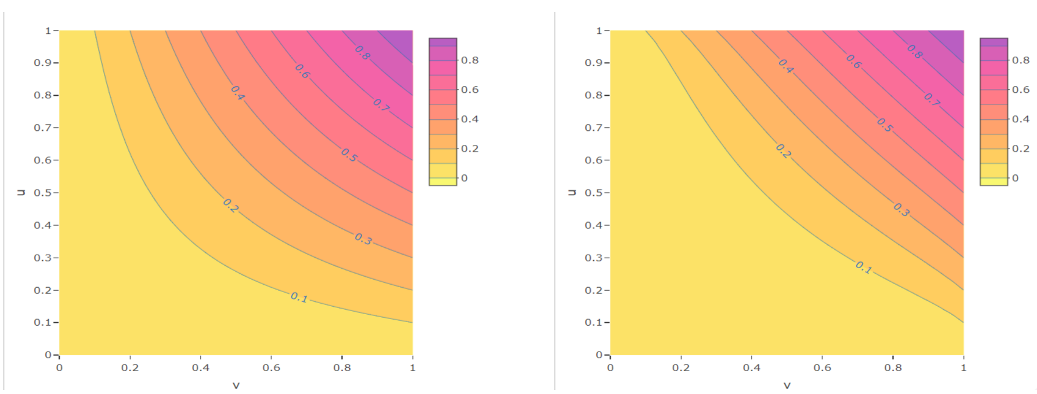

FIGURE 4. Plots of the contours of the SNRT copula for $a = 0.1$ (left) and $a = 0.9$ (right).

The validity of the SNRT copula is confirmed by the analysis of these figures. The typical semicircular shapes are observed, with increasing value trends into $[0, 1]$.

On the other hand, the copula density associated with the SNRT copula is as follows:

$$\begin{aligned} c_2(u, v; a) &= \frac{\partial^2}{\partial u \partial v} C_2(u, v; a) \\ &= uv \frac{a^2(2-u)(1-u)(2-v)(1-v) + auv[u(2v-3) - 3v + 3] + u^2v^2}{[uv + a(1-u)(1-v)]^3}, \\ &\quad (u, v) \in [0, 1]^2. \end{aligned}$$

To comprehend the modeling capacity of the SNRT copula, we must investigate the forms of this function. In Figure 5, we conduct a graphical analysis of $c_2(u, v; a)$ for two distinct values of a with $a \in [0, 1]$.

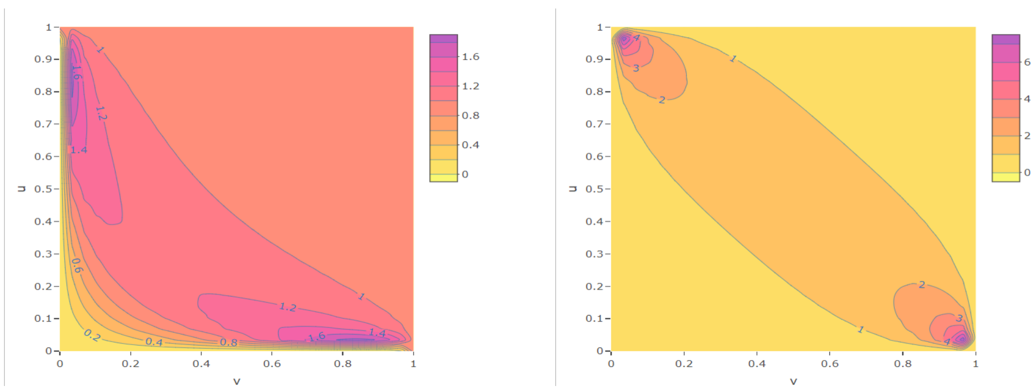

FIGURE 5. Plots of the contours of the SNRT copula density for $a = 0.1$ (left) and $a = 0.9$ (right).

Different contours for the SNRT copula density are visible in this figure. They are particularly notable at the top left and bottom right, in accordance with the demonstrated tail dependence properties.

On the other hand, the two conditional SNRT copulas are given by

$$C_{\sqcap}(u, v; a) = \frac{\partial}{\partial u} C_2(u, v; a) = uv^2 \frac{a(2-u)(1-v) + uv}{[uv + a(1-u)(1-v)]^2}$$

and

$$C_{\sqcup}(u, v; a) = \frac{\partial}{\partial v} C_2(u, v; a) = C_{\sqcap}(v, u; a) = vu^2 \frac{a(2-v)(1-u) + uv}{[uv + a(1-u)(1-v)]^2}, \quad (u, v) \in [0, 1]^2.$$

Similarly to the FNRT copula, the generation of a couple of values (u_ι, v_ι) from a random vector (U, V) having the SNRT copula as a CDF is as follows:

- Generate a couple of independent values (u_ι, w) based on the uniform distribution over $[0, 1]$.
- Find v_ι satisfying $C_{\sqcap}(u_\iota, v_\iota; a) = w$.
- Take into account (u_ι, v_ι).

Figure 6 depicts the scatter plots of 500 such generated couples of values for two different values of a with $a \in [0, 1]$.

