

HOLDER SOLVABILITY OF QUASI-LINEAR PARABOLIC SYSTEMS IN THE DIVERGENT FORM

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ABSTRACT. In this article, we consider a quasilinear parabolic system under fair general conditions with form-boundary conditions on the singular coefficients. We establish the requirements on the structural coefficients of the parabolic system under which the quasilinear system has the unique solution in the Holder functional space.

1. INTRODUCTION

The main inquiry of the theory of quasilinear parabolic equations and systems is to establish functional classes to which belong their solutions. In 1957, Nash showed that the Holder norm $|u|^{(\alpha)}$ of solutions $u(x, t)$ to a parabolic equation

$$\frac{\partial}{\partial t} u - \sum_{i=1, \dots, l} \nabla_i (a_{ij}(x, t) \nabla_i u) = 0$$

can be estimated as

$$|u(x, t) - u(y, t)| \leq c \left(|x - y| |t - s|^{-\frac{1}{2}} \right)^\alpha \|u\|_{L^\infty}, t \geq s$$

by a constant c depending only on ellipticity constants ν and μ given by

$$\nu \xi^2 \leq \sum_{ij=1, \dots, l} a_{ij} \xi_i \xi_j \leq \mu \xi^2$$

and the dimension of the Euclidean space $l > 2$. A year previous, a similar result was obtained by De Giorgi for the case of the elliptic equation

$$\sum_{i=1, \dots, l} \nabla_i (a_{ij}(x, t) \nabla_i u) = 0.$$

Since this time, the problem of regularity has played a fundamental role in the theory of parabolic equations and systems many classical and modern works have been published. The crucial moment in the investigation of the regularity of solutions to the quasilinear equations and systems is obtaining a priori estimations for the solutions and derivatives of these solutions. In the linear case, the Nash-De Giorgi fundamental ideas brought a new comprehensiveness to the theory of linear elliptic and parabolic equations and systems. A resurgence of interest in quasilinear parabolic systems can be explained not only by new results obtained in recent years in the linear theory but also by their widespread applications to signal processing and quantum information theory. Some applications are considered by Qi. S. Zhang, L. Guo, G. Karniadakis in [49] where the open problems of scientific computing for the long-time integration of nonlinear stochastic

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partial differential equations are studied. In [10], M. Kassmann and M. Weidner propose the local regularity program for weak solutions to linear parabolic nonlocal equations with bounded coefficients, they prove the parabolic Harnack inequality and obtain Hölder regularity estimates. In [26], H. Dong, S. Kim, and S. Lee search fundamental solutions of second-order parabolic equations in non-divergence form the Dini coefficients, and in [27] under similar conditions, authors prove a Harnack inequality for nonnegative adjoint solutions, and upper and lower Gaussian bounds for fundamental solutions. In [52], Zhen-Qing Chen, T. Kumagai, and J. Wang prove the stability of two-sided heat kernel estimates and heat kernel upper bounds, obtaining Faber-Krahn inequalities, also. The list of references consists of 52 selected works [1- 52].

1.1. The problem statement. We consider a quasilinear parabolic system in the general form

$$(1.1) \quad \frac{\partial}{\partial t} \vec{u} - \sum_{i=1, \dots, l} \frac{d}{dx_i} \vec{a}_i(x, t, \vec{u}, \nabla \vec{u}) + \vec{b}(x, t, \vec{u}, \nabla \vec{u}) = 0,$$

where $\vec{u}(x, t) = (u^1(x, t), \dots, u^N(x, t))$ is an unknown N -dimensional vector-function over $\text{clos}(D_T)$, the domain $D_T = \Omega \times (0, T)$, $\Omega \subset \mathbb{R}^l$, $l \geq 3$ and $\vec{b} : \Omega \times [0, T] \times \mathbb{R}^N \times \mathbb{R}^l \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a given vector-function.

The main goal of this paper is to establish the weakest possible conditions under which the system (1.1) has a solution with certain properties of regularity. Assume that functions $\vec{a}_i(x, t, \vec{u}, \vec{k})$ and $\vec{b}(x, t, \vec{u}, \vec{k})$ are correctly defined over $\text{clos}(D_T)$ and are continuous at \vec{u} and \vec{k} , and satisfy the conditions

$$(1.2) \quad \vec{a}_i(x, t, \vec{u}, \vec{k}) \vec{k}_i \geq \nu(|\vec{u}|) |\vec{k}|^2 - \gamma_1(x, t)$$

$$(1.3) \quad |\vec{a}_i(x, t, \vec{u}, \vec{k})| \leq \mu(|\vec{u}|) |\vec{k}| + \gamma_2(x, t)$$

$$(1.4) \quad |\vec{b}(x, t, \vec{u}, \vec{k})| \leq \tilde{\mu}(|\vec{u}|) |\vec{k}|^2 + \gamma_3(x, t)$$

where ν , μ , $\tilde{\mu}$ are positive functions of positive argument so that the function ν is monotone decreasing and the functions μ and $\tilde{\mu}$ are monotone increasing.

Definition 1.1. A vector-function $\vec{u} \in V_{1,0}^2(D_T)$ is called a generalized solution to the system (1.1) if it satisfies the following integral identity

$$(1.5) \quad \begin{aligned} & \int_{\Omega} \vec{u}(x, t) \vec{\phi}(x, t) dx \Big|_0^T - \int_{[0, T]} \int_{\Omega} \vec{u} \partial_t \vec{\phi} dx dt + \\ & + \int_{[0, T]} \int_{\Omega} \vec{a}_i \nabla_i \vec{\phi} dx dt + \\ & + \int_{[0, T]} \int_{\Omega} \vec{b} \vec{\phi} dx dt = 0 \end{aligned}$$

for all vector-functions $\vec{\phi} \in C_0^\infty(\Omega \times (0, T))$.

Definition 1.2. A real-value function $\vec{f} \in L_{loc}^1$ belongs to the parabolic form-boundary class $PK(\beta)$ if the integral inequality

$$(1.6) \quad \int_{[0, T]} \int_{\mathbb{R}^l} |\vec{f} \vec{\phi}|^2 dx dt \leq \beta \int_{[0, T]} \int_{\mathbb{R}^l} |\nabla \vec{\phi}|^2 dx dt + c(\beta) \int_{[0, T]} \int_{\mathbb{R}^l} |\vec{\phi}|^2 dx dt$$

holds for some positive constants β and for all $\vec{\phi}$ such that $\vec{\phi} \in C_0^\infty$.

The generalized solution $\vec{u} \in V_{1,0}^2(D_T)$ to system (1.1) is called bounded if the function \vec{u} is a generalized solution and $\text{ess max}_{D_T} |\vec{u}| < \infty$.

We denote

$$(1.7) \quad \|\vec{u}_n\|_{p,q,D_T} = \left(\int_{[0,T]} \left(\int_{\Omega} |\vec{u}(x,t)|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}},$$

so we have an inequality

$$(1.8) \quad \|\vec{u}_n\|_{p,q,D_T} \leq \theta \left(\int_{[0,T]} \|\vec{u}_n(t)\|_{2,\Omega}^{\alpha q} dt \right)^{\frac{1}{q}} \operatorname{ess\,max}_{[0,T]} \|\vec{u}_n(x,t)\|_{2,\Omega}^{1-\alpha}$$

for $q \in \left[2, \frac{2l}{l-2}\right]$, $l \geq 3$.

