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VARIATIONAL AND NUMERICAL ANALYSIS FOR AN ELECTRO-VISCOELASTIC UNILATERAL CONTACT PROBLEM WITH ADHESION

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ABSTRACT. In this paper we investigate a mathematical model for the unilateral contact between an electro-viscoelastic body and a conductive foundation. The Signorini conditions are used to model the contact, and adhesion between the contact surfaces is also taken into account. We establish the existence and uniqueness of a weak solution to this problem. Furthermore, we propose a fully discrete numerical scheme to approximate the solution, and we present and prove the main result concerning the estimation of errors.

1. INTRODUCTION

Contact problems involving deformable bodies are widely encountered in both industrial applications and everyday life, particularly in the case of piezoelectric materials that exhibit an interaction between mechanical and electrical properties. When these materials possess elastic or viscoelastic characteristics, they are referred to as electro-elastic or electro-viscoelastic materials (see [3, 12]). The complexities arising from the different behavior laws associated with such materials necessitate the development of new models. The study of contact processes within the framework of variational inequalities has received considerable attention in research, see [7, 11, 13]. To obtain a thorough understanding of variational and numerical analysis related to adhesive material models and piezoelectric effect models, both with and without friction., readers are encouraged to reference the provided citations [1, 2, 5, 6, 10] and the additional works mentioned within.

Our main concern in this paper is the investigation of an electro-viscoelastic material in quasistatic contact with a deformable conductive foundation. The contact is assumed to be frictionless and is governed by the frictionless Signorini conditions. We use the adhesion field as an additional dependent variable, similar to [8, 9], whose evolution is modeled by an ordinary differential equation. Then, we proceed to derive a variational formulation, denoted as Problem (PV), for the mechanical problem and establish the existence and uniqueness of a weak solution under appropriate regularity assumptions on the provided data. Furthermore, a significant addition of this study is the numerical approximation of the weak solution for the suggested problem. By utilizing a fully discrete approximation of the problem (PV), we define the problem (PV^{hk}), which admits a unique solution. Subsequently, under appropriate regularity conditions, we provide valuable error estimates to ensure the convergence of the algorithm.

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The remaining sections of the paper are organized as follows. In section 2, we state the mechanical model, introduce some notations, and establish a variational formulation. Section 3 is devoted to giving the result concerning the existence and uniqueness of the solution. Finally, in section 4, we perform the numerical analysis of the problem (PV), and prove the main result of error estimation and convergence.

2. THE MODEL AND ITS VARIATIONAL FORMULATION

Denote by $\mathbf{S}_d$ the space of second order symmetric tensors on $\mathbb{R}^d(d=2,3)$, while '.' and $\|.\|$ represent respectively, the inner product and the Euclidean norm on $\mathbf{S}_d$ and $\mathbb{R}^d$. We use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, given by $v_\nu = v.\nu = v_i\nu_i$, $v_\tau = v - v_\nu\nu$, $\sigma_\nu = \sigma\nu.\nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$.

Assume a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ $(d=2,3)$ occupied by a piezoelectric body. The boundary Γ of Ω is partitioned into three disjoint measurable parts, namely $\overline{\Gamma}_1$, $\overline{\Gamma}_2$, $\overline{\Gamma}_3$, such that $meas\,(\Gamma_1) > 0$. Additionally, it is partitioned into two disjoint measurable parts, denoted as $\overline{\Gamma}_a$ and $\overline{\Gamma}_b$ on other hand, such that $meas\,(\Gamma_a) > 0$ and $\overline{\Gamma}_3 \subset \overline{\Gamma}_b$. Let $[0,T], T>0$ the time interval of interest. The body under consideration is subjected to volume forces with a density of f_0, tractions with a density of f_2 on Γ_2, an electric charge with a density of q_0 on Ω and a surface electric charge with a density of q_b on Γ_b. The body is clamped on Γ_1 and the electric potential vanishes on Γ_a. On Γ_3, we assume that the body is in adhesive frictionless unilateral contact with a reactive and conductive foundation. We denote by $u=(u_i)$ the displacement field, $D=(D_1,...,D_d)$ the electric displacement, and $\sigma=\sigma_{ij}$ is the stress tensor. The equilibrium equations governing the displacement and electric displacement can be expressed as follows:

$$div\,\sigma = -f_{_0} \ \text{ in}\,\Omega\times(0,T)\,, \tag{2.1}$$

$$\text{Div}\,D = -q_0 \ \text{ in}\,\Omega\times(0,T)\,, \tag{2.2}$$

The behavior of electro-viscoelastic materials is described by the following constitutive law

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{F}\varepsilon(u) - \mathcal{E}^*E(\varphi) \ \text{ in}\,\Omega\times(0,T)\,, \tag{2.3}$$

$$D = \mathcal{E}\varepsilon(u) + \mathcal{C}E(\varphi) \ \text{ in}\,\Omega\times(0,T)\,, \tag{2.4}$$

where $\varepsilon\,(u) = (\varepsilon_{ij}\,(u))\,,\ \varepsilon_{ij}\,(u) = \frac{1}{2}\,(\partial_j u_i + \partial_i u_j)$. $\mathcal{A}$ and $\mathcal{F}$ are the viscosity and elasticity operators, respectively, $\mathcal{E}$ is the piezoelectric tensor, $\mathcal{E}^*$ is its transpose. $E(\varphi) = -\nabla\varphi$ is the electric field, and $\mathcal{C}$ is the permittivity tensor. According to the physical setting we prescribe boundary conditions.

$$u = 0 \ \text{ on}\,\Gamma_1\times(0,T)\,,\ \ \sigma\nu = f_2 \text{ on}\,\Gamma_2\times(0,T)\,, \tag{2.5}$$

$$\varphi = 0 \text{ on}\,\Gamma_a\times(0,T)\,,\ \ D\nu = q_b \text{ on}\,\Gamma_b\times(0,T)\,, \tag{2.6}$$

$$D\cdot v = \psi(u_v - g)\phi_L(\varphi-\varphi_\Gamma) \ \text{ on }\ \Gamma_3\times(0,T). \tag{2.7}$$

In (2.7) ψ is a Lipschitz continuous function, and φ_L is the truncation function

$$\varphi_L(s) = \begin{cases} -L_\varphi & \text{if } \ s < -L_\varphi \\ -s & \text{if } \ -L_\varphi \le s \le L_\varphi \\ L_\varphi & \text{if } \ s > L_\varphi \end{cases}$$

where L_φ is a positive constant.
On Γ_3, we use the Signorini's conditions with non-zero gap and adhesion,

$$\begin{gathered} u_\nu \le g, \quad \big(\sigma_\nu + p(u_\nu) - \gamma_\nu\beta^2 R_\nu(u_\nu)\big) \le 0, \\ \big(\sigma_\nu + p(u_\nu) - \gamma_\nu\beta^2 R_\nu(u_\nu)\big)\,(u_\nu - g) = 0. \end{gathered} \tag{2.8}$$

$$\dot{\beta} = -\left[\beta(\gamma_\nu(R_\nu u_\nu)^2) - \epsilon_a\right]_+ \text{on}\,\Gamma_3\times(0,T)\,. \tag{2.9}$$

In the provided equations, the function $g \geq 0$ represents the gap between Γ_3 and the foundation. p and $-\gamma_\nu \beta^2 R_\nu(u_\nu)$ are the normal contact functions, while γ_ν denotes a given adhesion which dependent on $x \in \Gamma_3$. R_ν is a truncation operator defined by:

$$R_\nu(s) = \begin{cases} L & \text{if } s < L \\ -s & \text{if } -L \leq s \leq 0 \\ 0 & \text{if } s > L \end{cases}$$

where $L > 0$ is the characteristic length of the bond.
In addition, we assume that the contact is frictionless:

$$\sigma_\tau = 0 \text{ on } \Gamma_3 \times (0,\ T), \tag{2.10}$$

To complete, we prescribe the following initial condition

$$u(0) = u_0, \ \ \beta(0) = \beta_0. \tag{2.11}$$

To summarize, we consider the following problem:

Problem (P). *Find a displacement field* $u : \Omega \times [0,T] \to \mathbb{R}^d$, *a stress field* $\sigma : \Omega \times [0,T] \to \mathbf{S}_d$, *an electric potential* $\varphi : \Omega \times [0,T] \to \mathbb{R}$, *an electric displacement field* $D : \Omega \times [0,T] \to \mathbb{R}^d$ *and a bonding field* $\beta : \Gamma_3 \times [0,T] \to \mathbb{R}$ *such that* (2.1)-(2.11) hold.