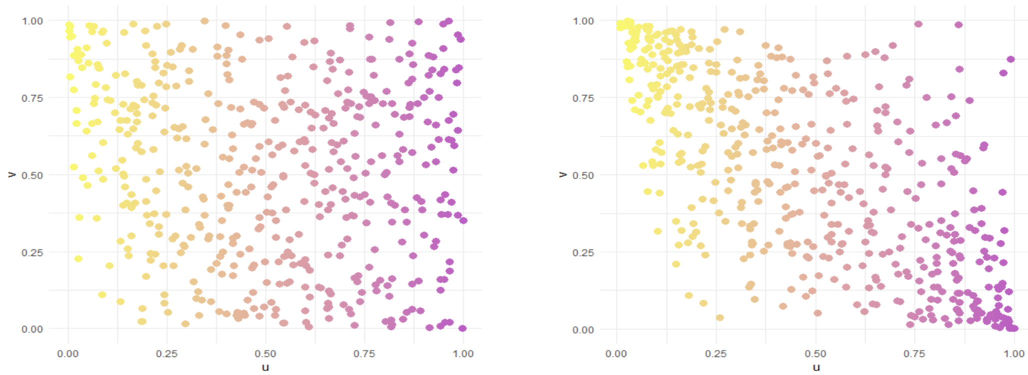

FIGURE 6. Scatter plots of 500 generated couples of values derived from the SNRT copula for $a = 0.1$ (left) and $a = 0.9$ (right).

This figure shows several kinds of dependence features; it has a reasonably well-defined structure with a few distinct clusters of points mainly concentrated on the top left and bottom right corners. This is logical in view of the demonstrated tail dependence properties of the SNRT copula.

3.4. **Related copulas.** Like for the FNRT copula, the flipping and survival techniques described in [27] can be applied to the SNRT copula. In particular, the u-flipping SNRT copula is obtained as

$$\begin{aligned} C_\wedge(u, v; a) &= v - C_2(1-u, v; a) \\ &= v - (1-u)v\frac{(1-u)v}{(1-u)v + au(1-v)}, \quad (u, v) \in [0,1]^2, \end{aligned}$$

the v-flipping SNRT copula is indicated as

$$\begin{aligned} C_\vee(u, v; a) &= u - C_2(u, 1-v; a) \\ &= u - u(1-v)\frac{u(1-v)}{u(1-v) + a(1-u)v}, \quad (u, v) \in [0,1]^2, \end{aligned}$$

and the survival SNRT copula is given by

$$\begin{aligned} C_\curlywedge(u, v; a) &= u + v - 1 + C_2(1-u, 1-v; a) \\ &= u + v - 1 + (1-u)(1-v)\frac{(1-u)(1-v)}{(1-u)(1-v) + auv}, \quad (u, v) \in [0,1]^2. \end{aligned}$$

The power function technique elaborated in [11] can also be applied. Let us consider $b \in [0, 1]$ and $c \in [0, 1]$. Then a valid copula based on the FNRT copula is given by

$$\begin{aligned} C_\diamond(u, v; a, b, c) &= u^b v^c C_2(u^{1-b}, v^{1-c}; a) \\ &= uv\frac{u^{1-b}v^{1-c}}{u^{1-b}v^{1-c} + a(1-u^{1-b})(1-v^{1-c})}, \quad (u, v) \in [0,1]^2. \end{aligned}$$

Like for the FNRT copula, although the parameters b and c provide the SNRT copula more flexibility, they also complicate the mathematical method.

On the other hand, the mixture technique provides

$$\begin{aligned} C_\triangle(u, v) &= \int_0^1 C_2(u, v; a)da = \int_0^1 uv\frac{uv}{uv + a(1-u)(1-v)}da \\ &= uv\frac{\log\left[1 + (u^{-1}-1)(v^{-1}-1)\right]}{(u^{-1}-1)(v^{-1}-1)}. \end{aligned}$$

We thus obtain a new exchangeable ratio-type logarithmic copula. Since $a \in [0,1]$, the duplication-parameter technique described in [7] can eventually be used to extend it parametrically. It gives

$$\begin{aligned} C_{\nabla}(u,v;a) &= \int_0^1 C_2(u,v;ab)db = \int_0^1 uv\frac{uv}{uv+ab(1-u)(1-v)}db \\ &= uv\frac{\log\left[1+a(u^{-1}-1)(v^{-1}-1)\right]}{a(u^{-1}-1)(v^{-1}-1)}. \end{aligned}$$

When $a \to 0$, the independence copula is obtained. The originality, generality, and coverage of the independence situation are hence advantages of this single-parameter exchangeable copula.

3.5. **Additional properties.** First, let us investigate the Archimedean nature of the SNRT copula through its potential associative property by considering a special example. By choosing $a = 0.05$, we have

$$C_2\left[\frac{1}{5}, C_2\left(\frac{1}{4},\frac{1}{3};a\right);a\right] = 0.0033 \neq 0.0025 = C_2\left[C_2\left(\frac{1}{5},\frac{1}{4};a\right),\frac{1}{3};a\right].$$

This contradicts the associative property; the SNRT copula is not Archimedean. This is a notable difference from the AMH copula.

In addition, the SNRT, FNRT, and AMH copulas are related by the following nonlinear equations:

$$\frac{uv}{1-uv}\left(\frac{1}{C_{\ddagger}(u,v;a)} - \frac{uv}{C_2(u,v;a)}\right) = 1$$

and

$$\frac{uv}{1-v}\left(\frac{1}{C_1(u,v;a)} - \frac{v}{C_2(u,v;a)}\right) = 1.$$

As for any valid copula, such as the AMH and FNRT copulas, the SNRT copula is 1-Lipschitz, i.e., for any $(u_1,u_2,v_1,v_2) \in [0,1]^4$, we have

$$|C_2(u_1,v_1;a) - C_2(u_2,v_2;a)| \leq |u_2-u_1| - |v_2-v_1|,$$

so

$$\left|u_1v_1\frac{u_1v_1}{u_1v_1+a(1-u_1)(1-v_1)} - u_2v_2\frac{u_2v_2}{u_2v_2+a(1-u_2)(1-v_2)}\right| \leq |u_2-u_1| - |v_2-v_1|.$$

This bivariate inequality may be of independent interest.

