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ON A CERTAIN CLASS OF BI-UNIVALENT FUNCTIONS IN CONNECTION WITH GEGENBAUER POLYNOMIALS

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ABSTRACT. Recent direction of studies shows that there is a kin connection between regular functions and orthogonal polynomials. In this paper, we study a new class of regular and bi-univalent functions that involve the familiar Gegenbauer polynomials. Some achieved results include some early coefficient bounds and the upper estimates for the Fekete-Szegö inequalities with real and complex parameters.

1. INTRODUCTION AND PRELIMINARIES

The study of orthogonal polynomials came into existence in the late 19th century through the study of continued fractions by Pafnuty Lvovich Chebyshev [13,33]. These set of polynomials have been very useful in the solution of many differential equations [13], Fourier series [15], random matrix theory, least square approximations of a function, interpolation, and quadrature [14]; and in many numerical computations, inferences and interpretations. In addition, some orthogonal polynomials such as the Zernike polynomials and Rogers-Szegö polynomials have been considered for some curves in the complex plane and in particular, the unit circle.

Explorations in geometric function theory show that regular functions have kin connections with orthogonal polynomials. Actually, some purposeful investigations have been carried out on regular functions in connection with the Legendre polynomials [12], Laguerre polynomials [35], Chebyshev polynomials [7], Jacobi polynomials [10], Horadam polynomials [32], Hermite polynomials [1], and Gegenbauer polynomials [26]. For more instances, see [2,3,5,9,22–25,29].

Specifically, let the generating function of the Gegenbauer polynomials be defined by

$$\mathfrak{G}_\alpha(t,z)=\frac{1}{(1-2tz+z^2)^\alpha} \tag{1.1}$$

where $\alpha\in\mathbb{R}-\{0\}$ is a constant, $t\in[-1,1]$ and $|z|<1$. Thus, for fixed t we have the Taylor's series representation of (1.1) as

$$\mathfrak{G}_\alpha(t,z)=\sum_{m=0}^{\infty}C_m^\alpha(t)z^m, \tag{1.2}$$

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where $C_m^\alpha(t)$ $(m \geqslant 0)$ are known as Gegenbauer polynomials of degree m. Now, suppose $\alpha = 0$, then (1.1) is set as

$$\mathfrak{G}_m^0(t) = 1 - \log(1 - 2tz + z^2) = \sum_{m=0}^{\infty} C_m^0(t) z^m.$$

We note that the recurrent relation for Gegenbauer polynomials can also be expressed as

$$C_m^\alpha(t) = \frac{1}{m}\Big[2t(m+\alpha-1)C_{m-1}^\alpha(t) - (m+2\alpha-2)C_{m-2}^\alpha(t)\Big] \tag{1.3}$$

while some early values are presented as

$$\begin{cases} C_0^\alpha(t) = 1, \\ C_1^\alpha(t) = 2\alpha t, \\ C_2^\alpha(t) = 2\alpha(1+\alpha)t^2 - \alpha, \\ C_3^\alpha(t) = -2\alpha(1+\alpha)t + \frac{4}{3}\alpha(1+\alpha)(2+\alpha)t^3. \end{cases} \tag{1.4}$$

From (1.4), we note that

(1) if $\alpha = 1$, then we will obtain the well-known Chebyshev polynomials and
(2) if $\alpha = \frac{1}{2}$, then we will obtain the well-known Legendre polynomials.

For more details on Gegenbauer polynomials see [2, 22, 26].

In this study, let $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{C}$ is the complex numbers' field, and let

$$\mathfrak{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

be the unit disk. Let $\mathfrak{A}$ represent the class of complex-valued functions that are regular (holomorphic or analytic) in the unit disk and let $\mathfrak{S}$ represent the class of regular and univalent functions in $\mathfrak{U}$; so that functions in $\mathfrak{S}$ are normalized by the conditions $F(0) = F'(0) - 1 = 0$ and are of the Taylor's series representation

$$F(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_m z^m + \cdots, \quad z \in \mathfrak{U}. \tag{1.5}$$