To establish a variational formulation of Problem (P), we need additional notations and preliminaries. For $T > 0$ and a real Hilbert space X, we use the usual notation for the spaces $L^p(0,T;X)$, $p \in [0,\infty]$, $W^{k,p}(0,T;X)$, $k = 1,2$ and the spaces of continuous functions $C([0,T];X)$, $C^1([0,T];X)$.
We will use the real Hilbert spaces:

$$H = L^2(\Omega)^d, \ Q = \left\{\tau = (\tau_{ij}), \ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\},$$
$$H_1 = \left\{u = (u_i) \,|u_i \in H^1(\Omega)\right\}, \ H_d = \left\{\sigma \in Q \,|\text{div}\, \sigma \in H\right\},$$

endowed with the respective inner products:

$$(u,v)_H = \textstyle\int_\Omega u_i v_i dx, \ \ \langle \sigma, \tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx,$$
$$(u,v)_{H_1} = \langle u,v\rangle_H + (\varepsilon(u),\varepsilon(v))_Q, \ \ (\sigma,\tau)_{H_d} = \langle \sigma,\tau\rangle_Q + (div\,\sigma, div\tau)_H.$$

We recall that the following Green's formula holds:

$$\langle \sigma, \varepsilon(v)\rangle_Q + (div\,\sigma, v)_H = \int_\Gamma \sigma\nu.v da \quad \forall v \in H_1. \tag{2.12}$$

To accommodate the condition (2.5), we will seek the displacement fields within the space

$$V = \{v \in H_1 : v = 0 \ \ a.e. \text{ on } \Gamma_1\}.$$

Since $meas(\Gamma_1) > 0$, there is a constant $C_\Omega > 0$ such that $\|\varepsilon(v)\|_Q \geqslant C_\Omega \|v\|_{H_1}$ for any $v \in V$ and V is a Hilbert space with the inner product $(u,v)_V = (\varepsilon(u),\varepsilon(v))_Q$ and the associated norm $\|.\|_V$. For $v \in H_1$ we use the same symbol v for its trace on Γ. From the Sobolev trace theorem, there is a constant $d_\Omega > 0$ such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leqslant d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.13}$$

Considering the Signorini's condition, we define the closed convex set of admissible displacements as follows:

$$U_{ad} = \{v \in V : v_\nu \leq g \ \text{ a.e. on } \Gamma_3\}.$$

For the electric field, we require the following two Hilbert spaces:

$$W = \{\xi \in H_1 \,|\xi = 0 \ \ a.e \text{ on } \Gamma_a\}, \ W_a = \left\{D = (D_i) \,\middle|D_i \in L^2(\Omega),\ \text{Div } D \in L^2(\Omega)\right\},$$

endowed respectively with the inner products

$$(\xi,\phi)_W = (\nabla\xi,\nabla\phi)_H\,,(D,E)_{Wa} = (D,E)_H + (\text{Div } D, \text{Div } E)_{L^2(\Omega)}\,.$$

Since $meas(\Gamma_a) > 0$, then the Friedrichs-Poincaré inequality holds, that there exists a constant $C_F > 0$ such that:

(2.14) $$\|\nabla\xi\|_W \geq C_F\,\|\xi\|_{H^1(\Omega)} \quad \forall\xi \in W.$$

Moreover, if $D \in W_d$ is a sufficiently regular, the following Green's formula holds:

(2.15) $$(D,\nabla\xi)_H + (\text{Div } D,\xi)_{L^2(\Omega)} = \int_{\Gamma_b} D\nu.\xi da \quad \forall\xi \in W.$$

We will also use the Banach space of fourth-order tensors

$$Q_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkh})\,;\ \mathcal{E}_{ijkh} = \mathcal{E}_{jikh} = \mathcal{E}_{khij} \in L^\infty(\Omega)\}\,,$$

endowed with the norm $\|\mathcal{E}\|_{Q_\infty} = \max\limits_{0\leq i,j,k,h\leq d} \|\mathcal{E}_{ijkh}\|_{L^\infty(\Omega)}$.
Finally, we introduce the space of bonding field denoted by B, given by

$$B = \left\{\beta : [0,T] \longrightarrow L^2(\Gamma_3);\ 0 \leq \beta(t) \leq 1 \ \ \forall t \in [0,T],\ \ a.e. \text{ on } \Gamma_3\right\}.$$

For the study of Problem (P), we impose the following assumptions on the data :

Operators $\mathcal{A},\ \mathcal{F}$, as well as the tensors $\mathcal{C}$, $\mathcal{E}$, and $\mathcal{E}^*$ satisfy the following hypotheses:

(2.16) $$\begin{cases} \text{(a) } \mathcal{A} = (a_{ijkl}) : \Omega\times S_d \longrightarrow S_d,\ a_{ijkl} = a_{ijlk} = a_{lkij} \in L^\infty(\Omega) \text{ , and} \\ \text{there exists } M_{\mathcal{A}} > 0 \text{ such that: } \|\mathcal{A}(x,\xi_1) - \mathcal{A}(x,\xi_2)\| \leq M_{\mathcal{A}}\,\|\xi_1 - \xi_2\| \\ \quad \forall\xi_1,\xi_2 \in S_d \text{ a.e. in } \Omega, \\ \text{(b) there exists } m_{\mathcal{A}} > 0 \text{ such that: } \mathcal{A}\xi.\xi \geqslant m_{\mathcal{A}}\,|\xi|^2 \quad \forall\xi \in S_d \text{ a.e. in } \Omega, \end{cases}$$

(2.17) $$\begin{cases} \text{(a) } \mathcal{F} = (b_{ijkl}) : \Omega\times S_d \longrightarrow S_d,\ b_{ijkl} = b_{ijlk} = b_{lkij} \in L^\infty(\Omega)\,, \text{ and} \\ \text{there exists } M_{\mathcal{F}} > 0 \text{ such that: } \|\mathcal{F}(x,\xi_1) - \mathcal{F}(x,\xi_2)\| \leq M_{\mathcal{F}}\,\|\xi_1 - \xi_2\| \\ \quad \forall\xi_1,\xi_2 \in S_d \text{ a.e. in } \Omega, \\ \text{(b) there exists } m_{\mathcal{F}} > 0 \text{ such that: } \mathcal{F}\xi.\xi \geqslant m_{\mathcal{F}}\,\|\xi\|^2 \quad \forall\xi \in S_d \text{ a.e. in } \Omega, \end{cases}$$