As the FNRT copula, the SNRT copula benefits from interesting tail increasing characteristics. Indeed, for any $(u,v) \in (0,1)^2$, we have

$$\frac{\partial}{\partial u}\left(\frac{C_2(u,v;a)}{u}\right) = \frac{a(1-v)v^2}{[uv+a(1-u)(1-v)]^2} \geq 0,$$

implying that the SNRT copula is left tail increasing (for both variables u and v since it is exchangeable), and

$$\frac{\partial}{\partial u}\left(\frac{v - C_2(u,v;a)}{1-u}\right) = \frac{a^2v(1-v)^2}{[uv+a(1-u)(1-v)]^2} \geq 0,$$

implying that the SNRT copula is right tail increasing (for both variables u and v since it is exchangeable).

After tedious differentiation steps and factorizations, we obtain

$$\frac{\partial^2}{\partial u^2}C_2(u,v;a) + \frac{\partial^2}{\partial v^2}C_2(u,v;a) = 2a^2\frac{(1-u)^2u^2+(1-v)^2v^2}{[uv+a(1-u)(1-v)]^3} \geq 0.$$

Hence, the SNRT copula is subharmonic.

The SNRT copula is involved in interesting copula orders, as investigated below. First, based on the following exponential inequality: $\exp(t) \geq 1 + t$ for $t \in \mathbb{R}$, for any $(u, v) \in [0, 1]^2$, we get $\exp[a(u^{-1} - 1)(v^{-1} - 1)] \geq 1 + a(u^{-1} - 1)(v^{-1} - 1)$, which implies that

$$C_2(u, v; a) = uv\frac{uv}{uv + a(1-u)(1-v)} \geq uv\exp[-a(u^{-1} - 1)(v^{-1} - 1)] = C_{\curlywedge}(u, v; a),$$

where $C_{\curlywedge}(u, v; a)$ is a variant of the Celebioglu-Cuadras copula (see [5]).

In addition, for any $v \in [0, 1]$, we have $1 - v \geq v(1 - v)$, implying that $v^{-1} - 1 \geq 1 - v$. Therefore, we have $1 + a(u^{-1} - 1)(v^{-1} - 1) \geq 1 + a(u^{-1} - 1)(1 - v)$, and

$$C_2(u, b; a) = uv\frac{uv}{uv + a(1-u)(1-v)} \leq uv\frac{1}{1 + a(u^{-1} - 1)(1 - v)} = C_1(u, v; a),$$

where $C_1(u, v; a)$ is the FNRT copula.

The two above results thus give the following copula orders:

$$C_{\dagger}(u, v; a) \leq C_1(u, b; a) \leq C_1(u, v; a) \leq C_{\ddagger}(u, v; a),$$

where $C_{\ddagger}(u, v; a)$ is the AMH copula as described in Equation (1.1) (the last inequality being explained in Equation (2.2)). We thus position the SNRT copula into some introduced copulas.

Under some parameter assumptions, the weighted harmonic mean transformation of two SNRT copulas with different parameters is still a SNRT copula with a special parameter. Indeed, for any $a_1 \in [0, 1]$, $a_2 \in [0, 1]$ and $b \in [0, 1]$, we have

$$\begin{aligned} C_{\smile}(u, v; a_1, a_2, b) &= \frac{1}{b/C_2(u, v; a_1) + (1 - b)/C_2(u, v; a_2)} \\ &= \frac{uv}{b[1 + a_1(u^{-1} - 1)(v^{-1} - 1)] + (1 - b)[1 + a_2(u^{-1} - 1)(v^{-1} - 1)]} \\ &= \frac{uv}{1 + [ba_1 + (1 - b)a_2](u^{-1} - 1)(v^{-1} - 1)} = C_2[u, v; ba_1 + (1 - b)a_2]. \end{aligned}$$

Based on Proposition 3.1, we thus obtain the SNRT copula with parameter $ba_1 + (1 - b)a_2$, which is valid if $ba_1 + (1 - b)a_2 \in [0, 1]$.

Applications of our copula findings to various bivariate logistic distributions are performed in the next section.

4. Applications: new bivariate logistic distributions

Let us begin this section with an overview of the famous logistic distribution.

4.1. **On the logistic distribution.** First, the logistic distribution is a continuous distribution defined on the entire real line, similarly to the normal distribution. It is of considerable importance in various research fields. Its main functions, i.e., the CDF and probability density function (PDF), have found numerous applications, particularly in statistical methodologies such as logistic regression, logit-type models, and neural networks. We recall that the CDF of the (standard) logistic distribution is given by

$$F(x) = [1 + \exp(-x)]^{-1}, \quad x \in \mathbb{R},$$

also known in statistics under the name of sigmoid function, and the associated PDF is indicated as

$$f(x) = \exp(-x)[1 + \exp(-x)]^{-2}, \quad x \in \mathbb{R}.$$

Furthermore, it is a symmetric distribution around 0, its quantile function is given by $Q(y) = \log[y/(1-y)]$, $y \in [0, 1]$, its failure rate function corresponds to its CDF, i.e., $r(x) = f(x)/[1 - F(x)]$ is equal to $F(x)$, its mean is 0, its variance is $\pi^2/3$, its skewness is 0, its kurtosis is $6/5$ and its characteristic function has a closed-form expression.