Next, we remind definitions of functional classes B_2 and $B_2^{N_1}$.

Definition 1.3. The class B_2 consist of all functions $u(x,t) \in V_{1,0}^2(D_T)$ such that

$$1) \quad \operatorname{ess\,max}_{D_T} |u| \leq M_1,$$

2) both functions $u(x,t)$ and $-u(x,t)$ satisfy the following inequalities

$$(1.9) \quad \begin{aligned} & \max_{t \in [\tilde{t}, \tilde{t}+h]} \|\pm u_n(x,t)\|_{2,B(\rho-\sigma_1\rho)}^2 \leq \\ & \leq \|\pm u_n(x,\tilde{t})\|_{2,B(\rho)}^2 + \\ & + \vartheta \left((\sigma_1\rho)^{-2} \|\pm u_n\|_{2,B(\rho) \times [\tilde{t}, \tilde{t}+h]}^2 + \left(\int_{[\tilde{t}, \tilde{t}+h]} \operatorname{mes}^{\frac{a}{p}} \Lambda_n(\rho,t) dt \right)^x \right) \end{aligned}$$

and

$$(1.10) \quad \begin{aligned} & |\pm u_n(x,t)|_{D(\rho-\sigma_1\rho, h-\sigma_2\rho)}^2 \leq \\ & \leq + \vartheta \left(\left((\sigma_1\rho)^{-2} + (\sigma_2\rho)^{-1} \right) \|u_n\|_{2,B(\rho) \times [\tilde{t}, \tilde{t}+h]}^2 + \left(\int_{[\tilde{t}, \tilde{t}+h]} \operatorname{mes}^{\frac{a}{p}} \Lambda_n(\rho,t) dt \right)^x \right) \end{aligned}$$

where $u_n(x,\tilde{t}) = \max\{u(x,t) - n, 0\}$, $\sigma_1, \sigma_2 \in (0,1)$.

A N_1 -dimensional vector-function $\vec{u}(x,t) = (u^1(x,t), \dots, u^{N_1}(x,t))$ correctly defined and measurable on $\operatorname{clos}(D_T)$ belongs to the class $B_2^{N_1}$ if we can construct the family of N_1 functions $\eta^1(u^1, \dots, u^{N_1}), \dots, \eta^{N_1}(u^1, \dots, u^{N_1})$ continuous and continuous differentiable in the domain $|\vec{u}| \leq M_1$ and such that functions $w^m(x,t) = \eta^m(u^1(x,t), \dots, u^{N_1}(x,t))$, $m = 1, \dots, N_1$ satisfy the conditions:

$$1) \quad \operatorname{ess\,max}_{D_T} |w^m(x,t)| \leq M_1, \quad w^m \in V_{1,0}^2(D_T);$$

2) for any cylinder $D = B(2\rho) \times [\tilde{t}, \tilde{t}+h]$ and each $t_0 \in [\tilde{t}, \tilde{t}+h]$ there is a number p such that

$$\operatorname{osc}\{w^p(x,t), D_T\} \geq \delta_1 \max_{k=1, \dots, N} \operatorname{osc}\{u^k(x,t), D_T\}$$

and

$$\begin{aligned} & \operatorname{mes} \left\{ x \in B(\rho) : w^p(x,t_0) \leq \operatorname{ess\,max}_D w^p(x,t) - \delta_2 \operatorname{osc}\{w^p(x,t), D\} \right\} \geq \\ & \geq \chi_l \rho^l (1 - \delta_3) \end{aligned}$$

where $\delta_1, \delta_2, \delta_3 > 0$ and $\delta_2, \delta_3 < 1$; balls $B(\rho)$ and $B(2\rho)$ are concentric;

3) all functions $w^m(x,t)$, $m = 1, \dots, N_1$ satisfy (1.9), (1.10) for all $D \subset D_T$ and $0 < \sigma_1, \sigma_2 < 1$ and n so that $\operatorname{ess\,max}_d w^m(x,t) - n \leq \delta$.

Functions of a class $B_2^{N_1}$ belong to Holder functional space $H^{\alpha, \frac{\alpha}{2}}$.

The main goals of this article are to establish the general conditions under which the quasilinear systems are solvable and to prove the existence of the solution in the Holder class of functions. We consider the first boundary problem for the system (1.1) under the condition

$$\vec{u}|_{\{x \in \partial\Omega, t \in [0,T]\} \cup \{(x,t) : x \in \Omega, t=0\}} = \vec{\phi}|_{\{x \in \partial\Omega, t \in [0,T]\} \cup \{(x,t) : x \in \Omega, t=0\}},$$

which contains the initial and boundary conditions. The boundary condition must satisfy the agreement condition in the vector form

$$\partial_t \vec{\phi} - \frac{d}{dx_i} \vec{a}_i(x, t, \vec{\phi}, \nabla \vec{\phi}) + \vec{b}(x, t, \vec{\phi}, \nabla \vec{\phi}) = 0$$

for the given $\vec{\phi}$.

2. THE BOUNDEDNESS OF THE GENERALIZED SOLUTIONS TO THE PARABOLIC SYSTEM UNDER PARABOLIC FORM-BOUNDARY CONDITIONS

Now, assuming that \vec{u} is a solution, we formulate a relative result in the form of the following theorem.

Theorem 2.1. *Let functions $\vec{a}_i(x, t, \vec{u}, \vec{k})$ and $\vec{b}(x, t, \vec{u}, \vec{k})$ satisfy the conditions (1.2) – (1.3) with a monotone decreasing function ν and are monotone increasing functions μ and $\tilde{\mu}$, and $\gamma_1^{\frac{1}{2}}, \gamma_2, \gamma_3^{\frac{1}{2}} \in PK(\beta)$, and let function \vec{u} be grangerized bounded solution to the system (1.1). Then the solution \vec{u} belongs to some functional Holder space $H^{\alpha, \frac{\alpha}{2}}(D_T)$.*

Proof. We denote Steklov's averages in time of a function $f(x, t)$ by f_h and $f_{\bar{h}}$, given by formulae

$$f_{\bar{h}}(x, t) = \frac{1}{h} \int_{[t-h, t]} f(x, \tau) d\tau$$

and

$$f_h(x, t) = \frac{1}{h} \int_{[t, t+h]} f(x, \tau) d\tau.$$

We compose an integral identity

$$(2.1) \quad \begin{aligned} & \int_{[t_1, t_2]} \int_{\Omega} \partial_t \vec{u}_h \vec{\phi} dx dt + \\ & + \int_{[t_1, t_2]} \int_{\Omega} \vec{a}_i \nabla_i \vec{\phi}_{\bar{h}} dx dt + \\ & + \int_{[t_1, t_2]} \int_{\Omega} \vec{b} \vec{\phi}_{\bar{h}} dx dt = 0 \end{aligned}$$

where $h \leq t_1 \leq t_2 \leq T - h$. We can take

$\vec{\phi}(x, t) = \xi^2(x, t) \max \{u_h^k(x, t) - n, 0\} \equiv \xi^2 \vec{u}_h$ where ξ is a nonnegative, continuous function equal to zero on a cylindrical boundary. We denote a ball of radius ρ by $B(\rho) \subset \Omega$. Applying standard arguments, using the Cauchy inequality, we have