In [19], Lewin made history by introducing the class of bi-univalent functions of the form (1.5) and demonstrated that the upper bound of coefficient a_2 for every bi-univalent function is less than 1.51. Later, authors in [8,21,34] demonstrated that $|a_2| \leqslant \sqrt{2}$, $|a_2| \leqslant 1\frac{1}{3}$ and $|a_2| \leqslant 1.485$, respectively. Presently, we note that the bounds $|a_m|$ $(m = 3, 4, \ldots)$ for the whole class of *bi-univalent functions* are apparently yet unsolved. Duren [11] established that every bi-univalent function always has inverse function F^{-1} defined by

$$F^{-1}(F(z)) = z, \quad z \in \mathfrak{U}, \quad F(F^{-1}(\omega)) = \omega,\ \omega : |\omega| < r_0(F) \quad \text{and} \quad r_0(F) \geqslant 0.25;$$

where some calculations show that

$$F^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots = \mathfrak{F}(\omega). \tag{1.6}$$

A function $F \in \mathfrak{S}$ is said to be *bi-univalent* (or *bi-schlicht*) in $\mathfrak{U}$ if both F and its inverse function $\mathfrak{F}$ are univalent in $\mathfrak{U}$. We represent by $\mathfrak{B}$ the class of regular and bi-univalent functions in $\mathfrak{U}$. The class $\mathfrak{B}$ is non-void since we have some instances of functions

$$F(z) = z, \quad F(z) = z(1-z)^{-1}, \quad F(z) = \log(1-z)^{-1},$$

and others in it. We refer readers to the works in [7, 8, 16–18, 28–31] for more details on history, properties and definitions of some existing subclasses of $\mathfrak{B}$.

Let

$$X(z) = x_1 z + x_2 z^2 + x_3 z^3 + \cdots \in \Omega$$

be a regular function where Ω is the class of Schwarz functions such that for $z \in \mathfrak{U}$, $X(0) = 0$ and $|X(z)| < 1$. Suppose $F_1, F_2 \in \mathfrak{A}$, then $F_1 \prec F_2$ *if and only if* $F_1(z) = F_2(X(z))$ for $z \in \mathfrak{U}$. Should F_2 be univalent in $\mathfrak{U}$, then $F_1(z) \prec F_2(z)$ *if and only if* $F_1(0) = F_2(0)$ and $F_1(\mathfrak{U}) \subset F_2(\mathfrak{U})$. Note that the notation '$\prec$' means subordination.

In [27] (see also [4,6]), the Sălăgean differential operator $\mathfrak{D}^n$ $(n \in \mathbb{N}_0)$ for F in (1.5) is defined by

$$\begin{cases} \mathfrak{D}^0 F(z) = & F(z) & = z + a_2 z^2 + a_3 z^3 + \cdots + a_m z^m + \cdots \\ \mathfrak{D}^1 F(z) = & zF'(z) & = z + 2a_2 z^2 + 3a_3 z^3 + \cdots + m a_m z^m + \cdots \\ \mathfrak{D}^2 F(z) = & z(\mathfrak{D}^1 F(z))' & = z + 2^2 a_2 z^2 + 3^2 a_3 z^3 + \cdots + m^2 a_m z^m + \cdots \\ \vdots & \vdots & \vdots \\ \mathfrak{D}^{n+1} F(z) = & z(\mathfrak{D}^n F(z))' & = z + 2^{n+1} a_2 z^2 + 3^{n+1} a_3 z^3 + \cdots + m^{n+1} a_m z^m + \cdots . \end{cases} \tag{1.7}$$

2. Relevant Lemmas

Let $\mathfrak{P}$ be the class of regular functions whose real parts are positive in $\mathfrak{U}$ so that $\xi(z) \in \mathfrak{P}$ has series representation

$$\xi(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots + b_m z^m + \cdots, \quad z \in \mathfrak{U}$$

normalized such that $\xi(0) = 1$ and $\mathfrak{Re}\,\xi(z) > 0$. To establish our results, we shall need the following lemmas.

Lemma 2.1 ([11]). *If* $\xi(z) \in \mathfrak{P}$, *then* $|b_m| \leqslant 2$, $\forall m \in \mathbb{N}$.