Beyond statistical analyses, the logistic distribution has demonstrated its usefulness in the fields of physics, ecology, risk analysis, sports, and, more recently, finance. One of its main advantages is its characteristic of having wider tails and higher kurtosis compared to the (standard) normal distribution, i.e., $6/5$ versus 0. Therefore, the logistic distribution provides a valuable tool for researchers and practitioners seeking nuanced insights into the likelihood and implications of extreme events in various domains. For more information on this topic, we may refer to [19] and [2].

On the other hand, by extending the capabilities of the logistic distribution to bivariate scenarios, researchers can capture more complex relationships between two random variables, especially those whose data present wider tails and higher kurtosis compared to the bivariate normal distribution. The seminal work in this direction is in [16], which is based on the following CDF:

$$F(x,y) = \left[1+\exp(-x)+\exp(-y)\right]^{-1}, \quad (x,y) \in \mathbb{R}^2.$$

This function characterizes the so-called Gumbel bivariate logistic distribution. It has a univariate (standard) logistic distribution for both the marginal distributions and comprehensive properties. More detail on it can be found in [23] and [3]. Also, as a proof of its modern utility, its located and scaled version is implemented in the R package `VGAM` with the function `bilogistic` (see [37]).

The Gumbel bivariate logistic distribution was later flexibilized through the use of the AMH copula in [1]. Thus, the following single-parameter CDF was proposed:

$$F(x,y;a) = \left[1+\exp(-x)+\exp(-y)+(1+a)\exp(-x-y)\right]^{-1}, \quad (x,y) \in \mathbb{R}^2,$$

with $a \in [-1,1]$. It is clear that $F(x,y;a) = F(x,y)$ for $a = -1$. Another single-parametric extension was explored through the exponentiated scheme in [29], which considers the following CDF:

$$\begin{aligned} G(x,y;b) &= [F(x,y)]^b \\ &= \left[1+\exp(-x)+\exp(-y)\right]^{-b}, \quad (x,y) \in \mathbb{R}^2, \end{aligned}$$

with $b > 0$. A multifactorial model of disease transmission based on it was also discussed.

More technical or modern works on extended bivariate logistic extensions include [23], [15], [32], [21], [17] and [36].

4.2. **First new bivariate logistic distribution.** Based on the FNRT copula as defined in Equation (2.1), we define the FNRT logistic distribution by the following CDF:

$$\begin{aligned} F_1(x,y;a) &= C_1[F(x),F(y);a] \\ &= F(x)F(y)\frac{F(x)}{F(x)+a[1-F(x)][1-F(y)]} \\ &= \frac{\exp(2x+y)}{[\exp(x)+1][a+\exp(x+y)+\exp(x)]}, \quad (x,y) \in \mathbb{R}^2. \end{aligned}$$

By construction, the FNRT logistic distribution has a univariate (standard) logistic distribution for both marginal distributions, and it is non-exchangeable. Such non-exchangeable bivariate distributions are able to model the dependence between two random variables, taking into account the observation order. Their application is particularly valuable in fields such as finance, biology, and environmental sciences, where capturing nuanced and dynamic associations between ordered random variables is crucial for accurate modeling and interpretation.

Figures 7 and 8 present the CDF $F_1(x,y;a)$ for two different values of a with $a \in [0,1]$.

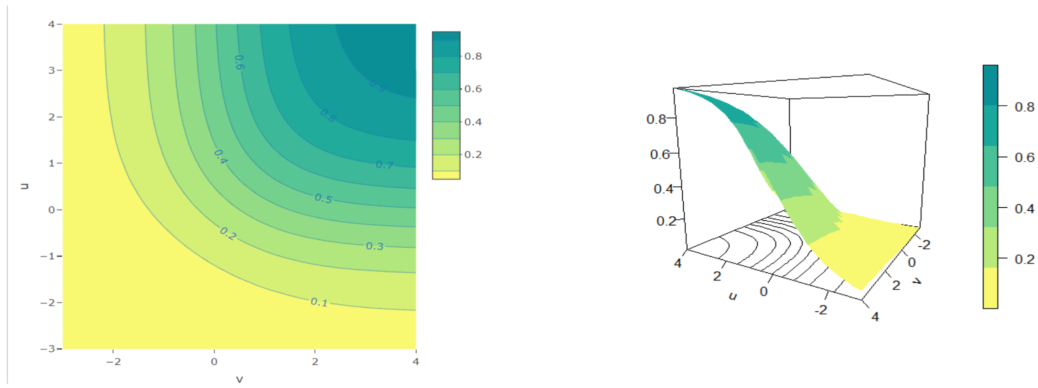

FIGURE 7. Plots of the CDF of the FNRT logistic distribution for $a = 0.1$: contours (left) and shapes (right).

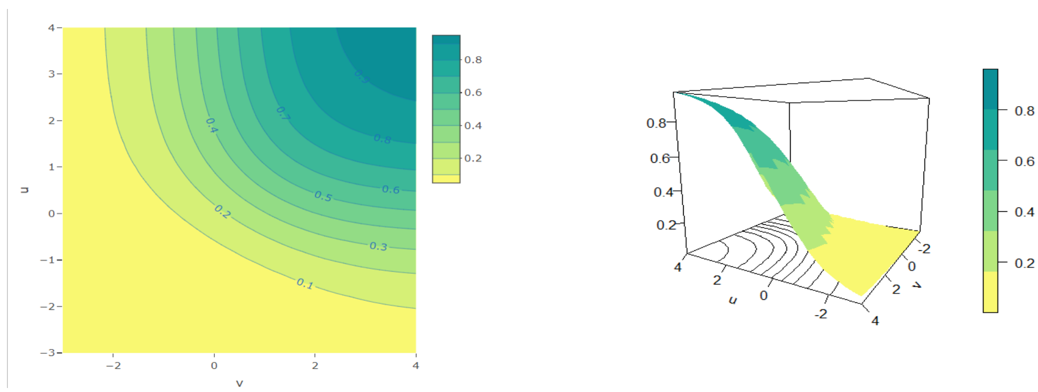

FIGURE 8. Plots of the CDF of the FNRT logistic distribution for $a = 0.9$: contours (left) and shapes (right).