$$\begin{aligned} & \frac{1}{2} \|\vec{u}_n(x, t) \xi(x, t)\|_{2, B(\rho)}^2 \Big|_{t_1}^{t_2} + \\ & + \nu \int_{[t_1, t_2]} \int_{B(\rho)} \xi^2 |\nabla \vec{u}_n|^2 dx dt \leq \\ & \leq \int_{[t_1, t_2]} \int_{\Lambda_n(\rho)} \xi^2 (\gamma_1 + \varepsilon_1 \mu |\nabla \vec{u}|^2 + \gamma_2^2) dx dt + \\ & + \int_{[t_1, t_2]} \int_{\Lambda_n(\rho)} \xi^2 (\tilde{\mu} |\nabla \vec{u}|^2 + \gamma_3) \left(\max_{B(\rho) \times [t_1, t_2]} |\vec{u}| - n \right) dx dt + \\ & + \int_{[t_1, t_2]} \int_{\Lambda_n(\rho)} |\nabla \xi|^2 \left(\left(\frac{\mu}{\varepsilon_1} + 1 \right) |\vec{u} - n|^2 \right) dx dt + \\ & + \int_{[t_1, t_2]} \int_{\Lambda_n(\rho)} \xi |\partial_t \xi| |\vec{u} - n|^2 dx dt, \end{aligned}$$

where $\Lambda_n(\rho) = \{x \in B(\rho) : |\vec{u}_h(x, t)| > n\}$. We can take a number n so that $\max_{B(\rho) \times [t_1, t_2]} |\vec{u}| - n \leq \varepsilon_1$, and obtain

$$\begin{aligned} & \|\vec{u}_n(x, t_2) \xi(x, t_2)\|_{2, B(\rho)}^2 + \\ & + \nu \int_{[t_1, t_2]} \int_{B(\rho)} \xi^2 |\nabla \vec{u}_n|^2 dx dt \leq \\ & \leq \|\vec{u}_n(x, t_1) \xi(x, t_1)\|_{2, B(\rho)}^2 + \hat{c} \int_{[t_1, t_2]} \int_{B(\rho)} |\nabla \vec{u}_n|^2 (|\nabla \xi|^2 + \xi |\partial_t \xi|) dx dt + \\ & + 2\check{c} \int_{[t_1, t_2]} \int_{\Lambda_n(\rho)} \xi^2 (\gamma_1 + \gamma_2^2 + \gamma_3) dx dt. \end{aligned}$$

The last terms deal with form-boundary conditions and obtain

$$\begin{aligned} & \|\vec{u}_n(x, t_2) \xi(x, t_2)\|_{2, B(\rho)}^2 + \\ & + \nu \int_{[t_1, t_2]} \int_{B(\rho)} \xi^2 |\nabla \vec{u}_n|^2 dx dt \leq \\ & \leq \|\vec{u}_n(x, t_1) \xi(x, t_1)\|_{2, B(\rho)}^2 + \hat{c} \int_{[t_1, t_2]} \int_{B(\rho)} |\nabla \vec{u}_n|^2 \left(|\nabla \xi|^2 + \xi |\partial_t \xi| \right) dx dt + \\ & + \beta \check{c} \int_{[t_1, t_2]} \int_{\Lambda_n(\rho)} |\nabla \xi|^2 dx dt + \bar{c}(\beta) \int_{[t_1, t_2]} \int_{\Lambda_n(\rho)} |\xi|^2 dx dt \end{aligned}$$

therefore, we obtain $\vec{u} \in B_2^{2N}$ so $\vec{u} \in H^{\alpha, \frac{\alpha}{2}}(D_T)$.

3. ESTIMATION OF MAXIMUM $|\nabla \vec{u}|$ AND EXISTENCE THEOREMS

We assume that functions $\vec{a}_i(x, t, \vec{u}, \vec{k})$ and $\vec{b}(x, t, \vec{u}, \vec{k})$ satisfy

$$(3.1) \quad \nu \xi^2 \leq \sum_{ij=1, \dots, l} \frac{\partial \vec{a}_i(x, t, \vec{u}, \vec{k})}{\partial k^k_j} \xi_i \xi_j \leq \mu \xi^2,$$

$$(3.2) \quad \left| \frac{\partial \vec{a}_i(x, t, \vec{u}, \vec{k})}{\partial u^k} \right| + \left| \vec{b}(x, t, \vec{u}, \vec{k}) \right| \leq \mu |\vec{k}| + \gamma_2(x, t),$$

$$(3.3) \quad \left| \frac{\partial \vec{a}_i(x, t, \vec{u}, \vec{k})}{\partial x_j} \right| \leq \mu |\vec{k}|^2 + \gamma_3(x, t),$$

$$(3.4) \quad \left| \vec{b}(x, t, \vec{u}, \vec{k}) \right| \leq \mu |\vec{k}|^2 + \gamma_4(x, t).$$

We show that if functions γ_i are form-bounded then the solution $\vec{u}(x, t)$ to the system (1.1) satisfies the condition for $\max_{\bar{D}} |\nabla \vec{u}|$. We denote $\max_{D_T} |\vec{u}| = M_1$. Let $\eta(x, t)$ be a smooth function such that $0 \leq \eta(x, t) \leq 1$ and equal zero on $\{(x, t) : x \in \partial\Omega, t \in [0, T]\} \cup \{(x, t) : x \in \Omega, t = 0\}$. We take a test function $\varphi(x, t) = u^k \exp(\lambda |\vec{u}|^2) \eta^2(x)$. We rewrite system (1.1) in the integral form

$$(3.5) \quad \begin{aligned} & \int_{\Omega} \vec{u}(x, t) \vec{\phi}(x, t) dx \Big|_0^T - \int_{[0, T]} \int_{\Omega} \vec{u} \partial_t \vec{\phi} dx dt + \\ & + \int_{[0, T]} \int_{\Omega} \frac{\partial \vec{a}_i}{\partial \nabla_j u^k} \nabla_j u^k \nabla_i \vec{\phi} dx dt + \\ & + \int_{[0, T]} \int_{\Omega} \frac{\partial \vec{a}_i}{\partial u^k} u^k \nabla_i \vec{\phi} dx dt + \\ & + \int_{[0, T]} \int_{\Omega} \frac{\partial \vec{a}_i}{\partial x_i} \nabla_i \vec{\phi} dx dt + \\ & + \int_{[0, T]} \int_{\Omega} \vec{b} \vec{\phi} dx dt = 0. \end{aligned}$$

