

MEASURE ON A LATTICE OF INFINITE TUPLES OF REAL NUMBERS

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ABSTRACT. We consider the analogue of measure on a ring of sets, by defining a lattice-measure on a lattice-ring of elements of a lattice of infinite tuples of real numbers. We obtain various results about the convergence of sequences of tuples in the lattice with respect to the lattice-measure and use these results to show that the limit of a convergent sequence in a lattice sigma-ring is also in the lattice sigma-ring under certain conditions.

1. INTRODUCTION

In classical Measure Theory, measures are defined on rings of sets. These measures have been extensively studied, but comparatively little study has been done on measures on lattices. A type of measure called a modular measure has been investigated. Guiseppina Barbieri [3] proved an extension theorem for modular measures on lattice ordered algebras and used this theorem to obtain analogues of the Nikodym theorem and other theorems. Avallone and Vitolo [1] obtained results about a modular measure on a pseudo-D-lattice, and Avallone, De Simone and Vitolo [2] proved a Caratheodory type extension theorem for σ -additive exhaustive modular measures on σ -continuous pseudo-D-lattices.

The concept of a modular measure is significantly different from the concept of a measure in classical Measure Theory. However, we define a measure called a lattice-measure or l-measure on a lattice of infinite tuples of real numbers in such a way that it has many similarities to a measure defined on a ring of subsets of a set. In order to do this, we utilize the concept of a lattice-ring, which was introduced by Sookoo [5]. A lattice-ring is the analogue of a ring of sets as defined in the study of measures in Functional Analysis, and a lattice sigma-ring is the analogue of a sigma-ring. We define and study an l-measure on a lattice ring, as well as an l-measure on a lattice sigma-ring. In so doing, we investigate relationships between the lattice structure and the l-measure and obtain properties of the l-measure.

Convergence in measure of functions has been investigated and continues to be of interest. Recent studies on this topic include work done by Unver and Sagioglu [6] and by Wilczynski [7]. We study convergence in measure for sequences of lattice elements, establishing that for certain such sequences, we can construct sequences that converge in measure to the upper bound of the original sequence, where each tuple of a sequence so constructed is made up of coordinate elements from possibly different tuples of the original sequence. Our approach in establishing most of these results is to consider the k -th coordinate of the infinite tuples, for each fixed, arbitrary natural number k . In doing so, we work with sequences of real

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Submitted on November 06, 2023.

2020 *Mathematics Subject Classification.* Primary 28A99; Secondary 06B99.

Key words and phrases. measure on a lattice, convergence in measure, limit.

numbers, and apply the methods of Real Analysis to establish convergence or whatever is required to prove the result in question.

These results are utilized in proving Theorem 5.2, the main theorem of the paper, which states that if a sequence of elements of a lattice-ring S converges in measure to some limit, then the limit also must be in S , provided that the sequence satisfies certain conditions. This is done by first establishing the result for sequences for which the values in each coordinate position are either monotone increasing or monotone decreasing.

The main aim of this paper is to investigate measures on lattices, starting with a lattice of tuples of real numbers, with the intention of stimulating the further development of the study of measures on lattices in general.

2. DEFINITIONS AND NOTATION

We will use the following notation:

\mathbb{R} = the set of real numbers,

\mathbb{N} = the set of natural numbers,

$\mathbb{U} = \{(a_1, a_2, \dots) \mid a_i \in \mathbb{R}, a_i \geq 0, \forall i \in \mathbb{N}\}$

$\bar{0}$ or $(0, 0, \dots)$ = the zero element of \mathbb{U} .

$\bar{a}_i = (a_{i1}, a_{i2}, \dots)$ for any element of \mathbb{U} denoted by \bar{a}_i

$(\bar{a} - \bar{b})_i$ = the i th component of $\bar{a} - \bar{b}$, for $\bar{a}, \bar{b} \in \mathbb{U}$

μ is the function defined on \mathbb{U} by

$$\mu(\bar{a}) = \sum_{i=1}^{\infty} a_i$$

for any element $\bar{a} \in \mathbb{U}$, where $\bar{a} = (a_1, a_2, \dots)$.

Definition 2.1 to Definition 2.4 are similar to definitions in [5].

Definition 2.1. A partial ordering on \mathbb{U} is given by

$$\bar{a} \leq \bar{b} \iff a_i \leq b_i, \forall i \in \mathbb{N}$$

for any elements $\bar{a}, \bar{b} \in \mathbb{U}$, where $\bar{a} = (a_1, a_2, \dots)$ and $\bar{b} = (b_1, b_2, \dots)$.