Lemma 2.2 ([20]). *If* $\xi(z) \in \mathfrak{P}$, *then* $2b_2 = b_1^2 + (4 - b_1^2)p$, *where* $|p| \leqslant 1$.

Proposition 2.3. *The implication of Lemma 2.2 is that for*

$$\left.\begin{array}{l} \xi(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \\ \zeta(\omega) = 1 + c_1 \omega + c_2 \omega^2 + c_3 \omega^3 + \cdots \end{array}\right\} \in \mathfrak{P}, \tag{2.1}$$

$$\left.\begin{array}{l} 2b_2 = b_1^2 + p(4 - b_1^2) \\ 2c_2 = c_1^2 + q(4 - c_1^2) \end{array}\right\} \implies 2(b_2 - c_2) = (4 - b_1^2)(p - q)$$

for some p, q *such that* $|p|, |q| \leqslant 1$ *and* $|b_1|, |c_1| \in [0, 2]$.

3. The Main Results

Definition 3.1. Henceforth, let

$$\begin{cases} \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \ 0 \leqslant \mu \leqslant 1, \ n \in \mathbb{N}_0, \ t \in \left(\frac{1}{2}, 1\right], \ \alpha \in \mathbb{R} - \{0\}, \\ \mathfrak{F}(\omega) = F^{-1}(\omega) \text{ in (1.6)}, \ \mathfrak{D}^{n+1} F(z) \text{ in (1.7) and } \mathfrak{G}_\alpha(t, z) \text{ in (1.1)}, \end{cases} \tag{3.1}$$

then a function $F \in \mathfrak{B}$ is said to be a member of class $\mathfrak{B}^n(\beta, \mu; \mathfrak{G})$ if it satisfies the geometric conditions:

$$(1 - e^{-2\beta i} \mu^2 z^2) \frac{\mathfrak{D}^{n+1} F(z)}{z} \prec \mathfrak{G}_\alpha(t, z) \tag{3.2}$$

and

$$(1 - e^{-2\beta i} \mu^2 \omega^2) \frac{\mathfrak{D}^{n+1} \mathfrak{F}(\omega)}{\omega} \prec \mathfrak{G}_\alpha(t, \omega). \tag{3.3}$$

Remark 3.2. If we set $\alpha = 1$ and represent the Chebyshev polynomials of the second kind by $\mathfrak{C}_n(t)$, then class $\mathfrak{B}^n(\beta, \mu; \mathfrak{G})$ will reduce to class $\mathfrak{B}^n(\beta, \mu; \mathfrak{C})$ studied by Ayinla and Opoola [7, Definition 3.1].

The following are the main results.

Theorem 3.3. *If $F \in \mathfrak{B}^n(\beta, \mu; \mathfrak{G})$, then*

$$|a_2| \leqslant \sqrt{\frac{2\alpha^2 t^3 + \mu^2|\alpha| t^2}{|3^{n+1}\alpha t^2 + 2^{2n}(2t - 2t^2 - 2\alpha t^2 + 1)|}}$$

$$|a_3| \leqslant \frac{\alpha^2 t^2}{2^{2n}} + \frac{2|\alpha| t}{3^{n+1}}$$

$$|a_4| \leqslant \frac{5\alpha^2 t^2}{2^n \cdot 3^{n+1}} + \frac{|\alpha| t}{2^{2n+1}} + \frac{2|\alpha| t + 2|\alpha| t^2 + 2\alpha^2 t^2 + |\alpha|}{2^{2n+1}} + \frac{4|\alpha| t^2 + 4\alpha^2 t^2 + 2|\alpha| + 2\alpha^2 t + \frac{8}{3}|\alpha| t^3 + 4\alpha^2 t^3 + \frac{4}{3}|\alpha|^3 t^3}{2^{2n+2}} + \frac{2\mu^2|\alpha| t}{2^{2n+2}}$$

where the declarations in (3.1) *hold.*

Proof. Let $F \in \mathfrak{B}^n(\beta, \mu; \mathfrak{G})$, then application of the subordination technique implies that (3.2) and (3.3) will transform to