We symbolically denote $a_{ij} = \frac{\partial \vec{a}_i(x, t, \vec{u}, \vec{k})}{\partial \nabla_j u^k}$ then for some large $\lambda > 1$, we sum k from one to N and obtain the integral equality

$$\begin{aligned} & \frac{1}{2\lambda} \int_{\Omega} \exp(\lambda |\vec{u}(x, t)|^2) \eta^2(x) dx \Big|_0^t + \\ & + \int_{[0, t]} \int_{\Omega} \exp(\lambda |\vec{u}|^2) \eta^2 a_{ij} \nabla_i \vec{u} \nabla_j \vec{u} dx dt + \\ & + 2\lambda \int_{[0, t]} \int_{\Omega} |\vec{u}|^2 \exp(\lambda |\vec{u}|^2) \eta^2 a_{ij} \nabla_i \vec{u} \nabla_j \vec{u} dx dt + \\ & + 2\lambda \int_{[0, t]} \int_{\Omega} |\vec{u}|^2 \exp(\lambda |\vec{u}|^2) \eta^2 a_{ij} \nabla_i \vec{u} \nabla_j \vec{u} dx dt + \\ & + 2 \int_{[0, t]} \int_{\Omega} \vec{u} \exp(\lambda |\vec{u}|^2) \eta a_{ij} \nabla_j \vec{u} \nabla_i \eta dx dt + \end{aligned}$$

$$\begin{aligned}
& + \int_{[0, T]} \int_{\Omega} \frac{\partial \vec{a}_i}{\partial u^k} u^k \exp \left(\lambda |\vec{u}|^2 \right) \eta^2 \nabla_i \vec{u} dx dt + \\
& + 2\lambda \int_{[0, T]} \int_{\Omega} \frac{\partial \vec{a}_i}{\partial u^k} |\vec{u}|^2 \exp \left(\lambda |\vec{u}|^2 \right) \eta^2 \nabla_i u^k dx dt + \\
& + 2 \int_{[0, T]} \int_{\Omega} \frac{\partial \vec{a}_i(x, t, \vec{u}, \vec{k})}{\partial u^k} |\vec{u}|^2 \exp \left(\lambda |\vec{u}|^2 \right) \eta \nabla_i \eta dx dt + \\
& + \int_{[0, T]} \int_{\Omega} \frac{\partial \vec{a}_i}{\partial x_j} \exp \left(\lambda |\vec{u}|^2 \right) \eta^2 \nabla_j \vec{u} dx dt + \\
& + 2\lambda \int_{[0, T]} \int_{\Omega} \frac{\partial \vec{a}_i}{\partial x_j} \exp \left(\lambda |\vec{u}|^2 \right) \eta^2 \nabla_i \vec{u} dx dt + \\
& + 2 \int_{[0, T]} \int_{\Omega} \frac{\partial \vec{a}_i}{\partial x_j} \vec{u} \exp \left(\lambda |\vec{u}|^2 \right) \eta \nabla_j \eta dx dt \\
& + \int_{[0, t]} \int_{\Omega} \exp \left(\lambda |\vec{u}|^2 \right) \eta^2 \vec{b} \vec{u} dx dt = 0,
\end{aligned}$$

where as usual we understand $\vec{b}\vec{u} = \sum_{k=1, \dots, N} b^k u^k$.

Now, applying our assumption on the structural coefficients and form-boundary conditions, we obtain inequality

$$\int_{[0, t]} \int_{\Omega} \sum_{k=1, \dots, N} \sum_{i=1, \dots, l} (\nabla_i u^k)^2 \eta^2 dx dt \leq \text{const}$$

with some constants.

Next, we take the test function $\varphi(x, t) = \nabla_m (\xi(x, t) \nabla_m u^k)$, where the function ξ and its first derivatives equal zero on

$$\{(x, t) : x \in \partial\Omega, t \in [0, T]\} \cup \{(x, t) : x \in \Omega, t = 0\}.$$

We put the test function in the integral identity

$$\begin{aligned}
& \int_{[0, T]} \int_{\Omega} \partial_t \vec{u} \vec{\phi} dx dt + \\
& + \int_{[0, T]} \int_{\Omega} \vec{a}_i \nabla_i \vec{\phi} dx dt + \\
& + \int_{[0, T]} \int_{\Omega} \vec{b} \vec{\phi} dx dt = 0
\end{aligned}$$

by summing m and k , we obtain

$$\begin{aligned}
& \frac{1}{2} \int_{[0, t]} \int_{\Omega} \xi \partial_t \left(\sum_{k=1, \dots, N} \sum_{i=1, \dots, l} (\nabla_i u^k)^2 \right) dx dt + \\
& + \int_{[0, t]} \int_{\Omega} \xi a_{ij} \nabla_i \nabla_m \vec{u} \nabla_m \nabla_j \vec{u} dx dt + \\
& + \int_{[0, t]} \int_{\Omega} a_{ij} \nabla_i \xi \nabla_m \vec{u} \nabla_m \nabla_j \vec{u} dx dt + \\
& + \int_{[0, T]} \int_{\Omega} \frac{\partial \vec{a}_i}{\partial u^k} \nabla_m u^k \xi \nabla_i \nabla_m \vec{u} dx dt + \\
& + \int_{[0, T]} \int_{\Omega} \frac{\partial \vec{a}_i}{\partial u^k} \nabla_m u^k \nabla_i \xi \nabla_m \vec{u} dx dt + \\
& + \int_{[0, T]} \int_{\Omega} \frac{\partial \vec{a}_i}{\partial x_m} \xi \nabla_i \nabla_m \vec{u} dx dt + \\
& + \int_{[0, T]} \int_{\Omega} \frac{\partial \vec{a}_i}{\partial x_m} \nabla_i \xi \nabla_m \vec{u} dx dt + \\
& + \int_{[0, t]} \int_{\Omega} \vec{b} \nabla_m \xi \nabla_m \vec{u} dx dt + \\
& + \int_{[0, t]} \int_{\Omega} \vec{b} \xi \nabla_m \nabla_m \vec{u} dx dt = 0,
\end{aligned}$$

where we understand $\frac{\partial \vec{a}_i(x, t, \vec{u}, \vec{k})}{\partial u^k} u^k = \sum_{k=1, \dots, N} \frac{\partial \vec{a}_i(x, t, \vec{u}, \vec{k})}{\partial u^k} u^k$.

We take the function $\xi(x, t) = 2 \left(\sum_{k=1, \dots, N} \sum_{i=1, \dots, l} (\nabla_i u^k)^2 \right)^s (v(x))^2$ where $v(x)$ is the cutoff for the ball $B(\rho) \subset \Omega$, $s \geq 0$. We denote

$$\sum_{k=1, \dots, N} \sum_{i=1, \dots, l} (\nabla_i u^k)^2 = \Theta$$

then $\xi = 2\Theta^s v^2$. Applying our conditions and form-boundary estimations we can write

$$\begin{aligned} & \frac{1}{s+1} \int_{\Omega} \Theta^{s+1} v^2 dx \Big|_0^t + \\ & + \nu \int_{[0, t]} \int_{\Omega} 2\Theta^s v^2 \sum_{i, m=1, \dots, l} |\nabla_i \nabla_m \tilde{u}|^2 dx dt + \\ & + s\nu \int_{[0, t]} \int_{\Omega} \Theta^{s-1} |\nabla \Theta|^2 v^2 dx dt \leq \\ & \leq c(s) \left(\int_{[0, t]} \int_{\Omega} \Theta^{s+1} |\nabla v|^2 dx dt + \right. \\ & + \int_{[0, t]} \int_{\Omega} \Theta^{s+2} v^2 dx dt + \\ & \left. + \int_{[0, t]} \int_{\Omega} |\nabla v|^2 dx dt \right). \end{aligned}$$