These figures give a visual validation of the CDF; we observe a clear increasing trend starting from 0 to 1, and also a slight deformation of the contours depending on the value of a. The non-exchangeability of the distribution is also clear.

To complete this functional analysis, upon differentiation of $F_1(x, y; a)$, the associated PDF is given as

$$\begin{aligned} f_1(x, y; a) &= \frac{\partial^2}{\partial x \partial y} F_1(x, y; a) \\ &= \exp(2x + y) \frac{a^2[\exp(x) + 2] - a\exp(x)[\exp(x + y) - \exp(x) - 3] + \exp(2x)[\exp(y) + 1]}{[\exp(x) + 1]^2[a + \exp(x + y) + \exp(x)]^3}, \\ & \quad (x, y) \in \mathbb{R}^2. \end{aligned}$$

This function is more able to determine the overall information behind the FNRT logistic distribution. It is, however, complex. For this reason, a graphical analysis is more appropriate than a mathematical analysis. Thus, Figures 9 and 10 depict this PDF for two different values of a with $a \in [0, 1]$.

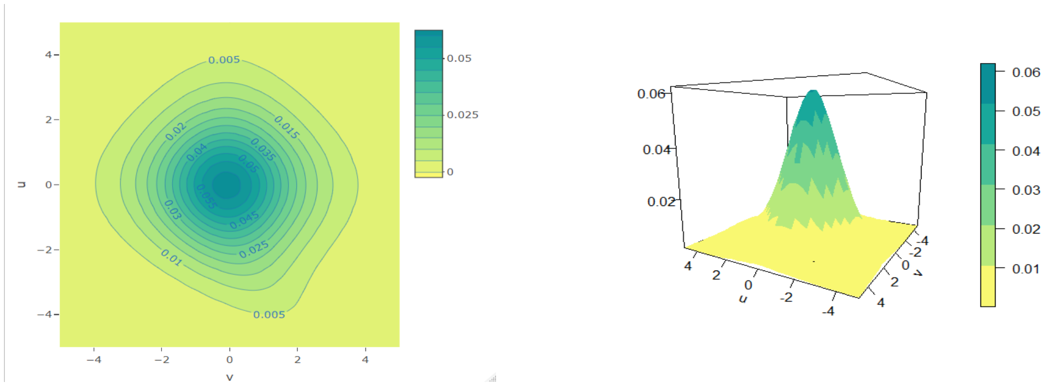

FIGURE 9. Plots of the PDF of the FNRT logistic distribution for $a = 0.1$: contours (left) and shapes (right).

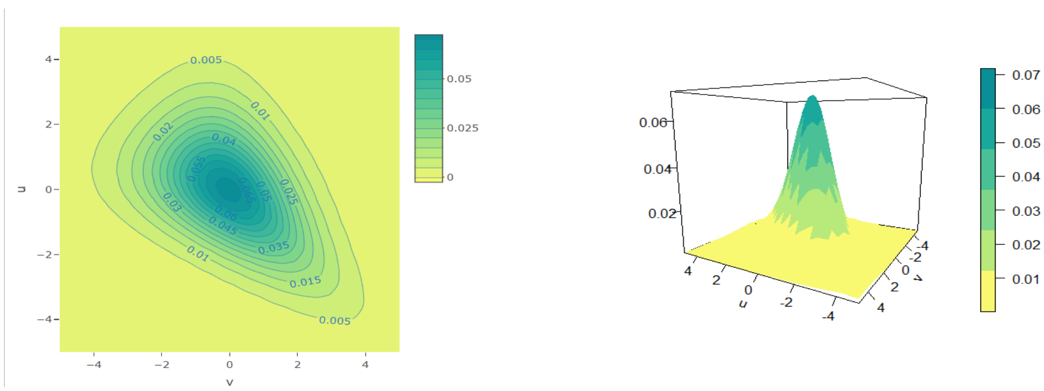

FIGURE 10. Plots of the PDF of the FNRT logistic distribution for $a = 0.9$: contours (left) and shapes (right).

Visually, the contours at the prime plan of this PDF are particularly versatile; they can adapt almost "losange shapes" or "trapezoid shapes", among others, with only one tuning parameter. The skewed ability of the overall bell shape is flexible. It is thus an interesting choice for applications with data from non-exchangeable random variables.

4.3. **Second new bivariate logistic distribution.** Based on the SNRT copula as given in Equation (3.1), another bivariate logistic distribution can be constructed. We define the SNRT logistic distribution by the following CDF:

$$
\begin{aligned}
F_2(x, y; a) &= C_2[F(x), F(y); a] \\
&= F(x)F(y)\frac{F(x)F(y)}{F(x)F(y) + a[1 - F(x)][1 - F(y)]} \\
&= \frac{\exp[2(x + y)]}{[\exp(x) + 1][\exp(y) + 1][a + \exp(x + y)]}, \quad (x, y) \in \mathbb{R}^2.
\end{aligned}
$$

By copula construction, like the FNRT logistic distribution, the SNRT logistic distribution has a univariate (standard) logistic distribution for both marginal distributions. Thanks to the exchangeable property of the SNRT copula, it is, however, exchangeable, contrary to the FNRT logistic distribution. It is thus particularly adapted to model systems involving two exchangeable components depending on a characteristic whose distribution is compatible with the logistic distribution, among others. More details on such a scenario are given in the next subsection.