$$(1 - e^{-2\beta i}\mu^2 z^2)\frac{\mathfrak{D}^{n+1}F(z)}{z} = \mathfrak{G}_\alpha(t, X(z)) \tag{3.4}$$

and

$$(1 - e^{-2\beta i}\mu^2 \omega^2)\frac{\mathfrak{D}^{n+1}\mathfrak{F}(\omega)}{\omega} = \mathfrak{G}_\alpha(t, Y(\omega)) \tag{3.5}$$

where $\omega, z \in \mathfrak{U}$,

$$\left.\begin{array}{l} X(z) = x_1 z + x_2 z^2 + x_3 z^3 + \cdots \\ Y(\omega) = y_1 \omega + y_2 \omega^2 + y_3 \omega^3 + \cdots \end{array}\right\} \in \Omega,$$

$X(0) = Y(0) = 0$, and $|X(z)|, |Y(\omega)| < 1$. It is well-known that for ξ and ζ in (2.1),

$$X(z) = \left(\frac{\xi(z) - 1}{\xi(z) + 1}\right) = \frac{1}{2}\left[b_1 z + \left(b_2 - \frac{b_1^2}{2}\right) z^2 + \left(\frac{b_1^3}{2^2} - b_1 b_2 + b_3\right) z^3 + \cdots\right] \tag{3.6}$$

and

$$Y(\omega) = \left(\frac{\zeta(\omega) - 1}{\zeta(\omega) + 1}\right) = \frac{1}{2}\left[c_1 \omega + \left(c_2 - \frac{c_1^2}{2}\right) \omega^2 + \left(\frac{c_1^3}{2^2} - c_1 c_2 + c_3\right) \omega^3 + \cdots\right] \tag{3.7}$$

so that by some calculations we have

$$\mathfrak{G}_\alpha(t, X(z)) = 1 + \frac{1}{2}C_1^\alpha(t) b_1 z + \left[\frac{1}{2}C_1^\alpha(t)\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}C_2^\alpha(t) b_1^2\right] z^2 + \left[\frac{1}{2}C_1^\alpha(t)\left(b_3 - b_1 b_2 + \frac{b_1^3}{4}\right) + \frac{1}{2}C_2^\alpha(t) b_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{8}C_3^\alpha(t) b_1^3\right] z^3 + \cdots \tag{3.8}$$

and

$$\mathfrak{G}_\alpha(t, Y(\omega)) = 1 + \frac{1}{2}C_1^\alpha(t) c_1 \omega + \left[\frac{1}{2}C_1^\alpha(t)\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}C_2^\alpha(t) c_1^2\right] \omega^2 + \left[\frac{1}{2}C_1^\alpha(t)\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \frac{1}{2}C_2^\alpha(t) c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{8}C_3^\alpha(t) c_1^3\right] \omega^3 + \cdots . \tag{3.9}$$

Using (1.7) with some calculations, then the LHS of (3.4) simplifies to

$$(1 - e^{-2\beta i}\mu^2 z^2)\frac{\mathfrak{D}^{n+1}F(z)}{z} = 1 + 2^{n+1}a_2 z + (3^{n+1}a_3 - e^{-2\beta i}\mu^2) z^2 + (4^{n+1}a_4 - 2^{n+1}e^{-2\beta i}\mu^2 a_2) z^3 + \cdots \tag{3.10}$$

and using (1.6) and (1.7) in (3.5) shows that LHS of (3.5) gives

$$(3.11)\quad (1-e^{-2\beta i}\mu^2\omega^2)\frac{\mathfrak{D}^{n+1}\mathfrak{F}(\omega)}{\omega}$$
$$=1-2^{n+1}a_2\omega+[3^{n+1}(2a_2^2-a_3)-e^{-2\beta i}\mu^2]\omega^2-[4^{n+1}(5a_2^3-5a_2a_3+a_4)-2^{n+1}e^{-2\beta i}\mu^2a_2]\omega^3+\cdots.$$