Assume $\alpha > 0$ and choose a small enough $\rho > 0$ then

$$\int_{B(\rho)} \Theta^{s+2} v^2 dx \leq \tilde{c} \rho^{\alpha} \int_{B(\rho)} \left(\Theta^s \sum_{i, m=1, \dots, l} |\nabla_i \nabla_m \tilde{u}|^2 v^2 + \Theta^{s+1} |\nabla v|^2 \right) dx$$

therefore we obtain

$$(3.6) \quad \max_t \int_{\tilde{\Omega}} \left(\sum_{k=1, \dots, N} \sum_{i=1, \dots, l} (\nabla_i u^k)^2 \right)^{s+1} dx \leq c(s).$$

Now, we consider a smooth function $\xi(x, t)$ in the cylinder $B(2\rho) \times (0, t_0)$ such that $0 \leq \xi(x, t) \leq 1$ and $\xi \in C_0^{\infty}$. We take a test function

$$\varphi = \nabla_m \left(|\nabla \tilde{u}|^{2s} \nabla_m u^k \xi^2 \right)$$

and obtain

$$\begin{aligned} & \frac{1}{2(s+1)} \int_{B(2\rho)} |\nabla \tilde{u}(x, t)|^{2(s+1)} \xi^2(x, t) dx - \\ & - \frac{1}{s+1} \int_{[0, t]} \int_{B(2\rho)} |\nabla \tilde{u}|^{2(s+1)} 2\xi \partial_t \xi dx dt + \\ & + \int_{[0, t]} \int_{B(2\rho)} \frac{\partial \tilde{a}_i}{\partial \nabla_j u^k} \nabla_m \nabla_j u^k \nabla_m \nabla_i \tilde{u} |\nabla \tilde{u}|^{2s} \xi^2 dx dt + \\ & + 2s \int_{[0, t]} \int_{B(2\rho)} \frac{\partial \tilde{a}_i}{\partial \nabla_j u^k} \nabla_m \nabla_j u^k \nabla_m \tilde{u} \nabla_n \tilde{u} \nabla_n \nabla_i \tilde{u} |\nabla \tilde{u}|^{2(s-1)} \xi^2 dx dt - \\ & - 2 \int_{[0, t]} \int_{B(2\rho)} \frac{\partial \tilde{a}_i}{\partial \nabla_j u^k} \nabla_m \nabla_j u^k |\nabla \tilde{u}|^{2s} \nabla_m \tilde{u} \xi \nabla_i \xi dx dt - \\ & - \int_{[0, t]} \int_{B(2\rho)} \left(\frac{\partial \tilde{a}_i}{\partial u^k} \nabla_m u^k + \nabla_m \tilde{a}_i \right) \nabla_i \left(|\nabla \tilde{u}|^{2s} \nabla_m \tilde{u} \xi^2 \right) dx dt + \\ & + \int_{[0, t]} \int_{B(2\rho)} \tilde{b} \delta_{im} \nabla_i \left(|\nabla \tilde{u}|^{2s} \nabla_m \tilde{u} \xi^2 \right) dx dt = 0 \end{aligned}$$

where we as usual denote $\tilde{b} \tilde{u} = \sum_{k=1, \dots, N} b^k u^k$ and the δ_{im} is Kronecker delta symbol. Applying the conditions and Cauchy inequality, we estimate

$$\begin{aligned} & \frac{1}{2(s+1)} \int_{B(2\rho)} |\nabla \tilde{u}(x, t)|^{2(s+1)} \xi^2(x, t) dx - \\ & - \frac{1}{s+1} \int_{[0, t]} \int_{B(2\rho)} |\nabla \tilde{u}|^{2(s+1)} 2\xi \partial_t \xi dx dt + \\ & + \int_{[0, t]} \int_{B(2\rho)} \frac{\partial \tilde{a}_i}{\partial \nabla_j u^k} \nabla_m \nabla_j u^k \nabla_m \nabla_i \tilde{u} |\nabla \tilde{u}|^{2s} \xi^2 dx dt + \\ & + 2s \int_{[0, t]} \int_{B(2\rho)} \frac{\partial \tilde{a}_i}{\partial \nabla_j u^k} \nabla_m \nabla_j u^k \nabla_m \tilde{u} \nabla_n \tilde{u} \nabla_n \nabla_i \tilde{u} |\nabla \tilde{u}|^{2(s-1)} \xi^2 dx dt \leq \end{aligned}$$

$$\begin{aligned}
&\leq \int_{[0, t]} \int_{B(2\rho)} \left(\frac{\mu^2}{\varepsilon} |\nabla \vec{u}|^{2(s+1)} |\nabla \xi|^2 + \varepsilon |\nabla \nabla \vec{u}|^2 |\nabla \vec{u}|^{2s} \xi^2 \right) dxdt + \\
&\quad + \int_{[0, t]} \int_{B(2\rho)} \left(\frac{\mu^2}{\varepsilon} |\nabla \vec{u}|^{2(s+1)} |\nabla \xi|^2 + \varepsilon |\nabla \nabla \vec{u}|^2 |\nabla \vec{u}|^{2s} \xi^2 \right) dxdt + \\
&\quad + \int_{[0, t]} \int_{B(2\rho)} \left(3\mu |\nabla \vec{u}|^2 + \gamma_2 |\nabla \vec{u}| + \gamma_3 + \gamma_4 \right) \sum_{m,i} \nabla_i \left(|\nabla \vec{u}|^{2s} \nabla_m \vec{u} \xi^2 \right) dxdt,
\end{aligned}$$

where we defined $|\nabla \nabla \vec{u}|^2 = \sum_{k=1, \dots, N} \sum_{n,m=1, \dots, l} (\nabla_n \nabla_m u^k)^2$.

Next, we will use the following lemma.

Lemma 3.1. *Let f be a bounded function from $W_1^{2(s+1)}(\Omega)$ and $\varphi \in C_0^\infty$ then the inequality*

$$\int_{\Omega} |\nabla f|^{2(s+1)} \varphi^2 dx \leq c \operatorname{osc} \{f, \Omega\} \int_{\Omega} \left(|\nabla f|^{2(s-1)} |\nabla \nabla f|^2 \varphi^2 + |\nabla f|^{2s} |\nabla \varphi|^2 \right) dx$$

holds with constant c dependent on the dimension of Euclidian space and on s .

Proof. We estimate

$$\begin{aligned}
&\int_{\Omega} |\nabla f|^{2(s+1)} \varphi^2 dx = \\
&= - \int_{\Omega} (f(x) - f(\tilde{x})) \left(\frac{\Delta f |\nabla f|^{2s} \varphi^2 + 2s |\nabla f|^{2s-2} \varphi^2 \nabla_i f \nabla_i \nabla_j f \nabla_j f + \right. \\
&\quad \left. + 2 \nabla_i f |\nabla f|^{2s} \varphi \nabla_i \varphi \right) dx \leq \\
&\leq \int_{\Omega} 3\varepsilon |\nabla f|^{2s+2} \varphi^2 dx + \\
&\quad + \int_{\Omega} \frac{l^2}{\varepsilon} (f(x) - f(\tilde{x}))^2 |\nabla f|^{2s-2} \varphi^2 |\nabla \nabla f|^2 dx + \\
&\quad + \int_{\Omega} \frac{1}{\varepsilon} (f(x) - f(\tilde{x}))^2 |\nabla f|^{2s} |\nabla \varphi|^2 dx + \\
&\quad + \int_{\Omega} \frac{s^2}{\varepsilon} (f(x) - f(\tilde{x}))^2 |\nabla f|^{2s-2} \varphi^2 |\nabla \nabla f|^2 dx,
\end{aligned}$$

where we choose a point \tilde{x} such that the product $(f(x) - f(\tilde{x})) \varphi(x)$ vanishes on the boundary. Therefore we can choose a constant, which satisfies the lemma.