(2.18) $$\begin{cases} \text{(a) } \ \mathcal{C} = (c_{ij}) : \Omega\times\mathbb{R}^d \longrightarrow \mathbb{R}^d,\ c_{ij} = c_{ji} \in L^\infty(\Omega), \\ \text{(b) there exists } m_{\mathcal{C}} > 0 \text{ such that: } c_{ij}(x)E_iE_j \geqslant m_{\mathcal{C}}\,\|E\|^2\,, \\ \ \forall\xi \in S_d \text{ a.e. in } \Omega. \end{cases}$$

(2.19) $$\mathcal{E} = (e_{ikj}),\ e_{ijk} = e_{ikj} \in L^\infty(\Omega),\ \mathcal{E}\sigma.v = \sigma.\mathcal{E}^*v \quad \forall\sigma \in \mathbf{S}_d,\ v \in \mathbb{R}^d.$$

Furthermore, we assume that the normal compliance function $p : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$ satisfies:

(2.20) $$\begin{cases} \text{(a) The function } x \to p(x,r) \text{ is measurable on } \Gamma_3 \text{ and is zero for all } r \leq 0 \\ \text{(c) there exists } M_{\mathfrak{p}} > 0, \text{ such that:} \\ \ |p(x,r_1) - p(x,r_2)| \leq M_{\mathfrak{p}}\,|r_1 - r_2| \ \ \forall r_1,r_2 \in \mathbb{R} \text{ a.e. in } \Gamma_3. \end{cases}$$

As an example, the function $p(r) = [r]_+$ satisfies condition (2.20).
Additionally, we assume that adhesion coefficients satisfy

(2.21) $$\gamma_\nu \in L^\infty(\Gamma_3)\,, \epsilon_a \in L^2(\Gamma_3)\,,\ \gamma_\nu,\ \epsilon_a \geqslant 0 \quad \text{a.e. on } \Gamma_3,$$

we suppose that:

(2.22)
$$f_0 \in C\left([0,T];L^2(\Omega)^d\right),\ f_2 \in C\left([0,T];L^2(\Gamma_2)^d\right),$$
$$q_0 \in C\left([0,T];L^2(\Omega)^d\right),\ q_b \in C\left([0,T];L^2(\Gamma_b)^d\right),$$

the initial data u_0 and β_0 satisfy

(2.23)
$$u_0 \in U_{ad},\ \ \beta_0 \in L^2(\Gamma_3),\ 0 \leq \beta_0 \leq 1 \ \text{ a.e. on } \Gamma_3,$$

and the surface electrical conductivity function $\psi : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$ satisfies

(2.24)
$$\begin{cases} \text{(a) The function } x \to \psi(x,r) \text{ is measurable on } \Gamma_3 \text{ and is zero for all } r \leq 0 \\ \text{and there exists } L_\psi > 0, \text{ such that: } |\psi(x,r)| \leq L_\psi\ \forall r \in \mathbb{R}, \text{ a.e. in } \Gamma_3, \\ \text{(c) there exists } m_\psi > 0, \text{ such that: } |\psi(x,r_1) - \psi(x,r_2)| \geq m_\psi\,|r_1 - r_2|\,, \\ \ \text{a.e. in } \Gamma_3, \end{cases}$$

Finally, the potential φ_Γ and the gap function g satisfy

(2.25)
$$\varphi_\Gamma \in L^2\left(0,T;L^2(\Gamma_3)\right),\ g \in L^2(\Gamma_3), g \geq 0 \ \text{ a.e. on } \Gamma_3.$$

Using the representation theorem of Riesz-Fréchet, for all $t \in [0,T]$, we define the elements $\boldsymbol{f}(t) \in V$ and $\boldsymbol{q}(t) \in W$ as follows

(2.26)
$$(\boldsymbol{f}(t),v)_V = \int_\Omega f_0(t).v dx + \int_{\Gamma_2} f_2(t).v da\ \forall v \in V,$$

(2.27)
$$(\boldsymbol{q}(t),\xi)_V = \int_\Omega q_0(t).\xi dx + \int_{\Gamma_2} q_b(t).\xi da\ \ \forall \xi \in W.$$

To simplify the writing, we denote by, $\boldsymbol{a} : V \times V \to \mathbb{R}$, $\boldsymbol{b} : V \times V \to \mathbb{R}$, $\boldsymbol{c} : W \times W \to \mathbb{R}$ and $\boldsymbol{e} : V \times W \to \mathbb{R}$

$$\begin{aligned} \boldsymbol{a}(u,\ v) &= (\mathcal{A}\varepsilon(u),\varepsilon(v))_Q;\ \boldsymbol{b}(u,\ v) = (\mathcal{F}\varepsilon(u),\varepsilon(v))_Q, \\ \boldsymbol{c}(\varphi,\ \xi) &= (\mathcal{C}\nabla\varphi,\nabla\xi))_H;\ \boldsymbol{e}(v,\ \xi) = (\mathcal{E}\varepsilon(v),\nabla\xi)_H = (\mathcal{E}^*\nabla\xi,\ \varepsilon(v))_Q, \end{aligned}$$

We note that according to equations (2.16)-(2.19), the forms $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are strongly monotone and Lipschitz continuous.

Let us denote by $j_a : L^\infty(\Gamma_3) \times V_0 \times V \longrightarrow \mathbb{R}$ and $j_e : V \times W \to W$ respectively the functionals given by

(2.28)
$$j_a(\beta,u,v) = \int_{\Gamma_3} (p(u_\nu) - \gamma_\nu \beta^2 R_\nu(u_\nu)v_\nu)da,$$

(2.29)
$$j_e(u,\varphi,\xi) = \int_{\Gamma_3} \psi(u_v - g)\phi_L(\varphi - \varphi_\Gamma)\xi da.$$

If u, σ , φ and D are regular and satisfy the equations and conditions (2.1)-(2.11), the Green's formula (2.12) and (2.15) enable us to derive the following variational formulation for problem P.

Problem (P_V). *Find a displacement field $u \in C([0,T];V)$, an electric potential $\varphi \in C([0,T];W)$ and a bonding field $\beta \in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right) \cap B$, such that*
$u(t) \in U_{ad}$ for all $t \in [0,T]$ and

(2.30)
$$\begin{aligned} &\boldsymbol{a}(\dot{u}(t),v-u(t)) + \boldsymbol{b}(u(t),v-u(t)) + \boldsymbol{e}\,(v-u(t),\varphi(t)) \\ &+ j_a(\beta(t),u(t),v-u(t)) = (\boldsymbol{f}(t),v-\dot{u}(t))_V\ \ \forall v \in U_{ad},\ t \in [0,T], \end{aligned}$$

(2.31)
$$\boldsymbol{b}\,(\varphi(t),\xi)_H - \boldsymbol{e}\,(u(t),\xi)_H + j_e(u,\varphi,\xi) = (\boldsymbol{q}(t),\xi)_W \quad \forall \xi \in W,\ t \in [0,T],$$

$$\dot{\beta}(t) = -\left[\beta(t)\left(\gamma_\nu (R_\nu u_\nu(t))^2\right) - \epsilon_a\right]_+, \text{ a.e. } t \in [0,T], \tag{2.32}$$

$$u(0) = u_0, \;\; \beta(0) = \beta_0. \tag{2.33}$$

3. Existence and uniqueness of solution

In this section, we establish the existence and uniqueness of the weak formulation (Problem (P_V)). The following theorem provides the main result.