If we compare the coefficients in (3.8), (3.9), (3.10), and (3.11); then it can easily be seen that

$$(3.12)\qquad 2^{n+1}a_2=\frac{1}{2}C_1^{\alpha}(t)b_1,$$

$$(3.13)\qquad 3^{n+1}a_3-e^{-2\beta i}\mu^2=\frac{1}{2}C_1^{\alpha}(t)\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}C_2^{\alpha}(t)b_1^2,$$

$$(3.14)\qquad 4^{n+1}a_4-2^{n+1}e^{-2\beta i}\mu^2a_2=\frac{1}{2}C_1^{\alpha}(t)\left(b_3-b_1b_2+\frac{b_1^3}{4}\right)+\frac{1}{2}C_2^{\alpha}(t)b_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{8}C_3^{\alpha}(t)b_1^3,$$

$$(3.15)\qquad -2^{n+1}a_2=\frac{1}{2}C_1^{\alpha}(t)c_1,$$

$$(3.16)\qquad 3^{n+1}(2a_2^2-a_3)-e^{-2\beta i}\mu^2=\frac{1}{2}C_1^{\alpha}(t)\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}C_2^{\alpha}(t)c_1^2,$$

and

$$(3.17)\quad -4^{n+1}(5a_2^3-5a_2a_3+a_4)+2^{n+1}e^{-2\beta i}\mu^2a_2$$
$$=\frac{1}{2}C_1^{\alpha}(t)\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)+\frac{1}{2}C_2^{\alpha}(t)c_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{8}C_3^{\alpha}(t)c_1^3.$$

Addition of (3.12) and (3.15) shows that

$$(3.18)\qquad b_1=-c_1\quad\text{and}\quad b_1^2=c_1^2$$

and the addition of the squares of (3.12) and (3.15) shows that

$$(3.19)\qquad 2^{2n+5}a_2^2=(C_1^{\alpha}(t))^2(b_1^2+c_1^2).$$

Putting (3.18) into (3.19) shows that

$$(3.20)\qquad b_1^2=\frac{2^{2n+4}a_2^2}{(C_1^{\alpha}(t))^2}\implies a_2^2=\frac{b_1^2(C_1^{\alpha}(t))^2}{2^{2n+4}}.$$

On the other hand, the addition of (3.13) and (3.16); and the substitution for b_1^2 from (3.20) show that

$$a_2^2=\frac{(C_1^{\alpha}(t))^3(b_2+c_2)+4e^{-2\beta i}\mu^2(C_1^{\alpha}(t))^2}{4\cdot 3^{n+1}(C_1^{\alpha}(t))^2+2^{2n+4}(C_1^{\alpha}(t)-C_2^{\alpha}(t))}$$

so using (1.4) gives

$$a_2^2=\frac{\alpha^2t^3(b_2+c_2)+2\alpha t^2e^{-2\beta i}\mu^2}{2\cdot 3^{n+1}\alpha t^2+2^{2n+1}(2t-2t^2-2\alpha t^2+1)}$$

and the application of Lemma 2.1 in the inequality

$$|a_2|^2\leqslant\frac{\alpha^2t^3|b_2+c_2|+2\mu^2|\alpha|t^2|e^{-2\beta i}|}{2|3^{n+1}\alpha t^2+2^{2n}(2t-2t^2-2\alpha t^2+1)|}$$

yields the desired result.

If we subtract (3.16) from (3.13), then we have

$$(3.21)\qquad a_3=a_2^2+\frac{C_1^{\alpha}(t)(b_2-c_2)}{2^2\cdot 3^{n+1}}$$

so that by substituting for a_2^2 in (3.21) from (3.20) gives

$$a_3 = \frac{(C_1^\alpha(t))^2 b_1^2}{2^{2n+4}} + \frac{C_1^\alpha(t)(b_2 - c_2)}{2^2 \cdot 3^{n+1}} \tag{3.22}$$

and

$$|a_3| \leqslant \frac{(C_1^\alpha(t))^2 |b_1|^2}{2^{2n+4}} + \frac{|C_1^\alpha(t)||b_2 - c_2|}{2^2 \cdot 3^{n+1}} \tag{3.23}$$

so that putting (1.4) into (3.23) and using Lemma 2.1 yield the desired result.