From lemma 3.1 and previous our estimations, we obtain that there is a constant \tilde{c} such that

$$\begin{aligned}
&\int_{B(2\rho)} |\nabla \vec{u}|^{2(s+2)} \xi^2 dx \leq \\
&\leq \tilde{c} \rho^{\tilde{\alpha}} \int_{B(2\rho)} \left(|\nabla \vec{u}|^{2s} |\nabla \nabla \vec{u}|^2 \xi^2 + |\nabla \vec{u}|^{2s+2} |\nabla \xi|^2 \right) dx
\end{aligned}$$

therefore we have

$$\begin{aligned}
&\frac{1}{2(s+1)} \int_{B(2\rho)} |\nabla \vec{u}(x, t)|^{2(s+1)} \xi^2(x, t) dx + \\
&\quad + \frac{\nu}{2} \int_{[0, t]} \int_{B(2\rho)} |\nabla \vec{u}|^{2s} |\nabla \nabla \vec{u}|^2 \xi^2 dxdt + \\
&\quad + \nu s \int_{[0, t]} \int_{B(2\rho)} |\nabla \vec{u}|^{2(s-1)} \xi^2 \sum_i \left(\sum_m \nabla_m \vec{u} \nabla_m \nabla_i \vec{u} \right)^2 dxdt \leq \\
&\leq c \frac{2}{s+1} \int_{[0, t]} \int_{B(2\rho)} |\nabla \vec{u}|^{2(s+1)} \xi \partial_t \xi dxdt + \\
&\quad + c \rho^{\tilde{\alpha}} (s+1) \int_{B(2\rho)} |\nabla \vec{u}|^{2s+2} |\nabla \xi|^2 dx + \\
&\quad + \int_{[0, t]} \int_{B(2\rho)} (\gamma_2 |\nabla \vec{u}| + \gamma_3 + \gamma_4) \sum_{m,i} \nabla_i \left(|\nabla \vec{u}|^{2s} \nabla_m \vec{u} \xi^2 \right) dxdt,
\end{aligned}$$

assuming smoothness of solutions and applying the conditions, we obtain the estimation

$$(3.7) \quad \max_{t \in [0, T]} \|\xi \nabla \vec{u}\|_{2, B(2\rho)}^2 + \int_{[0, t]} \int_{B(2\rho)} \left(|\nabla \vec{u}|^4 + |\nabla \nabla \vec{u}|^2 \right) \xi^2 dxdt \leq c$$

so that $\|\xi \partial_t \vec{u}\|_{2, B(2\rho)}^2 \leq c$ for some positive constants c dependent on the structural coefficients.

Finally, taking a test function $\varphi(x, t) = \nabla_m \xi(x, t)$ where $\xi \in C_0^\infty$ we obtain a system

$$\begin{aligned} & \int_{B(\rho)} \nabla_m \vec{u}(x, t) \xi(x, t) dx - \\ & - \int_{[0, T]} \int_{\Omega} \nabla_m \vec{u} \partial_t \xi dx dt + \\ & + \int_{[0, t]} \int_{B(\rho)} \frac{\partial \vec{a}_i}{\partial \nabla_j u^k} \nabla_m \nabla_j u^k \nabla_i \xi dx dt + \\ & + \int_{[0, t]} \int_{B(\rho)} \frac{\partial \vec{a}_i}{\partial u^k} \nabla_m u^k \nabla_i \xi dx dt + \\ & + \int_{[0, t]} \int_{B(\rho)} \nabla_m \vec{a}_i \nabla_i \xi dx dt - \\ & - \int_{[0, t]} \int_{B(\rho)} \vec{b} \nabla_m \xi dx dt = 0. \end{aligned}$$

We can consider the function $\nabla_m \vec{u} = \vec{v}_m$ as a solution to the linear parabolic system

$$\frac{\partial \vec{v}_m}{\partial t} - \frac{\partial (a_{ij} \nabla_j \vec{v}_m)}{\partial x_i} - \frac{\partial \Xi_{im}}{\partial x_i} = 0,$$

where we denote $a_{ij}(x, t) = \frac{\partial \vec{a}_i(x, t, \vec{u}(x, t), \nabla \vec{u}(x, t))}{\partial \nabla_j u^k}$ and

$$\begin{aligned} \Xi_{im} &= \frac{\partial \vec{a}_i(x, t, \vec{u}(x, t), \nabla \vec{u}(x, t))}{\partial u^k} \nabla_m u^k + \\ &+ \frac{\partial \vec{a}_i(x, t, \vec{u}(x, t), \nabla \vec{u}(x, t))}{\partial x_m} - \vec{b}(x, t, \vec{u}(x, t), \nabla \vec{u}(x, t)) \delta_{im}. \end{aligned}$$

Since functions a_{ij} and Ξ_{im} satisfy conditions that guarantee the regularity of a solution to the linear system, we finally can formulate the following theorems.

Theorem 3.2. Let function $\vec{u} \in C^{2,1}(D_T)$ be a solution to (1.1). If functions \vec{a}_i and \vec{b} satisfy (3.1)-(3.4). Then, for every subdomain $\tilde{D} \subset D_T$, there is $\alpha > 0$ such that $|\vec{u}|_{\tilde{D}}^{(\alpha)} \leq c$ where constant c depends only on the structural coefficients of the system (1.1).

Theorem 3.3. Let functions \vec{a}_i and \vec{b} satisfy (3.1)-(3.4), and the estimation

$$\vec{b}(x, t, \vec{u}, 0) \vec{u} - \frac{\partial \vec{a}_i(x, t, \vec{u}, 0)}{\partial u^k} \nabla_i u^k \vec{u} - \frac{\partial \vec{a}_i(x, t, \vec{u}, 0)}{\partial x_i} \vec{u} \geq -c |\vec{u}|^2 - \tilde{c},$$

holds for $(x, t) \in D_T$, $|\vec{u}| \leq M_1$ where c and \tilde{c} are some nonnegative constants and

$$M_1 = \min_{\hat{c} > c} \exp(\hat{c}T) \left(\max_{\Omega} |\vec{u}(x, 0)| + \left(\tilde{c}(\hat{c} - c)^{-\frac{1}{2}} \right) \right).$$