Figures 11 and 12 present this CDF for two different values of a with $a \in [0, 1]$.

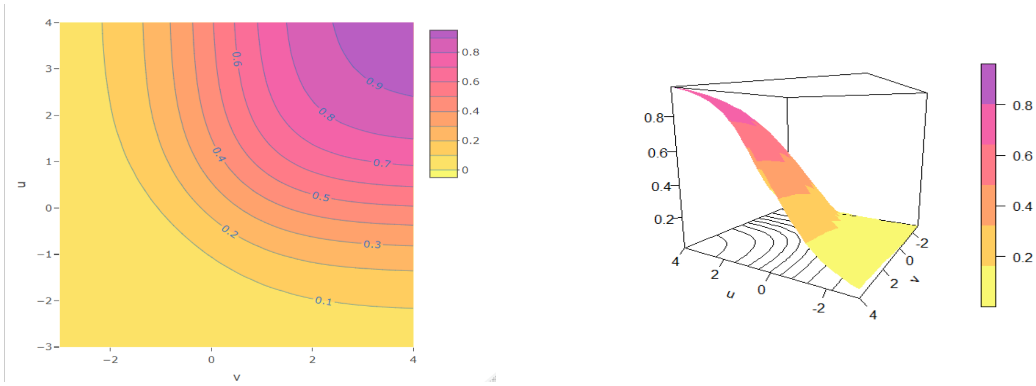

FIGURE 11. Plots of the CDF of the SNRT logistic distribution for $a = 0.1$: contours (left) and shapes (right).

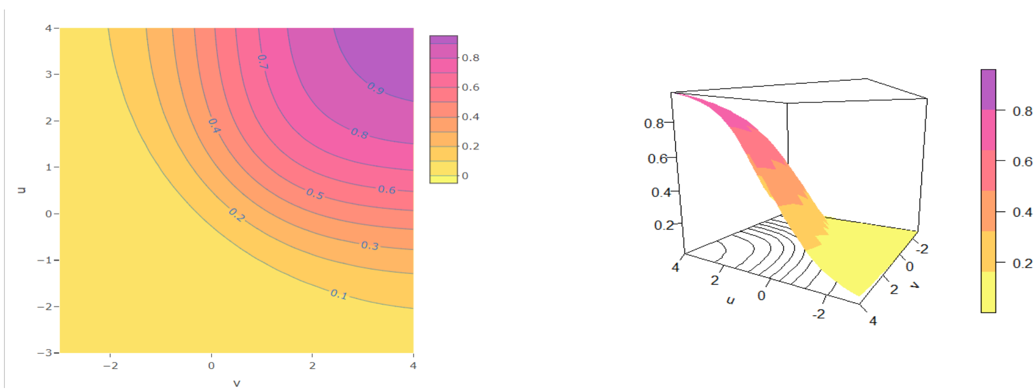

FIGURE 12. Plots of the CDF of the SNRT logistic distribution for $a = 0.9$: contours (left) and shapes (right).

These figures give a visual sketch of the validity of the CDF and its exchangeable properties; the main graphical objects are symmetric with respect to the diagonal line $v = u$.

Upon differentiation of $F_2(x, y; a)$, the associated PDF is given as

$$f_2(x, y; a) = \frac{\partial^2}{\partial x \partial y} F_2(x, y; a)$$
$$= \exp[2(x+y)] \frac{a^2[\exp(x)+2][\exp(y)+2] - a\exp(x+y)[\exp(x+y)-3] + \exp[2(x+y)]}{[\exp(x)+1]^2[\exp(y)+1]^2[a+\exp(x+y)]^3},$$
$$(x, y) \in \mathbb{R}^2.$$

Figures 13 and 14 show this PDF for two different values of a with $a \in [0, 1]$.

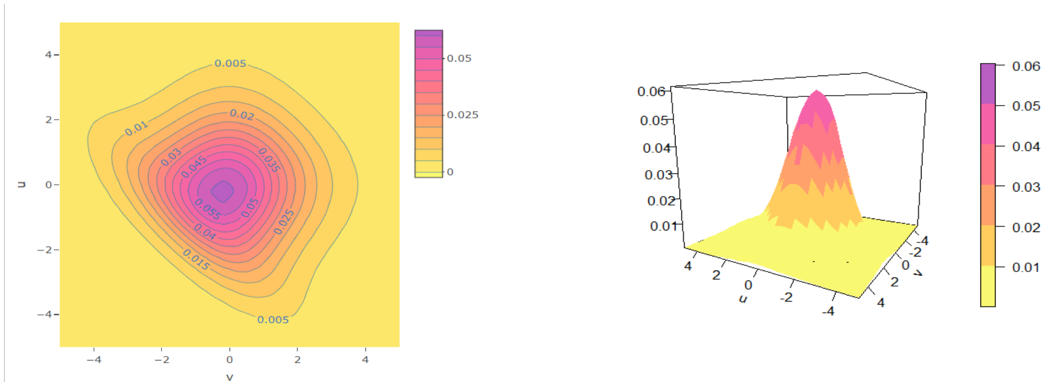

FIGURE 13. Plots of the PDF of the SNRT logistic distribution for $a = 0.1$: contours (left) and shapes (right).