To find the bound on a_4, we subtract (3.17) from (3.14) to get

$$a_4 = \frac{5 \cdot 2^n (C_1^\alpha(t))^2 (b_2 - c_2) b_1}{2^{2n+5} \cdot 3^{n+1}} + \frac{C_1^\alpha(t)(b_3 - c_3)}{2^{2n+4}} - \frac{[C_1^\alpha(t) - C_2^\alpha(t)](b_2 + c_2) b_1}{2^{2n+4}} + \frac{[C_1^\alpha(t) - 2C_2^\alpha(t) + C_3^\alpha(t)] b_1^3}{2^{2n+5}} + \frac{C_1^\alpha(t) e^{-2\beta i} \mu^2 b_1}{2^{2n+3}}$$

and

$$|a_4| \leqslant \frac{5 \cdot 2^n (C_1^\alpha(t))^2 |b_2 - c_2||b_1|}{2^{2n+5} \cdot 3^{n+1}} + \frac{|C_1^\alpha(t)||b_3 - c_3|}{2^{2n+4}} + \frac{|C_1^\alpha(t) - C_2^\alpha(t)||b_2 + c_2||b_1|}{2^{2n+4}} + \frac{|C_1^\alpha(t) - 2C_2^\alpha(t) + C_3^\alpha(t)||b_1|^3}{2^{2n+5}} + \frac{|C_1^\alpha(t)||e^{-2i\beta}|\mu^2 |b_1|}{2^{2n+3}} \tag{3.24}$$

so that putting (1.4) into (3.24) and using Lemma 2.1 yield the desired result. □

Remark 3.4. If we set $\alpha = 1$ in Theorem 3.3, then we will arrive at the results of Ayinla and Opoola [7, Theorem 3.2].

Theorem 3.5. *If $F \in \mathfrak{B}^n(\beta, \mu; \mathfrak{G})$, then for a real value ρ,*

$$|a_3 - \rho a_2^2| \leqslant \begin{cases} \frac{2|\alpha|t}{3^{n+1}} & when \quad 0 \leqslant |\Upsilon(\rho)| \leqslant \frac{1}{3^{n+1}} \\ 2|\alpha|t|\Upsilon(\rho)| & when \quad |\Upsilon(\rho)| \geqslant \frac{1}{3^{n+1}} \end{cases}$$

where

$$\Upsilon(\rho) = \frac{\alpha t(1-\rho)}{2^{2n+2}}.$$

Proof. Using (3.19) and (3.22) implies that

$$\begin{aligned} a_3 - \rho a_2^2 &= \frac{(C_1^\alpha(t))^2 (b_1^2 + c_1^2)}{2^{2n+5}} + \frac{C_1^\alpha(t)(b_2 - c_2)}{2^2 \cdot 3^{n+1}} - \rho \frac{(C_1^\alpha(t))^2 (b_1^2 + c_1^2)}{2^{2n+5}} \\ &= \frac{C_1^\alpha(t)(b_2 - c_2)}{2^2 \cdot 3^{n+1}} + \frac{(1-\rho)(C_1^\alpha(t))^2 (b_1^2 + c_1^2)}{2^{2n+5}} \\ &= \frac{C_1^\alpha(t)}{2^2} \left\{ \frac{(b_2 - c_2)}{3^{n+1}} + \frac{(1-\rho) C_1^\alpha(t)(b_1^2 + c_1^2)}{2^{2n+3}} \right\} \end{aligned} \tag{3.25}$$

so that

$$|a_3 - \rho a_2^2| \leqslant \frac{|C_1^\alpha(t)|}{2^2} \left\{ \frac{|b_2 - c_2|}{3^{n+1}} + |b_1^2 + c_1^2||\Upsilon(\rho)| \right\} \tag{3.26}$$

for

$$\Upsilon(\rho) = \frac{(1-\rho) C_1^\alpha(t)}{2^{2n+3}}.$$

So, using (1.4) and Lemma 2.1 yields the desired result. □

Theorem 3.6. *If $F \in \mathfrak{B}^n(\beta, \mu; \mathfrak{G})$, then for a complex value ϱ,*

$$|a_3 - \varrho a_2^2| \leqslant \begin{cases} \frac{2|\alpha|t}{3^{n+1}} & when \quad |1-\varrho| \in \left[0,\ \frac{2^{2n+1}}{3^{n+1}|\alpha|t}\right) \\ \frac{\alpha^2 t^2 |1-\varrho|}{2^{2n}} & when \quad |1-\varrho| \in \left[\frac{2^{2n+1}}{3^{n+1}|\alpha|t},\ 0\right). \end{cases}$$