Let functions \vec{a}_i and \vec{b} satisfy the Lipschitz condition at the time variable

$$\begin{aligned} & \left| \frac{\vec{a}_i(x, t+h, \vec{u}, \vec{k}) - \vec{a}_i(x, t, \vec{u}, \vec{k})}{h} \right| \leq \gamma(x, t), \\ & \left| \frac{\vec{b}(x, t+h, \vec{u}, \vec{k}) - \vec{b}(x, t, \vec{u}, \vec{k})}{h} \right| \leq \gamma(x, t) \end{aligned}$$

and let

$$\begin{aligned} & \left| \frac{\partial \vec{a}_i(x, t, \vec{u}, \vec{k})}{\partial u^k} \right| \leq \gamma(x, t), \\ & \left| \frac{\partial \vec{b}(x, t, \vec{u}, \vec{k})}{\partial k^k_j} \right| \leq \gamma(x, t), \\ & \left| \frac{\partial \vec{b}(x, t, \vec{u}, \vec{k})}{\partial u^k} \right| \leq \gamma(x, t), \end{aligned}$$

where function γ satisfies the form-bounded condition. Let the inequality

$$\sum_{i=1,\dots,l} (|\vec{a}_i| + |\frac{\partial \vec{a}_i}{\partial u^k}|) (1 + |\vec{k}|) + \sum_{i,j=1,\dots,l} |\frac{\partial \vec{a}_i}{\partial x_j}| + |\vec{b}| \leq \mu (1 + |\vec{k}|)^2 (|\frac{\partial \vec{a}_i}{\partial u^k}|)$$

holds for all arguments. Let

$$\vec{u}|_{\{x \in \partial\Omega, t \in [0, T]\} \cup \{(x, t) : x \in \Omega, t=0\}} = \vec{\phi}|_{\{x \in \partial\Omega, t \in [0, T]\} \cup \{(x, t) : x \in \Omega, t=0\}}$$

such that

$$\partial_t \vec{\phi} - \frac{d}{dx_i} \vec{a}_i(x, t, \vec{\phi}, \nabla \vec{\phi}) + \vec{b}(x, t, \vec{\phi}, \nabla \vec{\phi}) = 0,$$

where $\vec{\phi} \in H^{\chi+2, 1+\frac{\chi}{2}}(\text{clos}(D_T))$. Then there exists a unique solution $\vec{u} \in H^{\chi+2, 1+\frac{\chi}{2}}(\text{clos}(D_T))$ to the system (1.1) under the boundary condition

$$\vec{u}|_{\{x \in \partial\Omega, t \in [0, T]\} \cup \{(x, t) : x \in \Omega, t=0\}} = \vec{\phi}|_{\{x \in \partial\Omega, t \in [0, T]\} \cup \{(x, t) : x \in \Omega, t=0\}}.$$

Proof. The theorem will be proven by the Leray-Schauder method. The system (1.1) can be rewritten in the form

$$(3.8) \quad \Lambda \vec{u} = \frac{\partial \vec{u}}{\partial t} - \frac{\partial \vec{a}_i(x, t, \vec{u}, \nabla \vec{u})}{\partial \nabla_j u^k} \nabla_i \nabla_j u^k + \vec{Y}(x, t, \vec{u}, \nabla \vec{u}) = 0,$$

where the operator Λ is a nonlinear differential operator, and we denote

$$\vec{Y}(x, t, \vec{u}, \nabla \vec{u}) = \vec{b}(x, t, \vec{u}, \nabla \vec{u}) - \frac{\partial \vec{a}_i(x, t, \vec{u}, \nabla \vec{u})}{\partial u^k} \nabla_i u^k - \frac{\partial \vec{a}_i(x, t, \vec{u}, \nabla \vec{u})}{\partial x_i}.$$

We consider a set of linear problems

$$(3.9) \quad \begin{aligned} & \frac{\partial \vec{v}}{\partial t} - \tau \left(\frac{\partial \vec{a}_i(x, t, \vec{w}, \nabla \vec{w})}{\partial \nabla_j w^k} + (1 - \tau) \delta_{ij} \right) \nabla_i \nabla_j v^k + \\ & + \tau \vec{Y}(x, t, \vec{w}, \nabla \vec{w}) - (1 - \tau) \left(\frac{\partial \vec{\phi}}{\partial t} - \Delta \vec{\phi} \right) = 0, \\ & \vec{v}|_{\{x \in \partial\Omega, t \in [0, T]\} \cup \{(x, t) : x \in \Omega, t=0\}} = \\ & = \vec{\phi}|_{\{x \in \partial\Omega, t \in [0, T]\} \cup \{(x, t) : x \in \Omega, t=0\}}, \tau \in [0, 1], \end{aligned}$$

where the function \vec{v} is unknown and the function \vec{w} is considered given.

We introduce a functional Banach space Ξ_δ , which consists of all smooth, continuous functions \vec{w} , the norm $\|\cdot\|_{\Xi_\delta}$ is defined by

$$\|\vec{w}\|_{\Xi_\delta} = |\vec{w}|_{D_T}^{(\delta)} + |\nabla \vec{w}|_{D_T}^{(\delta)}.$$

In Ξ_δ exists a nonlinear operator P , which maps each element \vec{w} of Ξ_δ to a solution \vec{v} of the linear problem (3.9) by $\vec{v} = P(\vec{w}, \tau)$, the nonlinear operator P depends on the parameter τ .

The fixed point of the operator P at value $\tau = 1$ is a solution to the boundary problem for the system (1.1).

Assume \vec{u}^τ is a fixed point of nonlinear operator P so that $\vec{u}^\tau = P(\vec{u}^\tau, \tau)$ then function \vec{u}^τ solves the following problem

$$(3.10) \quad \begin{aligned} & \Lambda^\tau \vec{u} = \frac{\partial \vec{u}}{\partial t} - \frac{d}{dx_i} (\tau \vec{a}_i(x, t, \vec{u}, \nabla \vec{u}) + (1 - \tau) \nabla_i \vec{u}) \\ & + \tau \vec{b}(x, t, \vec{u}, \nabla \vec{u}) - (1 - \tau) \left(\frac{\partial \vec{\phi}}{\partial t} - \Delta \vec{\phi} \right) = 0, \\ & \vec{u}|_{\{x \in \partial\Omega, t \in [0, T]\} \cup \{(x, t) : x \in \Omega, t=0\}} = \\ & = \vec{\phi}|_{\{x \in \partial\Omega, t \in [0, T]\} \cup \{(x, t) : x \in \Omega, t=0\}}, \tau \in [0, 1]. \end{aligned}$$