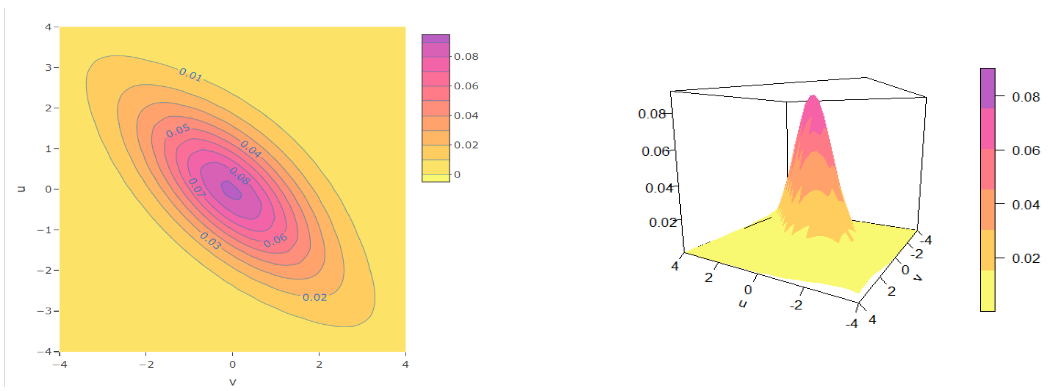

FIGURE 14. Plots of the PDF of the SNRT logistic distribution for $a = 0.9$: contours (left) and shapes (right).

We see that, with only one parameter, the PDF can accommodate diverse skewed bell shapes with a base of a trapezoid-like or ellipse-like form.

4.4. **Application: reliability analysis.** Following the studies in [26], [13] and [35], exchangeable bivariate distributions can be involved in random systems, depending on various minimum or maximum random variable scenarios. Thus, we consider a random vector (X, Y) that follows the SNRT logistic distribution, and we define the random variables S and T as follows:

$$S = \min(X, Y), \quad T = \max(X, Y).$$

Such minima or maxima of not necessarily independent random variables from a random vector naturally appear in many applied scenarios. See [26] and [13], in which bivariate lifetime distributions are considered, and [35] for the use of the Gumbel bivariate logistic distribution.

In our SNRT logistic distribution setting, by introducing the probability operator $\mathbb{P}$, the reliability function of T is given by

$$\begin{aligned} R_2(t; a) = \mathbb{P}(T > t) = 1 - F_2(t, t; a) &= 1 - \frac{\exp(4t)}{[\exp(t) + 1]^2[a + \exp(2t)]} \\ &= \frac{2a\exp(t) + a\exp(2t) + a + \exp(2t) + 2\exp(3t)}{[\exp(t) + 1]^2[a + \exp(2t)]}, \quad t \in \mathbb{R}. \end{aligned}$$

Also, the reliability function of S is obtained as

$$R_1(t;a) = \mathbb{P}(S > t) = 2[1 - F(t)] - R_2(t;a) = \frac{a + (1-a)\exp(2t)}{[\exp(t)+1]^2[a+\exp(2t)]}, \quad t \in \mathbb{R}.$$

To the best of our knowledge, these ratio-type reliability functions are new in the literature.

Based on these results, upon differentiation, the PDF of T is indicated as

$$f_2(t;a) = -[R_2(t;a)]' = 2\exp(4t)\frac{a[\exp(t)+2] + \exp(2t)}{[\exp(t)+1]^3[a+\exp(2t)]^2}, \quad t \in \mathbb{R}.$$

Furthermore, the PDF of S is given by

$$\begin{aligned} f_1(t;a) &= -[R_1(t;a)]' \\ &= \exp(t)\frac{2a^2[\exp(t)+1] - 2a\exp(2t)[\exp(2t)-2] + 2\exp(4t)}{[\exp(t)+1]^3[a+\exp(2t)]^2}, \quad t \in \mathbb{R}. \end{aligned}$$

In order to analyze the modeling capabilities of the related distributions, Figure 15 displays these PDFs for three different values of a with $a \in [0,1]$.

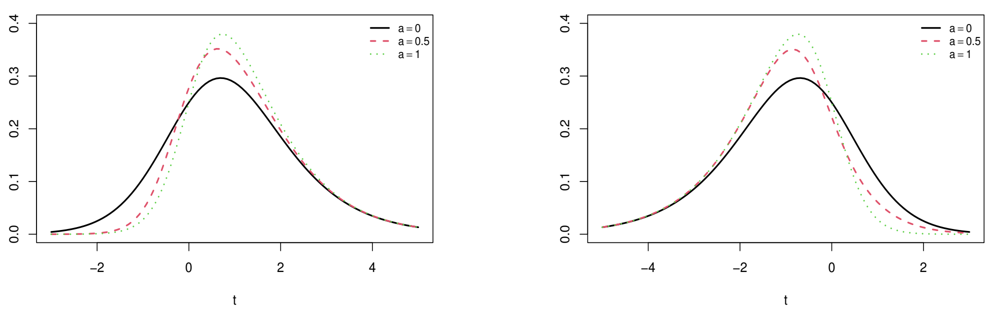

FIGURE 15. Plots of the PDF of T (left) and S (right).

We observe slightly skewed PDFs, which vary between the mesokurtic and leptokurtic states depending on the value of a.