Definition 2.2. Subtraction is defined on \mathbb{U} as follows:

Given elements $\bar{a}, \bar{b} \in \mathbb{U}$, where $\bar{a} = (a_1, a_2, \dots)$ and $\bar{b} = (b_1, b_2, \dots)$

$$\bar{a} - \bar{b} = \bar{c}$$

where $\forall i \in \mathbb{N}$

$$c_i = \begin{cases} a_i - b_i, & \text{if } a_i \geq b_i \\ 0, & \text{if } a_i < b_i \end{cases}$$

and

$$\bar{c} = (c_1, c_2, \dots)$$

Definition 2.3. An l-ring or lattice ring of lattice elements is a non-empty subset R of $\mathbb{U} \ni$ for any two elements \bar{a} and \bar{b} of R , $\bar{a} \vee \bar{b} \in R$ and $\bar{a} - \bar{b} \in R$.

Definition 2.4. An l σ -ring is a non-empty subset S of $\mathbb{U} \ni$

(1) If $\bar{a}, \bar{b} \in S$, then $\bar{a} - \bar{b} \in S$

(2) If $\bar{a}_i \in S, \forall i \in \mathbb{N}$, then $\bigvee_{i=1}^{\infty} \bar{a}_i \in S$

The definitions below are analogues of definitions from classical Measure Theory [c.f.[4]].

Definition 2.5. A function η defined on a subset E of \mathbb{U} is called additive if for any two elements \bar{a} and \bar{b} in $E \ni \bar{a} \vee \bar{b}$ is also in E and $\bar{a} \wedge \bar{b} = \bar{0}$,

$$\eta(\bar{a} \vee \bar{b}) = \eta(\bar{a}) + \eta(\bar{b})$$

Definition 2.6. A function η defined on a subset E of \mathbb{U} is called finitely additive if

$$\eta(\bar{a}_1 \vee \bar{a}_2 \vee \dots \vee \bar{a}_n) = \sum_{i=1}^n \eta(\bar{a}_i)$$

for any finite set $\{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n\}$ of distinct elements of \mathbb{U} satisfying the condition that for any two distinct elements $i, j \in \{1, 2, \dots, n\}$, $\bar{a}_i \wedge \bar{a}_j = \bar{0}$.

Definition 2.7. A function η defined on a subset E of \mathbb{U} is called countably additive if

$$\eta\left(\bigvee_{i=1}^{\infty} \bar{a}_i\right) = \sum_{i=1}^{\infty} \eta(\bar{a}_i)$$

for any countably set $\{\bar{a}_1, \bar{a}_2, \dots\}$ of distinct elements of \mathbb{U} satisfying the condition that for any two distinct elements $i, j \in \mathbb{N}$, $\bar{a}_i \wedge \bar{a}_j = \bar{0}$.

Definition 2.8. A lattice-measure or l-measure is an extended, real-valued, non-negative and countably additive function η defined on an l-ring $R \ni \eta(\bar{0}) = 0$, where $\bar{0}$ is the zero element of the lattice.

Definition 2.9. If η is an l-measure on the l-ring R , then an element \bar{e} of the l-ring R is said to have finite measure if $\eta(\bar{e}) < \infty$.

Definition 2.10. A hereditary set E of elements of U is a subset of U that has the property that if $\bar{f} \in E$ and $\bar{e} \leq \bar{f}$, then \bar{e} is also an element of E .

Definition 2.11. If E is any set of lattice elements, then the hereditary l σ -ring generated by E is the smallest hereditary l σ -ring containing E . It is denoted by $lH(E)$.

Definition 2.12. An extended, real-valued function μ^* defined on a set E of lattice elements is subadditive if

$$\mu^*(\bar{a} \vee \bar{b}) \leq \mu^*(\bar{a}) + \mu^*(\bar{b}), \forall \bar{a}, \bar{b} \in E \ni \bar{a} \vee \bar{b} \in E.$$

Definition 2.13. An extended, real-valued function μ^* defined on a set E of lattice elements is finitely subadditive if for any finite subset $\{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n\}$ of $E \ni \bigvee_{i=1}^n \bar{a}_i \in E$,

$$\mu^*\left(\bigvee_{i=1}^n \bar{a}_i\right) \leq \sum_{i=1}^n \mu^*(\bar{a}_i)$$