Assume that any fixed $\tau \in [0, 1]$, operator Λ^τ satisfies the conditions of theorems of the existence of solutions then operators Λ^τ satisfy these conditions for all $\tau \in [0, 1]$. We have obtained the estimation

$$\max_{D_T} |\vec{u}^\tau| \leq M_1, \max_{D_T} |\nabla \vec{u}^\tau| \leq M_2$$

and

$$\|\vec{u}^\tau\|_{\Xi_\alpha} = |\vec{u}^\tau|_{D_T}^{(\alpha)} + |\nabla \vec{u}^\tau|_{D_T}^{(\alpha)} \leq M_3$$

with some $0 < \alpha < 1$ for all $\vec{u}^\tau \in C^{2,1}(\text{clos}(D_T))$ problems for the systems (3.10). The linear theory guarantees that the solution $\vec{v} = P(\vec{w}, \tau)$ to the linear problem (3.9) belongs to the class $H^{\chi\delta+2, 1+\frac{\chi\delta}{2}}(\text{clos}(D_T))$. Let's take $\delta = \alpha < 1$ all fixed points of the operator $P(\vec{w}, \tau)$ belonging to the set Θ consisting of elements $\vec{w} \in \Xi_\delta$ such that

$$\max_{D_T} |\vec{w}| \leq M_1 + \varepsilon, \max_{D_T} |\nabla \vec{w}| \leq M_2 + \varepsilon$$

and

$$\|\vec{w}^\tau\|_{\Xi_\alpha} \leq M_3 + \varepsilon.$$

All fixed points \vec{u}^τ of $P(\vec{w}, \tau)$ belong to the set Θ . The set $P(\vec{w}, \tau)$ on $\Theta \times [0, 1]$ is uniformly continuous at \vec{w} and τ , and uniformly compact operators. To show uniform continuity $P(\vec{w}, \tau)$ on $\Theta \times [0, 1]$, we take two near elements \vec{w}_1 and \vec{w}_2 that belong to Θ so that $\vec{v}_1 = P(\vec{w}_1, \tau)$ and $\vec{v}_2 = P(\vec{w}_2, \tau)$. We subtract from the system (3.9) for \vec{v}_1 system (3.9) for \vec{v}_2 and obtain

$$\begin{aligned} (3.11) \quad & \frac{\partial \vec{v}}{\partial t} - \tau \left(\frac{\partial \vec{a}_i(x, t, \vec{w}_1, \nabla \vec{w}_1)}{\partial \nabla_j w^k} + (1 - \tau) \delta_{ij} \right) \nabla_i \nabla_j v^k = \\ & = \tau \left(\frac{\partial \vec{a}_i(x, t, \vec{w}_1, \nabla \vec{w}_1)}{\partial \nabla_j w^k} + \frac{\partial \vec{a}_i(x, t, \vec{w}_2, \nabla \vec{w}_2)}{\partial \nabla_j w^k} \right) \nabla_i \nabla_j v^k - \\ & - \tau \left(\vec{Y}(x, t, \vec{w}_1, \nabla \vec{w}_1) - \vec{Y}(x, t, \vec{w}_2, \nabla \vec{w}_2) \right) = 0, \end{aligned}$$

where $\vec{v} = \vec{v}_1 - \vec{v}_2$ and condition $\vec{v}|_{\{x \in \partial\Omega, t \in [0, T]\} \cup \{(x, t) : x \in \Omega, t=0\}} = 0$. If the norm $\|\vec{w}_1 - \vec{w}_2\|_{\Xi_\alpha}$ is small uniformly at $\tau \in [0, 1]$ then the right part of (3.11) is uniformly small as a function of (x, t) in the norm of $H^{\chi\alpha, \frac{\chi\alpha}{2}}(\text{clos}(D_T))$ spaces, therefore, $\|\vec{v}\|_{\Xi_\alpha}$ is small. Similar arguments establish uniform continuity $P(\vec{w}, \tau)$ on $\Theta \times [0, 1]$ at $\tau \in [0, 1]$.

Thus, we establish that for each $\tau \in [0, 1]$ there exists a fixed point $\vec{u}^\tau \in H^{\chi+2, \frac{\chi}{2}+1}(\text{clos}(D_T))$ of $P(\vec{w}, \tau)$ that is a solution to the problem (3.10). However, the solution $\vec{u}^\tau \in H^{\chi+2, \frac{\chi}{2}+1}(\text{clos}(D_T))$ to the problem (3.10) is a solution to the linear system (3.9) when $\vec{w} = \vec{u}^\tau$ therefore $\vec{v} = \vec{u}^\tau$. Thus, we proved that there exists a solution $\vec{u} \in H^{\chi+2, \frac{\chi}{2}+1}(\text{clos}(D_T))$ to the boundary problem for system (1.1).

Applying the linear theory, we can straightforwardly prove the uniqueness of the solution to the problem for the system (1.1) by contradiction. Assume that there are two different solutions \vec{u}_1 and \vec{u}_2 to the boundary problem for the system (1.1) then they must satisfy the integral identity

$$\begin{aligned} & \int_{[0, T]} \int_{\Omega} \vec{\phi} \partial_t \vec{u} dx dt + \\ & + \int_{[0, T]} \int_{\Omega} \vec{a}_i \nabla_i \vec{\phi} dx dt + \\ & + \int_{[0, T]} \int_{\Omega} \vec{b} \vec{\phi} dx dt = 0 \end{aligned}$$

for all $\vec{\phi} \in C_0^\infty$. Subtract from identity for \vec{u}_1 the identity for \vec{u}_2 , we have

$$\begin{aligned} (3.12) \quad & \int_{[0, T]} \int_{\Omega} \vec{\phi} \partial_t \vec{v} dx dt + \\ & + \int_{[0, T]} \int_{\Omega} \left(\vec{a}_{ij} \nabla_j \vec{v} + \vec{a}_i \vec{v} \right) \nabla_i \vec{\phi} dx dt + \\ & + \int_{[0, T]} \int_{\Omega} \left(\vec{b}_i \nabla_i \vec{v} + \vec{b} \vec{v} \right) \vec{\phi} dx dt = 0, \end{aligned}$$

where we denote

$$\begin{aligned} & \vec{a}_i(x, t, \vec{u}_1, \nabla \vec{u}_1) - \vec{a}_i(x, t, \vec{u}_2, \nabla \vec{u}_2) = \\ &= \nabla_j v^k \int_{[0, 1]} \frac{\partial \vec{a}_i(x, t, \tau \vec{u}_1 + (1-\tau) \vec{u}_2, \tau \nabla \vec{u}_1 + (1-\tau) \nabla \vec{u}_2)}{\partial \nabla_j u^k} d\tau + \\ &+ v^k \int_{[0, 1]} \frac{\partial \vec{a}_i(x, t, \tau \vec{u}_1 + (1-\tau) \vec{u}_2, \tau \nabla \vec{u}_1 + (1-\tau) \nabla \vec{u}_2)}{\partial u^k} d\tau = \\ &= \vec{a}_{ij} \nabla_j \vec{v} + \vec{a}_i \vec{v} \end{aligned}$$

and

$$\begin{aligned} & \vec{b}(x, t, \vec{u}_1, \nabla \vec{u}_1) - \vec{b}(x, t, \vec{u}_2, \nabla \vec{u}_2) = \\ &= \nabla_j v^k \int_{[0, 1]} \frac{\partial \vec{b}(x, t, \tau \vec{u}_1 + (1-\tau) \vec{u}_2, \tau \nabla \vec{u}_1 + (1-\tau) \nabla \vec{u}_2)}{\partial \nabla_j u^k} d\tau + \\ &+ v^k \int_{[0, 1]} \frac{\partial \vec{b}(x, t, \tau \vec{u}_1 + (1-\tau) \vec{u}_2, \tau \nabla \vec{u}_1 + (1-\tau) \nabla \vec{u}_2)}{\partial u^k} d\tau = \\ &= \vec{b}_i \nabla_i \vec{v} + \vec{b} \vec{v}, \end{aligned}$$

and $\vec{v} = \vec{u}_1 - \vec{u}_2$. The system of (3.12) is a linear parabolic system from the linear theory we obtain $\vec{v} \equiv 0$. Theorem 3.2 has been proven.

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