Theorem 3.1. *Assume that the assumptions (2.1)–(2.25) hold. Then, the problem (P_V) has a unique solution (u, φ, β). Moreover, the solution satisfies*

$$\begin{aligned} &u \in C^1([0,T]\,; V), \\ &\varphi \in C([0,T]\,; W), \\ &\beta \in W^{1,\infty}(0,T; L^\infty(\Gamma_3)) \cap B. \end{aligned} \tag{3.1}$$

Proof. In order to prove Theorem 3.1, we provide a proof by defining intermediate problems and demonstrating their unique solvability. Subsequently, we construct a contraction mapping such that its unique fixed point corresponds to the weak solution of the problem (P).
To begin, let $l \in C([0,T]\,; V)$ and consider the following intermediate problem:

Problem (P_l^1). *Find* $u_l \in C([0,T]\,; X)$ such that

$$\begin{aligned} &\boldsymbol{a}(\dot{u}_l(t), v - u_l(t)) + (l(t), v - u_l(t))_{V\times V} = (\boldsymbol{f}(t), v - u_l(t))_V \\ &\forall v \in U_{ad}, \; t \in [0,T]. \\ &\qquad\qquad u_l(0) = u_0. \end{aligned} \tag{3.2}$$

For this first problem, we have the following result.

Lemma 3.2. *Problem (P_l^1) has a unique solution $u_l \in C^1([0,T]\,; V)$.*

Proof. Let us denote the operator $\tilde{\boldsymbol{a}} : V \longrightarrow V$ such that $(\tilde{\boldsymbol{a}}u, v) = \boldsymbol{a}(u,v)$. Since $\tilde{\boldsymbol{a}}$ is a strongly monotone and Lipschitz continuous operator on V, it is invertible and its inverse $\boldsymbol{a}^{-1}$ is also a strongly monotone and Lipschitz continuous operator. Therefore, using the regularity (2.18), we can conclude that there exists a unique function u_l which satisfies

$$\begin{aligned} &w_l \in C([0,T]\,; X). \\ &\tilde{\boldsymbol{a}} w_l(t) + l(t) = f(t), \quad \forall t \in [0,T]. \end{aligned} \tag{3.3}$$

Let $u_l : [0,T] \longrightarrow V$ defined by $u_l(t) = u_0 + \int_0^t w_l(s)ds$. Then, u_l is the unique solution to the problem, and it also belongs to the set $U_{ad} \cap C^1([0,T]\,; V)$. □

In the next step, we use the solution $u_l \in C^1([0,T]\,; V)$ obtained in Lemma 3.2 to formulate an intermediate problem for the electrical potential.

Problem (P_l^2). *Find* $\varphi_l \in C([0,T]\,; W)$ such that

$$\begin{aligned} &\boldsymbol{c}\left(\varphi_l(t), \xi\right) - \boldsymbol{e}\left(u_l(t), \xi\right) + j_e(u_l(t), \varphi_l(t), \xi) = (\boldsymbol{q}(t), \xi)_W \\ &\qquad \forall \xi \in W, \; t \in [0,T]. \end{aligned} \tag{3.4}$$

We establish the following lemma.

Lemma 3.3. *Problem* (P_l^2) *has a unique solution* $\varphi_l \in C([0,T]\,;W)$.

Proof. We define the operator $\Psi : W \longrightarrow W$ by

$$\langle \Psi(t)\varphi, \xi\rangle = \boldsymbol{c}\,(\varphi(t), \xi) - \boldsymbol{e}\,(u_l(t), \xi) + j_e(u_l(t), \varphi(t), \xi).$$

For $\varphi_1, \varphi_2 \in W$, by using (2.18), (2.19), (2.24) and (2.29), we show

$$\langle \Psi(t)\varphi_1 - \Psi(t)\varphi_2, \varphi_1 - \varphi_2\rangle \geq m_{\mathcal{C}} \left\|\varphi_1 - \varphi_2\right\|^2, \tag{3.5}$$

and there exist $C > 0$ such that

$$\langle \Psi(t)\varphi_1 - \Psi(t)\varphi_2, \xi\rangle \leq C \left\|\varphi_1 - \varphi_2\right\|_W^2 \left\|\xi\right\|_W. \tag{3.6}$$

The inequalities (3.5) and (3.6) show that operator $\Psi(t)$ is strongly monotone and Lipschitz continuous on W. Therefore, there exists a unique element $\varphi_l(t) \in W$ which satisfies(3.4).

For $t_1, t_2 \in [0,T]$, we use (2.13), (2.18), (2.19), (2.24) (2.29) and the Lipschitz continuity of the functions ψ and φ to establish the following estimate:

$$\left\|\varphi_l(t_1) - \varphi_l(t_2)\right\|_W \leq C\left(\left\|u_l(t_1) - u_l(t_2)\right\|_V + \left\|q(t_1) - q(t_2)\right\|_W\right) \tag{3.7}$$

Since $u_l \in C^1([0,T];V)$ and $C([0,T];W)$, it implies that $\varphi_l \in C([0,T]\,;W)$, which concludes the proof.□

Proceeding to the next step, we once again use the previously obtained solution u_l obtained in Lemma 3.2, and we consider the following Cauchy problem.

Problem (P_l^3) *Find a bonding field* $\beta_l : [0,T] \longrightarrow L^\infty(\Gamma_3)$ *such that:*

$$\dot{\beta}_l(t) = -\left[\beta_l(t)\left(\gamma_\nu(R_\nu u_{l\nu}(t))^2\right) - \epsilon_a\right]_+ \ \text{ a.e. } t \in [0,T], \tag{3.8}$$

$$\beta_l(0) = \beta_0. \tag{3.9}$$

We have:

Lemma 3.4. *Problem* (P_l^3) *has a unique solution* β_l *which satisfies*
$\beta_l \in W^{1,\infty}([0,T]\,;L^\infty(\Gamma_2)) \cap B$.

Proof. Consider the mapping $\Phi : [0,T] \times L^2(\Gamma_3) \longrightarrow L^2(\Gamma_3)$ defined by

$$\Phi_l(t,\beta) = -\left[\beta\left(\gamma_\nu(R_\nu u_{l\nu}(t))^2\right) - \epsilon_a\right]_+$$

For all $t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$, it follows from the properties of the operator R_ν that Φ_β is Lipschitz continuous with respect to its second argument, uniformly in time. Additionally, for any $\beta \in B$, the function $t \to \Phi_\beta(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus, by using a version of Cauchy-Lipschitz theorem (see[5]), Problem (P_l^3) possesses a unique solution $\beta_l \in W^{1,\infty}(0,T;L^\infty(\Gamma_3))$. Since the restriction $0 \leq \beta_\beta \leq 1$ a.e. on Γ_3 is included in the variational problem P_V, it follows that β belongs to B, which completes the proof of Lemma 3.4. □

Now, for all l$\in C([0,T]\,;V)$, we note by u_l the solution of problem (P_l^1) provided in Lemma 3.2, by φ_l the solution of the problem (P_l^2) provided in Lemma 3.3 and by β_l the solution of the problem (P_l^3) provided in Lemma 3.4. In addition, we apply the Riesz representation theorem to define the operator $\Lambda : [0,T] \longrightarrow V$

$$(\Lambda l(t), v)_{V\times V} = \boldsymbol{b}\,(u_l(t), v) + \boldsymbol{e}\,(v, \varphi_l(t)) + j_a(\beta_l(t), u_l(t), v) \quad \forall v \in V,\ t \in [0,T]. \tag{3.10}$$

For this operator, we have the following lemma

Lemma 3.5. *The operator* Λ *admits a unique fixed point* $l^* \in C([0,T]\,;V)$.