Definition 2.14. An extended, real-valued function μ^* defined on a set E of lattice elements is countably subadditive if for any countable subset $\{\bar{a}_i\}$ of $E \ni \bigvee_{i=1}^{\infty} \bar{a}_i \in E$,

$$\mu^*\left(\bigvee_{i=1}^{\infty} \bar{a}_i\right) \leq \sum_{i=1}^{\infty} \mu^*(\bar{a}_i)$$

Definition 2.15. A function η on a set $E \subseteq \mathbb{U}$ is monotone increasing if

$$\eta(\bar{e}) \leq \eta(\bar{f}), \forall \bar{e}, \bar{f} \in E \ni \bar{e} \leq \bar{f}$$

Definition 2.16. A function η on a set $E \subseteq \mathbb{U}$ is monotone decreasing if

$$\eta(\bar{e}) \geq \eta(\bar{f}), \forall \bar{e}, \bar{f} \in E \ni \bar{e} \leq \bar{f}$$

Definition 2.17. A function η on a set $E \subseteq \mathbb{U}$ is subtractive if

$$\eta(\bar{f} - \bar{e}) = \eta(\bar{f}) - \eta(\bar{e})$$

$\forall \bar{e}, \bar{f} \in E \ni \bar{e} \leq \bar{f}$ and $\bar{f} - \bar{e} \in E$.

Definition 2.18. A sequence $\{\bar{e}_j\}; j = 1, 2, \dots$ in \mathbb{U} is said to converge in measure to an element \bar{v} of \mathbb{U} if, given $\epsilon > 0, \exists$ a natural number $M \ni$

$$\max\{\mu(\bar{e}_j - \bar{v}), \mu(\bar{v} - \bar{e}_j)\} < \epsilon, \forall j > M$$

3. PROPERTIES OF L-MEASURES

The following theorem is obvious.

Theorem 3.1. Let R be an l -ring of elements of \mathbb{U} . Then μ is an l -measure on R .

The following theorem is the analogue of Theorem A, page 37, [4].

Theorem 3.2. μ is monotone and subtractive.

Proof. Clearly, μ is monotone. We now show that μ is also subtractive. Let $\bar{e}, \bar{f} \in \mathbb{U} \ni \bar{e} \leq \bar{f}$ where $\bar{e} = (e_1, e_2, \dots), \bar{f} = (f_1, f_2, \dots)$. Then

$$(3.1) \quad \mu(\bar{f} - \bar{e}) = \mu(f_1 - e_1, f_2 - e_2, \dots)$$

$$(3.2) \quad = \sum_{i=1}^{\infty} (f_i - e_i)$$

$$(3.3) \quad = \sum_{i=1}^{\infty} f_i - \sum_{i=1}^{\infty} e_i$$

$$(3.4) \quad = \mu(\bar{f}) - \mu(\bar{e})$$

□

The following theorem is the analogue of Theorem B, page 37, [4].

Theorem 3.3. Let S be an $l\sigma$ -ring of elements of \mathbb{U} and let $\bar{e}_0 \in S$. If $\{\bar{e}_n\}$ is a sequence of elements of $S \ni \bar{e}_0 \leq \bigvee_{i=1}^{\infty} \bar{e}_i$, where each component of $\bigvee_{i=1}^{\infty} \bar{e}_i$ is finite, then $\mu(\bar{e}_0) \leq \sum_{i=1}^{\infty} \mu(\bar{e}_i)$.

Proof. Let $\bigvee_{i=1}^{\infty} \bar{e}_i = (\hat{e}_1, \hat{e}_2, \dots)$.

Then

$$(3.5) \quad \mu(\bar{e}_0) = \sum_{i=1}^{\infty} e_{0i}$$

$$(3.6) \quad \leq \sum_{i=1}^{\infty} \sup\{e_{1i}, e_{2i}, \dots\}$$

$$(3.7) \quad = \sum_{i=1}^{\infty} \hat{e}_i$$

$$(3.8) \quad \leq \sum_{i=1}^{\infty} \left(e_{ki} + \frac{\epsilon}{2^i} \right), \text{ for some } k \in \mathbb{N}$$

$$(3.9) \quad \leq \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} e_{ji} + \frac{\epsilon}{2^i} \right)$$

$$(3.10) \quad = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} e_{ji} + \sum_{i=1}^{\infty} \frac{\epsilon}{2^i}$$

$$(3.11) \quad = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} e_{ji} + \epsilon \sum_{i=1}^{\infty} \frac{1}{2^i}$$

$$(3.12) \quad = \sum_{j=1}^{\infty} \mu(\bar{e}_j) + \epsilon$$

Since ϵ is arbitrary

$$\mu(\bar{e}_0) = \sum_{i=1}^{\infty} \mu(\bar{e}_i)$$

□

Theorem 3.4. Let $\bar{a}, \bar{b} \in \mathbb{U}$. Then

$$\mu(\bar{a} \vee \bar{b}) = \mu(\bar{a} - \bar{b}) + \mu(\bar{a} \wedge \bar{b}) + \mu(\bar{b} - \bar{a})$$

Proof. Let $i \in \mathbb{N}$. We consider the i th components of \bar{a} and \bar{b} .