From the above reliability functions and PDFs, we derive the failure rate of T by

$$\begin{aligned} r_2(t;a) &= \frac{f_2(t;a)}{R_2(t;a)} \\ &= 2\exp(4t)\frac{a[\exp(t)+2]+\exp(2t)}{[\exp(t)+1][a+\exp(2t)][2a\exp(t)+a\exp(2t)+a+\exp(2t)+2\exp(3t)]}, \\ &\quad t \in \mathbb{R}. \end{aligned}$$

Also, the failure rate of S is given by

$$\begin{aligned} r_1(t;a) &= \frac{f_1(t;a)}{R_1(t;a)} \\ &= \exp(t)\frac{2a^2[\exp(t)+1] - 2a\exp(2t)[\exp(2t)-2] + 2\exp(4t)}{[\exp(t)+1][a+\exp(2t)][(1-a)\exp(2t)+a]}, \quad t \in \mathbb{R}. \end{aligned}$$

They are of moderate complexity and new in reliability analysis. Figure 16 depicts these functions for three different values of a with $a \in [0,1]$.

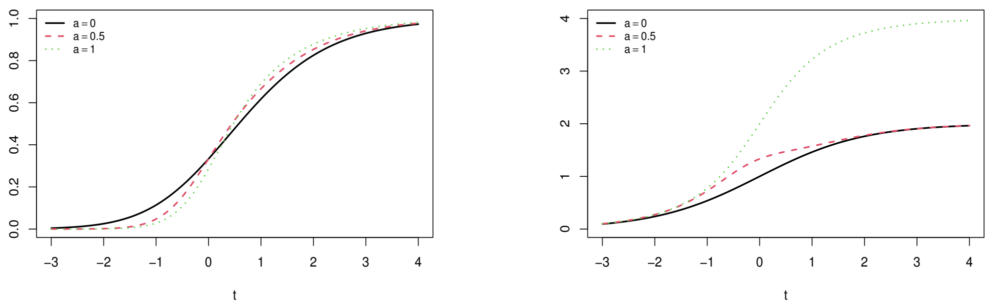

FIGURE 16. Plots of the failure rates of T (left) and S (right).

Based on this figure, it is clear that S and T have increasing failure rates, sharing similar characteristics to the normal and logistic distributions. Surprisingly, it graphically appears that the failure rate of T seems to have the properties of a CDF, i.e., $\lim_{t\to-\infty} r_2(t;a) = 0$, $\lim_{t\to\infty} r_2(t;a) = 1$, and $r_2(t;a)$ increasing.

In this section, we have thus provided the mathematical basis for a new modeling horizon in bivariate data and reliability analysis. Applications with modern and interesting data are left as perspectives for another complete work.

5. CONCLUSION

In conclusion, this research paper presents two modifications of the AMH copula, aiming to overcome some of its limitations while preserving a similar ratio-like form and a single tuning parameter. Unlike previous approaches, these modifications directly alter the functional form of the copula instead of the associated strict generating function. The resulting copulas exhibit interesting features and provide advantages in different modeling scenarios. In particular, the first proposed copula is not exchangeable, allowing for significant negative dependence correlations and flexible tail dependence properties. Meanwhile, the second copula retains a similar exchangeability to the original AMH copula but wins in providing a broader range of negative dependence correlations and more flexible tail dependence properties. However, our modification strategy involves tradeoffs; the positive dependence nature of the AMH copula and its Archimedean identity are sacrificed in some sense. For a direct view, our main copula findings in comparison to known information about the AMH copula are summarized in Table 1.

Copula	Expression	Values of a	Exchangeable	Archimedean	Tail dependence	Values of M	Values of ρ	New
AMH	$uv\dfrac{1}{1+a(1-u)(1-v)}$	$a \in [-1,1]$	Yes	Yes	No (except $a=-1$)	$[-0.2, 0.3333]$	$[-0.2711, 0.4784]$	No
FNRT	$uv\dfrac{u}{u+a(1-u)(1-v)}$	$a \in [0,1]$	No	No	Yes	$[-0.3333, 0]$	$[-0.4784, 0]$	Yes
SNRT	$uv\dfrac{uv}{uv+a(1-u)(1-v)}$	$a \in [0,1]$	Yes	No	Yes	$[-0.5, 0]$	$[-0.7011, 0]$	Yes

TABLE 1. A brief summary of the main findings on copula

In addition to that, two new single-parameter variations of the Gumbel bivariate logistic distribution are established using the modified copulas as applications. We discuss their implications in the context of random systems with two components.

Overall, our comprehensive exploration of the proposed modified AMH copulas, their characteristics, and applications contributes to a broader understanding of copula theory and its implications in probability modeling. Ambitious work in this direction includes the following aspects:

- the bivariate data fits based on our modified AMH copulas or variants of the Gumbel bivariate logistic distribution; in view of the applicability of similar models in the literature, some interesting studies can benefit from the offered alternative options,
- always for data analysis purposes, the construction of regression models based on these copulas, which are still a very demanded approach (see [20] and [30]),
- the multivariate extension of our findings, beyond the bivariate case; for any integer $d \geq 3$, a logical expression for an adapted d-variate copula candidate is as follows:

$$C_{\natural}(u_1, \ldots, u_d; a) = \left(\prod_{i=1}^{d} u_i\right) \frac{1}{1 + a \prod_{i=1}^{d}[(1-u_i)/u_i^{b_i}]}, \quad (u_1, \ldots, u_d) \in [0,1]^d,$$

where, for any $i = 1, \ldots, d$, we have fixed $b_i \in \{0, 1\}$.

These aspects need more investigation, and we leave them for the future.

Conflict of interest statement. The author declares that there is no conflict of interests.

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