Proof. For $i = 1, 2$, let denote u_{l_i}, φ_{l_i} and β_{l_i}, the solutions to previous intermediate problems with $l_i \in C([0,T]\,;V)$. For all $t \in [0,T]$, it follows from (3.2) that

$$\boldsymbol{a}\left(\dot{u}_{l_1}(t) - \dot{u}_{l_2}(t), \dot{u}_{l_1}(t) - \dot{u}_{l_2}(t)\right) + \left(l_2(t) - l_1(t), \dot{u}_{l_1}(t) - \dot{u}_{l_2}(t)\right) = 0,$$

using the properties (2.16), we find

$$\|\dot{u}_{l_1}(t) - \dot{u}_{l_2}(t)\|_V \leq C\,\|l_1(t) - l_2(t)\|_V\,,\ \forall t \in [0,T]. \tag{3.11}$$

Since $\|u_{l_1}(t) - u_{l_2}(t)\|_V \leq \int\limits_0^t \|\dot{u}_{l_1}(s) - \dot{u}_{l_2}(s)\|_V\, ds$, it follows from inequality (3.11)

$$\|u_{l_1}(t) - u_{l_2}(t)\|_V \leq C\,\|l_1(t) - l_2(t)\|_V\,,\ \forall t \in [0,T]. \tag{3.12}$$

Also, using similar arguments to (3.7), we find

$$\|\varphi_l(t_1) - \varphi_l(t_2)\|_W \leq C\,\|u_l(t_1) - u_l(t_2)\|_V \tag{3.13}$$

Writing (3.8) with $\beta_l = \beta_{l_1}$ and with $\beta_l = \beta_{l_2}$, and using the assumption on the properties of the operators R_ν, we can perform some elementary calculations to demonstrate the existence of a constant $C > 0$ such that

$$\begin{aligned}\|\beta_{l_1}(t) - \beta_{l_2}(t)\|_{L^2(\Gamma_3)} \leq C\Big(&\textstyle\int_0^t \|\beta_{l_1}(s) - \beta_{l_2}(s)\|_{L^2(\Gamma_3)}\, ds \\ &+ \textstyle\int_0^t \|u_{l_1}(s) - u_{l_2}(s)\|_{L^2(\Gamma_3)^d}\, ds\Big).\end{aligned} \tag{3.14}$$

Using the conditions (2.13) and applying a Gronwall-type argument, we find

$$\|\beta_{l_1}(t) - \beta_{l_2}(t)\|_V \leq C\int\limits_0^t \|u_{l_1}(s) - u_{l_2}(s)\|_V\, ds. \tag{3.15}$$

On the other hand, we have the following equality

$$\begin{aligned}(\Lambda l_1(t) - \Lambda l_2(t), v)_{V\times V} &= \boldsymbol{b}\left(u_{l_1}(t) - u_{l_2}(t), v\right) + \boldsymbol{e}\left(v, \varphi_{l_1}(t) - \varphi_{l_2}(t)\right) \\ &+ j_a(\beta_{l_1}(t), u_{l_1}(t), v) - j_a(\beta_{l_2}(t), u_{l_2}(t), v) \quad \forall v \in V.\end{aligned} \tag{3.16}$$

Keeping in mind (2.19), (2.20) and (2.28), there exists a constant $C > 0$ such that:

$$\begin{aligned}\|\Lambda l_1(t) - \Lambda l_2(t)\|_V \leq C\Big(&\|\varphi_{l_1}(t) - \varphi_{l_2}(t)\|_Q + \|u_{l_{1\nu}}(t) - u_{l_{2\nu}}(t)\|_{L^2(\Gamma_3)} \\ &+ \left\|\beta^2_{l_1}(t)R_\nu(u_{l_1\nu}(t)) - \beta^2_{l_2}(t)R_\nu(u_{l_2\nu}(t))\right\|_{L^2(\Gamma_3)}\Big).\end{aligned} \tag{3.17}$$

Hence, using (2.13) and the properties of the operator R_ν, it follows that

$$\begin{aligned}\|\Lambda l_1(t) - \Lambda l_2(t)\|_V \leq C\Big(&\|u_{l_1}(t) - u_2(t)\|_V + \|\varphi_{l_1}(t) - \varphi_{l_2}(t)\|_Q \\ &+ + \|\beta_{l_1}(t) - \beta_{l_2}(t)\|_{L^2(\Gamma_3)}\Big).\end{aligned} \tag{3.18}$$

Now, combine (3.18), (3.15), (3.7), and (3.12), we obtain:

$$\|\Lambda l_1 - \Lambda l_2\|_{C([0,T];Q)} \leq C\int\limits_0^t \|l_1(s) - l_2(s)\|_V\, ds.$$

By iteration, we deduce that for any positive integer n :

$$\|\Lambda^n l_1 - \Lambda^n l_2\|_{C([0,T];V)} \leq \frac{C^n T^n}{n!}\,\|l_1 - l_2\|_{C([0,T];V)}\,.$$

Then, for a positive integer n sufficiently large, it follows that Λ is a contraction on the space $C([0,T];V)$. Therefore, by the Banach fixed point theorem, we deduce that Λ has a unique fixed point l^*.
For $t_1, t_2 \in [0,T]$, using similar arguments to the proof of (3.18), we find

$$\|\Lambda l(t_1) - \Lambda l(t_2)\|_V \leq C\,(\|u_l(t_1) - u_l(t_2)\|_V + \|\varphi_l(t_1) - \varphi_l(t_2)\|_Q + \|\beta_l(t_1) - \beta_l(t_2)\|_{L^2(\Gamma_3)}\Big), \tag{3.19}$$

since $u_l \in C^1([0,T];V)$, $\varphi_l \in C(0,T;W)$ and $\beta_l \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ we deduce from inequality (3.19) that $l^* = \Lambda l^* \in C([0,T];V)$ which completes the proof of Lemma 3.5.□

Now, we have all the necessary ingredients to conclude the proof of Theorem 3.1. Let l^* denote the unique fixed point of the operator Λ, and let u^*, φ^*, β^* be the respective solutions to problems $(P^1_{l^*})$, $(P^2_{l^*})$ and $(P^3_{l^*})$ i.e. $u^* = u_{l^*}$, $\varphi^* = \varphi_{l^*}$ and $\beta^* = \beta_{l^*}$.
Then, by (3.2), (3.4), (3.8) and (3.9), we conclude that the triple $(u^*, \varphi^*, \beta^*)$ is a solution to Problem (P_V). The uniqueness of this solution follows directly from the uniqueness of the fixed point of the operator Λ, and regularities (3.1) follow from Lemmas 3.2, 3.3 and 3.4. □

4. Numerical approach

In this section, we employ the finite element method to introduce a fully-discrete scheme that provides an approximation for the solution of Problem (PV). The parameter of discretization is denoted by $h > 0$, and we consider $\mathcal{T}^h$ as a regular finite element partition of the domain Ω, which is compatible with the boundary partition of Γ. To approximate the displacement field u, the electric potential φ, and the bonding field β, we introduce finite element spaces $V^h \subset V$, $W^h \subset W$, and $B^h \subset L^2(\Gamma_3)$ respectively.defined by

$$\begin{gathered} V^h = \Big\{ v^h \in \left[C(\overline{\Omega})\right]^d ;\ v^h_{|T_r} \in [\mathbb{P}_1(T_r)]^d ,\ T_r \in \mathcal{T}^h,\ v^h = 0 \text{ on } \overline{\Gamma}_1 \Big\}, \\ W^h = \left\{ \psi^h \in C(\overline{\Omega});\ v^h_{|T_r} \in \mathbb{P}_1(T_r),\ T_r \in \mathcal{T}^h, \varphi^h = 0 \text{ on } \overline{\Gamma}_a \right\}, \\ B^h = \left\{ \beta^h \in L^2(\Gamma_3);\ \beta^h_{|\gamma} \in \mathbb{R},\ \gamma \in \mathcal{T}^h_{\Gamma_3} \right\}, \end{gathered} \tag{4.1}$$

where $\mathbb{P}_1(T_r)$ represents the space of polynomials of global degree at most 1 in Tr.