Case 1 $a_i \geq b_i$

$$(3.13) \quad (\bar{a} \vee \bar{b})_i = a_i$$

$$(3.14) \quad (\bar{a} - \bar{b})_i = a_i - b_i$$

$$(3.15) \quad (\bar{a} \wedge \bar{b})_i = b_i$$

$$(3.16) \quad (\bar{b} - \bar{a})_i = 0$$

$$(3.17) \quad \text{Therefore, } (\bar{a} \vee \bar{b})_i = (\bar{a} - \bar{b})_i + (\bar{a} \wedge \bar{b})_i + (\bar{b} - \bar{a})_i$$

Case 2 $a_i < b_i$

$$(3.18) \quad (\bar{a} \vee \bar{b})_i = b_i$$

$$(3.19) \quad (\bar{a} - \bar{b})_i = 0$$

$$(3.20) \quad (\bar{a} \wedge \bar{b})_i = a_i$$

$$(3.21) \quad (\bar{b} - \bar{a})_i = b_i - a_i$$

$$(3.22) \quad \text{Therefore, } (\bar{a} \vee \bar{b})_i = (\bar{a} - \bar{b})_i + (\bar{a} \wedge \bar{b})_i + (\bar{b} - \bar{a})_i$$

From the above we can see that

$$\mu(\bar{a} \vee \bar{b}) = \mu(\bar{a} - \bar{b}) + \mu(\bar{a} \wedge \bar{b}) + \mu(\bar{b} - \bar{a})$$

□

Theorem 3.5. Let $\bar{a}, \bar{b} \in \mathbb{N}$. Then

$$\mu[(\bar{a} - \bar{b}) \vee (\bar{b} - \bar{a})] = \mu(\bar{a} - \bar{b}) + \mu(\bar{b} - \bar{a})$$

Proof. We will show that $\forall i \in \mathbb{N}$

$$[(\bar{a} - \bar{b}) \vee (\bar{b} - \bar{a})]_i = (\bar{a} - \bar{b})_i + (\bar{b} - \bar{a})_i$$

We consider all possible conditions on the relative sizes of a_i and b_i .

Case 1 $a_i \geq b_i$

$$(3.23) \quad [(\bar{a} - \bar{b}) \vee (\bar{b} - \bar{a})]_i = \bar{a}_i - \bar{b}_i$$

$$(3.24) \quad (\bar{a} - \bar{b})_i + (\bar{b} - \bar{a})_i = a_i - b_i$$

$$(3.25)$$

Case 2 $a_i < b_i$

$$(3.26) \quad [(\bar{a} - \bar{b}) \vee (\bar{b} - \bar{a})]_i = b_i - a_i$$

$$(3.27) \quad (\bar{a} - \bar{b})_i + (\bar{b} - \bar{a})_i = b_i - a_i$$

Hence $\forall i \in \mathbb{N}$

$$(3.28) \quad [(\bar{a} - \bar{b}) \vee (\bar{b} - \bar{a})]_i = (\bar{a} - \bar{b})_i + (\bar{b} - \bar{a})_i$$

$$(3.29) \quad \mu[(\bar{a} - \bar{b}) \vee (\bar{b} - \bar{a})] = \sum_{i=1}^{\infty} [(\bar{a} - \bar{b}) \vee (\bar{b} - \bar{a})]_i$$

$$(3.30) \quad = \sum_{i=1}^{\infty} [(a - b)_i + (b - a)_i]$$

$$(3.31) \quad = \mu(\bar{a} - \bar{b}) + \mu(\bar{b} - \bar{a})$$

□

The next theorem was suggested by part of Theorem A, page 42, [4].

Theorem 3.6. *let R be an l -ring of elements of the lattice (\mathbb{U}, \leq) and let μ_e be the function on $lH(R)$ defined by*

$$\mu_e(\bar{a}) = \inf \{ \mu(\bar{b}) \mid \bar{b} \in R, \bar{a} \leq \bar{b} \}, \forall \bar{a} \in lH(R)$$

Then μ_e is countably subadditive.