Proof. Using (3.18) in (3.25) yields

$$a_3 - \varrho a_2^2 = \frac{C_1^\alpha(t)(b_2 - c_2)}{2^2 \cdot 3^{n+1}} + \frac{2b_1^2(1-\varrho)(C_1^\alpha(t))^2}{2^{2n+5}}$$

so that the application of Proposition 2.3 gives

$$a_3 - \varrho a_2^2 = \frac{C_1^\alpha(t)(4-b_1^2)(p-q)}{2^3 \cdot 3^{n+1}} + \frac{b_1^2(1-\varrho)(C_1^\alpha(t))^2}{2^{2n+4}}. \tag{3.27}$$

Now without any restriction, let $b = b_1$ and by Lemma 2.1 we know that $b \in [0, 2]$. Also, for simplicity reason, let $P = |p| < 1$ and $Q = |q| < 1$; and with some rearrangement, (3.27) simplifies to

$$|a_3 - \varrho a_2^2| \leqslant |1-\varrho| \frac{b^2 |C_1^\alpha(t)|^2}{2^{2n+4}} + \frac{|C_1^\alpha(t)|(4-b^2)}{2^3 \cdot 3^{n+1}}(P+Q) = \Psi(P, Q).$$

Since $P, Q \in [0, 1]$, then

$$\max\{\Psi(P,Q)\} = \Psi(1,1) = |1-\varrho| \frac{b^2 |C_1^\alpha(t)|^2}{2^{2n+4}} + \frac{|C_1^\alpha(t)|(4-b^2)}{2^2 \cdot 3^{n+1}}$$

and with some simplifications we have

$$\Psi(1,1) = \frac{|C_1^\alpha(t)|}{3^{n+1}} + \frac{|C_1^\alpha(t)|^2}{2^{2n+4}} \left\{ |1-\varrho| - \frac{2^{2n+2}}{3^{n+1}|C_1^\alpha(t)|} \right\} b^2 = \Phi(b). \tag{3.28}$$

Also, since $b \in [0, 2]$, then

$$\Phi'(b) = \frac{|C_1^\alpha(t)|^2}{2^{2n+3}} \left\{ |1-\varrho| - \frac{2^{2n+2}}{3^{n+1}|C_1^\alpha(t)|} \right\} b$$

shows that there is a critical point at $b = 0$, hence for $\Phi'(b) < 0$,

$$|1-\varrho| \in \left[0,\ \frac{2^{2n+2}}{3^{n+1}|C_1^\alpha(t)|}\right).$$

This means that $\Phi(b)$ is strictly a decreasing function of $|1-\varrho|$, hence from (3.28)

$$\max\{\Phi(b) : b \in [0, 2]\} = \Phi(0) = \frac{|C_1^\alpha(t)|}{3^{n+1}}. \tag{3.29}$$

Also for $\Phi(b) \geqslant 0$,

$$|1-\varrho| \in \left[\frac{2^{2n+2}}{3^{n+1}|C_1^\alpha(t)|},\ 0\right)$$

which implies that $\Phi(b)$ is an increasing function of $|1-\varrho|$, hence from (3.28),

$$\max\{\Phi(b) : b \in [0, 2]\} = \Phi(2) = \frac{|1-\varrho||C_1^\alpha(t)|^2}{2^{2n+2}}. \tag{3.30}$$

Thus, putting (3.29) and (3.30) together and using (1.4) give the desired result. □

4. Conclusion

In this investigation, we studied a certain class of regular functions in relation to the subordination principle and the well-known Gegenbauer polynomials. This class consists of regular and bi-univalent functions that are defined in the unit disk. Some achieved results include some early coefficient bounds and the upper estimates for the Fekete-Szegö inequalities with real and complex parameters. The obtained results extend that of Ayinla and Opoola [7] and others.

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