It should be noted that $\mathcal{T}^h_{\Gamma_3}$ is the partition induced by the triangulation $\mathcal{T}^h$. Additionally, let $\mathcal{P}_{B^h} : L^2(\Gamma_3) \longrightarrow B^h$ be the orthogonal projection operator on B^h. Furthermore, we define U^h_{ad} as the discrete convex set of admissible displacement given by $U^h_{ad} = U_{ad} \cap V^h$.

To discretize the time derivatives, we utilize a uniform partition of $[0,T]$ denoted as $0 = t_0 < t_1 < ... < t_N = T$. Let $k = T/N$ be the time step. For a continuous function $w(t)$, we denote $w_n = w(t_n)$ to represent the values of w at the discrete time points. For a sequence $(w_n)^N_{n=0}$, we introduce the finite differences $\delta w_n = (w_n - w_{n-1})/k_n$.
Next, we introduce the finite differences

$$u^{hk}_n = \sum_{j=1}^n k_j \delta u^{hk}_j + u^h_0,\ \varphi^{hk}_n = \sum_{j=1}^n k_j \delta \varphi^{hk}_j + \varphi^h_0,\ \beta^{hk}_n = \sum_{j=1}^n k_j \delta \beta^{hk}_j + \beta^h_0,\ n \geq 1. \tag{4.2}$$

Here, no summation is assumed over a repeated index. Using the backward Euler scheme, the fully-discrete approximation of Problem (PV) is the following.

Problem (PV^{hk}). *Find* $u^{hk} = \left(u^{hk}_n\right)^N_{n=0} \subset U^h_{ad}$, $\varphi^{hk} = \left(\varphi^{hk}_n\right)^N_{n=0} \subset W^h$, *and* $\beta^{hk} = \left(\beta^{hk}_n\right)^N_{n=0} \subset L^2(\Gamma_3)$, *such that, for all* $n = 1, 2, ..., N$

$$\begin{aligned} &\boldsymbol{a}(\delta u^{hk}_n, v^h - u^{hk}_n) + \boldsymbol{b}(u^{hk}_n, v^h - u^{hk}_n) + \boldsymbol{e}(v^h - u^{hk}_n, \varphi^{hk}_n) \\ &+ j_a(\beta^{hk}_n, u^{hk}_n, v^h - u^{hk}_n) = (\boldsymbol{f}_n, v^h - u^{hk}_n), \end{aligned} \tag{4.3}$$

$$\boldsymbol{c}(\varphi^{hk}_n, \xi^h) - \boldsymbol{e}(u^{hk}_n, \xi^h) + j_e(u^{hk}_n, \varphi^{hk}_n, \xi^h) = (\boldsymbol{q}_n, \xi^h), \tag{4.4}$$

$$\delta\beta_n^{hk} = -\mathcal{P}_{B^h}\left[\beta_{n-1}^{hk}(\gamma_\nu R_\nu\left(u_{(n-1)\nu}^{hk}\right))^2 - \epsilon_a\right]_+, \tag{4.5}$$

$$u_0^{hk} = u_0^h, \tag{4.6}$$

$$\beta_0^{hk} = \beta_0^h. \tag{4.7}$$

where u_0^h and β_0^h are suitable approximations of u_0 and β_0 respectively.

We denote by $\varphi_0^{hk} = \varphi_0^h$ the unique solution of (4.4) in Problem (PV^h) for $n = 0$, and $(u_n^{hk}, \varphi_n^{hk}, \beta_n^{hk})$ denote the solutions of $\left(PV^{hk}\right)$, respectively. By employing the same arguments as in the proof of Theorem 3.1, we conclude that the Problem (PV^{hk}) possesses a unique solution $(u_n^{hk}, \varphi_n^{hk}, \beta_n^{hk})$.

To prove the convergence of the scheme, we have the following error estimate result.

Lemma 4.1. *Let the assumptions of Theorem 3.1 hold. Under the following regularity conditions:*

$$u \in W^{2,1}(0,T;V) \cap C([0,T]; H^2(\Omega)^d), \tag{4.8}$$

$$\sigma \in C([0,T]\,; H^1(\Omega)^{d\times d}) \cap W^{1,1}(0,T;Q), \tag{4.9}$$

$$\beta \in W^{2,1}(0,T; L^2(\Gamma_3)) \cap C^1([0,T]\,; H^1(\Gamma_3)), \tag{4.10}$$

$$\beta_0 \in H^1(\Gamma_3), \tag{4.11}$$

there exists a positive constant $C > 0$, independent of parameters h and k, such that for $v^h = (v_j^h)_{j=1}^N \in U_{ad}^h$, $\xi^h = (\xi_j^h)_{j=1}^N \in W^h$,

$$\begin{aligned} &\max_{1\le n\le N}\{\|u_n - u_n^{hk}\|_V + \|\varphi_n - \varphi_n^{hk}\|_W + \|\beta_n - \beta_n^{hk}\|_Q\} \\ &\le C \max_{1\le n\le N} \inf_{\substack{v^h\in V^h\\ \xi^h\in W^h\\ \beta^h\in B^h}} \left\{\|v^h - u_n\|_V + \|\xi^h - \varphi_n\|_W + \|v^h - u_{n\nu}\|_{L^2(\Gamma_3)}^{\frac{1}{2}} + h + k\right\} \end{aligned} \tag{4.12}$$

We now proceed to estimate the numerical errors on the displacement field, where we apply in some inequalities Young's inequality $ab \le \delta a^2 + \frac{1}{4\delta}b^2$.
Add (4.3) and (2.30) with $v = u_n^{hk}$ at $t = t_n$, we find

$$\begin{aligned} &\boldsymbol{a}(\delta u_n^{hk}, v^h - u_n^{hk}) + \boldsymbol{a}(\dot{u}_n, u_n^{hk} - u_n) + \boldsymbol{b}(u_n^{hk}, v^h - u_n^{hk}) + \boldsymbol{b}(u_n, u_n^{hk} - u_n) \\ &+\boldsymbol{e}(v^h - u_n^{hk}, \varphi_n^{hk}) + \boldsymbol{e}(u_n^{hk} - u_n, \varphi_n) + j_a(\beta_n, u_n, u_n^{hk} - u_n) \\ &+j_a(\beta_n^{hk}, u_n^{hk}, v^h - u_n^{hk}) = (\boldsymbol{f}_n, v^h - u_n^{hk}) + (\boldsymbol{f}_n, u_n^{hk} - u_n) \end{aligned} \tag{4.13}$$