Proof. Let $\{\bar{a}_i\}$ be a sequence in $lH(R)$. Given $\epsilon > 0$, for each $i \in \mathbb{N}$, $\exists \bar{b}_i \in R \ni \bar{a}_i \leq \bar{b}_i$ and

$$\mu(\bar{b}_i) \leq \mu_e(\bar{a}_i) + \frac{\epsilon}{2^i}$$

Now

$$(3.32) \quad \mu_e(\vee_{i=1}^{\infty} \bar{a}_i) = \inf \{ \mu(\bar{b}) \mid \bar{b} \in R, \vee_{i=1}^{\infty} \bar{a}_i \leq \bar{b} \}$$

$$(3.33) \quad \leq \mu(\vee_{i=1}^{\infty} \bar{b}_i), \text{ since } \vee_{i=1}^{\infty} \bar{a}_i \leq \vee_{i=1}^{\infty} \bar{b}_i$$

$$(3.34) \quad \leq \sum_{i=1}^{\infty} \mu(\bar{b}_i)$$

because if the first coordinate of $\vee_{i=1}^{\infty} \bar{b}_i$ tends to the limit L as $n \rightarrow \infty$, then the contribution to $\mu(\vee_{i=1}^{\infty} \bar{b}_i)$ from the first coordinate must be less than or equal to the contribution to $\sum_{i=1}^{\infty} \mu(\bar{b}_i)$ from the first coordinates of $\bar{b}_1, \bar{b}_2, \dots$ and if the first coordinate of $\vee_{i=1}^{\infty} \bar{b}_i$ is equal to the first coordinate of \bar{b}_h , for some $h \in \mathbb{N}$, then clearly in this case also the contribution to $\mu(\vee_{i=1}^{\infty} \bar{b}_i)$ from the first coordinate must be less than or equal to the contribution to $\sum_{i=1}^{\infty} \mu(\bar{b}_i)$ from all the first coordinates of $\bar{b}_1, \bar{b}_2, \dots$ and the same is true of the other coordinates. Also

$$\sum_{i=1}^{\infty} \mu(\bar{b}_i) \leq \sum_{i=1}^{\infty} \left[\mu_e(\bar{a}_i) + \frac{\epsilon}{2^i} \right] \leq \sum_{i=1}^{\infty} \mu_e(\bar{a}_i) + \epsilon$$

We have shown that

$$\mu_e(\bigvee_{i=1}^{\infty} \bar{a}_i) \leq \sum_{i=1}^{\infty} \mu_e(\bar{a}_i) + \epsilon$$

Since ϵ is arbitrary,

$$\mu_e(\bigvee_{i=1}^{\infty} \bar{a}_i) \leq \sum_{i=1}^{\infty} \mu_e(\bar{a}_i)$$

□

4. CONVERGENCE IN MEASURE

Theorem 4.1. Let $\{\bar{e}_k\}$ be a sequence of elements of $\mathbb{U} \ni e_{hi} = 0$ if $i > m, \forall h \in \mathbb{N}$, where m is a fixed, arbitrary natural number. If $(\hat{e}_1, \hat{e}_2, \dots) = \bar{e}_1 \vee \bar{e}_2 \vee \dots$ and \hat{e}_i is finite for each $i \in \mathbb{N}$, then \exists a sequence $\{(e_{k_{1j}1}, e_{k_{2j}2}, \dots, e_{k_{mj}m}, 0, 0, \dots)\}; j = 1, 2, \dots$ which converges to $(\hat{e}_1, \hat{e}_2, \dots)$ in measure.

Proof. Given $\epsilon > 0$, for each $i \leq m, \exists N_i \in \mathbb{N}$ and $k_{ij} \in \mathbb{N} \ni$

$$|\hat{e}_i - e_{k_{ij}i}| < \frac{\epsilon}{2^i}, \forall j > N_i$$

Let $N = \max\{N_1, N_2, \dots, N_m\}$. Then

$$\begin{aligned} & \max\{\mu[(e_{k_{1j}1}, e_{k_{2j}2}, \dots, e_{k_{mj}m}, 0, 0, \dots) - (\hat{e}_1, \hat{e}_2, \dots)], \\ & \mu[(\hat{e}_1, \hat{e}_2, \dots) - (e_{k_{1j}1}, e_{k_{2j}2}, \dots, e_{k_{mj}m}, 0, 0, \dots)]\} \\ (4.1) \quad & = |\hat{e}_1 - e_{k_{1j}1}| + |\hat{e}_2 - e_{k_{2j}2}| + \dots + |\hat{e}_m - e_{k_{mj}m}| \\ & < \frac{\epsilon}{2} + \frac{\epsilon}{2^2} + \dots + \frac{\epsilon}{2^m} < \epsilon, \forall j > N \end{aligned}$$