Using the relation,

$$\begin{aligned} \boldsymbol{b}(u_n - u_n^{hk}, u_n - u_n^{hk}) \;=\; &\boldsymbol{b}(u_n, u_n - v^h) - \boldsymbol{b}(u_n^{hk}, u_n - v^h) + \boldsymbol{b}(u_n, v^h - u_n^{hk}) \\ &-\boldsymbol{b}(u_n^{hk}, v^h - u_n^{hk}), \end{aligned}$$

and the fact that the functional j_a is linear with respect to the last argument, we obtain

$$\begin{aligned} &\boldsymbol{b}(u_n - u_n^{hk}, u_n - u_n^{hk}) = \mathcal{R}_n(v^h, u_n) + \boldsymbol{a}(\delta u_n^{hk} - \dot{u}_n, v^h - u_n^{hk}) \\ &+\boldsymbol{b}(u_n^{hk} - u_n, v^h - u_n) + \boldsymbol{e}(v^h - u_n^{hk}, \varphi_n^{hk} - \varphi_n) + j_a(\beta_n^{hk}, u_n^{hk}, v^h - u_n^{hk}) \\ &-j_a(\beta_n, u_n, v^h - u_n^{hk}) - (\boldsymbol{f}_n, v^h - u_n), \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} &\mathcal{R}_n(v^h, u_n) = \boldsymbol{a}(\dot{u}_n, v^h - u_n) + \boldsymbol{b}(u_n, v^h - u_n) + \boldsymbol{e}(v^h - u_n, \varphi_n) \\ &+j_a(\beta_n, u_n, v^h - u_n) - (\boldsymbol{f}_n, v^h - u_n) \end{aligned} \tag{4.15}$$

We then proceed in estimating error on the displacement field based on the equation (4.14). First, denote

$$\sigma_n = \mathcal{A}\varepsilon(\dot{u}_n) + \mathcal{F}\varepsilon(u_n) - \mathcal{E}^*E(\varphi_n)$$

Taking into account the relation (4.9) which implies that $\sigma_\nu \in C([0,T]\,; L^2(\Gamma_3))$, then using Green's formula and the boundary conditions (2.8) and (2.10), it follows that

$$\begin{aligned}
\left|\mathcal{R}_n(v^h, \dot{u}_n)\right| &\leq \int_{\Gamma_3} \left|\left(\sigma_{n\nu} + p(u_{n\nu}) - \gamma_\nu \beta_n^2 R_\nu(u_{n\nu})\right)(v_\nu^h - u_{n\nu})\right| da \\
&\leq C\|v^h - u_{n\nu}\|_{L^2(\Gamma_3)},
\end{aligned} \tag{4.16}$$

here and below C is a generic positive constant not dependent on n.
Next, by using (2.20) and the properties of the operator R_ν, we find

$$\begin{aligned}
& |j_a(\beta_n^{hk}, u_n^{hk}, v^h - u_n^{hk}) - j_a(\beta_n, u_n, v^h - u_n^{hk})| \\
\leq \quad & C\left(\|\beta_n - \beta_n^{hk}\|^2_{L^2(\Gamma_3)} + \|u_n - u_n^{hk}\|^2_V\right)\|v^h - u_n^{hk}\|^2_V,
\end{aligned}$$

and therefore

$$\begin{aligned}
& |j_a(\beta_n^{hk}, u_n^{hk}, v^h - u_n^{hk}) - j_a(\beta_n, u_n, v^h - u_n^{hk})| \\
& C\left(\|\beta_n - \beta_n^{hk}\|^2_{L^2(\Gamma_3)} + \|u_n - u_n^{hk}\|^2_V + \|v^h - u_n^{hk}\|^2_V\right)
\end{aligned} \tag{4.17}$$

Taking into account $\|v^h - u_n^{hk}\| \leq \|v^h - u_n\| + \|u_n - u_n^{hk}\|$, by combining (4.14)-(4.17) and using (2.16), (2.17) and (2.19), we conclude that

$$\begin{aligned}
& m_{\mathcal{F}}\|u_n - u_n^{hk}\|^2_V \leq C\left\{\|v^h - u_{n\nu}\|_{L^2(\Gamma_3)} + \|\dot{u}_n - \delta u_n^{hk}\|^2_V\right. \\
& \left. + \|u_n - u_n^{hk}\|^2_V + \|\varphi_n - \varphi_n^{hk}\|^2_W + \|\beta_n - \beta_n^{hk}\|^2_{L^2(\Gamma_3)} + \|v^h - u_n\|^2_V\right\}.
\end{aligned} \tag{4.18}$$

Now to obtain an error estimate on the electric potential, taking (2.31) at time $t = t_n$ with $\xi = \xi^h$ and subtracting it to (4.4) to get

$$\boldsymbol{c}(\varphi_n - \varphi_n^{hk}, \xi^h) - \boldsymbol{e}(u_n - u_n^{hk}, \xi^h) + j_e(u_n, \varphi_n, \xi^h) - j_e(u_n^{hk}, \varphi_n^{hk}, \xi^h) = 0. \tag{4.19}$$

Writing (4.19) with the test function $\xi^h = \varphi_n - \varphi_n^{hk}$ and with $\xi^h = \varphi_n - \xi^h$, yields to the equality

$$\begin{aligned}
& \boldsymbol{c}(\varphi_n - \varphi_n^{hk}, \varphi_n - \varphi_n^{hk}) - \boldsymbol{e}(u_n - u_n^{hk}, \varphi_n - \varphi_n^{hk}) + j_e(u_n, \varphi_n, \varphi_n - \varphi_n^{hk}) \\
& - j_e(u_n^{hk}, \varphi_n^{hk}, \varphi_n - \varphi_n^{hk}) = \boldsymbol{c}(\varphi_n - \varphi_n^{hk}, \varphi_n - \xi^h) - \boldsymbol{e}(u_n - u_n^{hk}, \varphi_n - \xi^h) \\
& + j_e(u_n, \varphi_n, \varphi_n - \xi^h) - j_e(u_n^{hk}, \varphi_n^{hk}, \varphi_n - \xi^h),
\end{aligned} \tag{4.20}$$

After performing some algebraic manipulation, we can rewrite the expression in the following form:

$$\begin{aligned}
& \boldsymbol{c}(\varphi_n - \varphi_n^{hk}, \varphi_n - \varphi_n^{hk}) = \boldsymbol{c}(\varphi_n - \varphi_n^{hk}, \varphi_n - \xi^h) + \boldsymbol{e}(u_n - u_n^{hk}, \xi^h - \varphi_n^{hk}) \\
& + j_e(u_n, \varphi_n, \varphi_n - \xi^h) - j_e(u_n^{hk}, \varphi_n^{hk}, \varphi_n - \xi^h) - j_e(u_n, \varphi_n, \varphi_n - \varphi_n^{hk}) \\
& + j_e(u_n^{hk}, \varphi_n^{hk}, \varphi_n - \varphi_n^{hk}).
\end{aligned} \tag{4.21}$$

Denote $J = j_e(u_n, \varphi_n, \varphi_n - \xi^h) - j_e(u_n^{hk}, \varphi_n^{hk}, \varphi_n - \xi^h) - j_e(u_n, \varphi_n, \varphi_n - \varphi_n^{hk})$
$+ j_e(u_n^{hk}, \varphi_n^{hk}, \varphi_n - \varphi_n^{hk})$

Using the assumptions (2.29), (2.24) and (2.29), we find

$$\begin{aligned}
|J| = \Big| &\int_{\Gamma_3} \psi(u_{nv}-g)\phi_L(\varphi_n-\varphi_0)\left(\varphi_n-\xi^h\right) da \\
&- \int_{\Gamma_3} \psi(u_{nv}^{hk}-g)\phi_L(\varphi_n^{hk}-\varphi_\Gamma)\left(\varphi_n-\xi^h\right) da \\
&+ \int_{\Gamma_3} \psi(u_{nv}-g)\phi_L(\varphi_n-\varphi_\Gamma)\left(\varphi_n-\varphi_n^{hk}\right) da \\
&- \int_{\Gamma_3} \psi(u_{nv}^{hk}-g)\phi_L(\varphi_n^{hk}-\varphi_\Gamma)\left(\varphi_n-\varphi_n^{hk}\right) da \Big| \\
\leq \quad &\int_{\Gamma_3} \left|\psi(u_{nv}-g)\phi_L(\varphi_n-\varphi_\Gamma)-\psi(u_{nv}^{hk}-g)\phi_L(\varphi_n^{hk}-\varphi_\Gamma)\right| \left|\left(\varphi_n-\varphi_n^{hk}\right)\right| da \\
&+ \int_{\Gamma_3} \left|\psi(u_{nv}-g)\phi_L(\varphi_n-\varphi_\Gamma)-\psi(u_{nv}^{hk}-g)\phi_L(\varphi_n^{hk}-\varphi_\Gamma)\right| \left|\left(\varphi_n^{hk}-\xi^h\right)\right| da.
\end{aligned}$$

Since, ψ and φ_L are Lipschitz continuous functions, we get

$$\begin{aligned}
&|J| \leq M_\psi \textstyle\int_{\Gamma_3} \left|\left(\varphi_n-\varphi_n^{hk}\right)\right|^2 da + L_\varphi L_\psi \int_{\Gamma_3} \left|\left(u_n-\varphi_n^{hk}\right)\right| \left|\left(u_n-\varphi_n^{hk}\right)\right| da \\
&+M_\psi \textstyle\int_{\Gamma_3} \left|\left(\varphi_n-\varphi_n^{hk}\right)\right| \left|\varphi_n^{hk}-\xi^h\right| da + L_\varphi L_\psi \int_{\Gamma_3} \left|\left(u_n-\varphi_n^{hk}\right)\right| \left|\varphi_n^{hk}-\xi^h\right| da.
\end{aligned}$$

Then, using (2.13), (2.14) leads to

$$|J| \leq C\{\|\varphi_n-\varphi_n^{hk}\|_W^2 + \|u_n-u_n^{hk}\|_V^2 + \|\varphi_n-\xi^h\|_W^2\}. \tag{4.22}$$

By combining equations (4.22) and (4.21) and applying the properties stated in (2.17) and (2.19), it follows after reordering the terms that

$$\|\varphi_n-\varphi_n^{hk}\|_W^2 \leq C\{\|u_n-u_n^{hk}\|_V^2 + \|\xi^h-\varphi_n\|_W^2\}. \tag{4.23}$$

To obtain an error estimate on the bonding field, let $\beta_0^{hk}=\beta_0^h=\mathcal{P}_{B^h}\beta_0$ be the orthogonal projection of β_0, then $\|\beta_0-\beta_0^h\|_{L^2(\Gamma_3)} \leq Ch$. Under the assumptions (4.10) and (4.11), we have the following lemma

Lemma 4.2. *There exist a constant $C>0$, such that*

$$\|\beta_n-\beta_n^{hk}\|_{L^2(\Gamma_3)}^2 \leq \sum_{j=1}^{n-1} C\left(\|u_j-u_j^{hk}\|_V + \|\beta_j-\beta_j^{hk}\|_{L^2(\Gamma_3)}\right) + h^2 + k^2. \tag{4.24}$$

The estimate described in (4.24) has been stated and proven in [13, page 61-64] using inequalities commonly employed in the numerical analysis of contact problems that involve adhesion.

Since $u(t)=u_0+\int\limits_0^t \dot{u}(s)ds$, under the regularity assumption (4.8), we have the following standard estimation, see [4]

$$\|u_n-u_n^{hk}\|_V^2 \leq c\left(\sum_{j=1}^{n-1} k\|\dot{u}_j-\delta u_j^{hk}\|_V^2 + h^2 + k^2\right).$$

Now, adding inequalities (4.19) and (4.23) and taking into account the estimation (4.24), we then obtain

$$\begin{aligned}
&\{\|u_n-u_n^{hk}\|_V^2 + \|\varphi_n-\varphi_n^{hk}\|_W^2 + \|\beta_n-\beta_n^{hk}\|_{L^2(\Gamma_3)}^2\} \\
&\leq C\Big\{\|v^h-u_{n\nu}\|_{L^2(\Gamma_3)} + \|v^h-u_n\|_V^2 + \|\xi^h-\varphi_n\|_W^2 + \sum_{j=1}^{n-1} k\|\dot{u}_j-\delta u_j^{hk}\|_V^2 \\
&+ \sum_{j=1}^{n-1} k\left(\|u_j-u_j^{hk}\|_V^2 + \|\beta_j-\beta_j^{hk}\|_{L^2(\Gamma_3)}^2\right) + h^2 + k^2.\Big\}
\end{aligned} \tag{4.25}$$

Applying discrete Gronwall's inequality and the arbitrariness of $v^h \in U_{ad}^h$ leads to the estimate (4.12). This concludes the proof. □

Now, we proceed to state and prove the main result of this section.

Theorem 4.3. *Let the assumptions of Theorem 3.1 and Lemma 4.1 hold. Assuming that the initial values* $u_0^h \in V^h$, $\varphi_0^h \in W^h$ *and* $\beta_0^h \in Q^h$ *are chosen in such a way that*

$$\|u_0 - u_0^h\|_V \longrightarrow 0,\ \|\varphi_0 - \varphi_0^h\|_V \longrightarrow 0 \textit{ and } \|\beta_0 - \beta_0^h\|_Q \longrightarrow 0 \ \textit{ as } \ h,\ k \to 0. \tag{4.26}$$

Then, the fully-discrete solution converges, i.e.,

$$\max_{1\le n\le N} \left\{ \|u - u_n^{hk}\|_V + \|\varphi - \varphi_n^{hk}\|_W + \|\beta - \beta_n^{hk}\|_Q \right\} \to 0, \quad \textit{as } \ h,\ k \to 0. \tag{4.27}$$

Proof. To prove Theorem 4.3, , we consider the standard finite element interpolation operator of u and φ, denoted by $\Pi^h u_n$, and $\Pi^h \varphi_n$ respectively. Then, the following approximation properties hold:

$$\begin{aligned}
&\max_{1\le n\le N} \inf_{v^h \in V^h} \|v^h - u_{n\nu}\|_{L^2(\Gamma_3)} \le Ch\|u\|_{C([0,T];H^2(\Gamma_3))},\\
&\max_{1\le n\le N} \inf_{v^h \in V^h} \|v^h - u_n\|_V \le Ch\|u\|_{C([0,T];H^2(\Gamma_3))},\\
&\max_{1\le n\le N} \inf_{\xi^h \in W^h} \|\xi^h - \varphi_n^{hk}\|_W \le Ch\|\dot{u}\|_{C([0,T];H^2(\Omega))},
\end{aligned} \tag{4.28}$$

where the constant C is independent of u, φ, and β. Finally, the convergence (4.27) is obtained by combining Lemma 4.1 and (4.28). □

In this paper, using a variational method, we have established the existence and uniqueness of the solution of the problem. Then, we introduced a fully discrete scheme to approximate the solution of the contact problem. The exploration of numerical simulations using the same algorithm represents an intriguing line of future research.

Competing interests. The authors declare no competing interests.

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