Hence $\{(e_{k_{1j}1}, e_{k_{2j}2}, \dots, e_{k_{mj}m}, 0, 0, \dots)\}; j = 1, 2, \dots$ converges to $(\hat{e}_1, \hat{e}_2, \dots)$ in measure. □

Theorem 4.2. Let $\{\bar{e}_h\}$ be a sequence of elements of \mathbb{U} . If $\hat{e} = (\hat{e}_1, \hat{e}_2, \dots) = \bar{e}_1 \vee \bar{e}_2 \vee \dots$, and $\mu(\hat{e})$ is finite, then \exists a sequence $\{\bar{g}_j\} = \{(e_{h_{1j}1}, e_{h_{2j}2}, \dots)\}; j = 1, 2, \dots$ which converges to $(\hat{e}_1, \hat{e}_2, \dots)$ in measure.

Proof. Choose a monotone-decreasing sequence $\{\epsilon_i\}$ which converges to zero. Given $\epsilon > 0, \exists \hat{N} > 0 \ni \sum_{i=\hat{N}+1}^{\infty} \hat{e}_i < \frac{\epsilon_1}{2}$. From the previous theorem, \exists a sequence $\{(e_{k_{1j}1}, e_{k_{2j}2}, \dots, e_{k_{\hat{N}j}\hat{N}}, 0, 0, \dots)\}; j = 1, 2, \dots$ which converges to $(\hat{e}_1, \hat{e}_2, \dots, \hat{e}_{\hat{N}}, 0, 0, \dots)$ in measure. Hence $\exists N_1 > 0 \ni$

$$\mu[(\hat{e}_1, \hat{e}_2, \dots, \hat{e}_{\hat{N}}, 0, 0, \dots) - (e_{k_{1j}1}, e_{k_{2j}2}, \dots, e_{k_{\hat{N}j}\hat{N}}, 0, 0, \dots)] < \frac{\epsilon_1}{2}, \forall j > N_1.$$

Choose $J > N_1$ and consider $(e_{k_{1J}1}, e_{k_{2J}2}, \dots, e_{k_{\hat{N}J}\hat{N}}, e_{1(\hat{N}+1)}, e_{1(\hat{N}+2)}, \dots)$, which we will call $(e_{h_{11}1}, e_{h_{21}2}, \dots)$. Then

$$\begin{aligned} & \mu[(\hat{e}_1, \hat{e}_2, \dots) - (e_{h_{11}1}, e_{h_{21}2}, \dots)] \\ & = \mu[(\hat{e}_1, \hat{e}_2, \dots, \hat{e}_{\hat{N}}, 0, 0, \dots) - (e_{k_{1J}1}, e_{k_{2J}2}, \dots, e_{k_{\hat{N}J}\hat{N}}, 0, 0, \dots)] \\ (4.2) \quad & + \mu[(0, 0, \dots, \hat{e}_{\hat{N}+1}, \hat{e}_{\hat{N}+2}, \dots) - (0, 0, \dots, e_{1(\hat{N}+1)}, e_{1(\hat{N}+2)}, \dots)] \\ & < \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} \\ & = \epsilon_1 \end{aligned}$$

Similarly \exists a tuple $(e_{h_{12}1}, e_{h_{22}2}, \dots) \ni$

$$\mu[(\hat{e}_1, \hat{e}_2, \dots) - (e_{h_{12}1}, e_{h_{22}2}, \dots)] < \epsilon_2$$

and in general, for each $p \in \mathbb{N} \ni$ a tuple $(e_{h_{1p}1}, e_{h_{2p}2}, \dots) \ni$

$$\mu[(\hat{e}_1, \hat{e}_2, \dots) - (e_{h_{1p}1}, e_{h_{2p}2}, \dots)] < \epsilon_p$$

Clearly the sequence $\{\bar{g}_j\} = \{(e_{h_{1j}1}, e_{h_{2j}2}, \dots); j = 1, 2, \dots\}$ converges to $(\hat{e}_1, \hat{e}_2, \dots)$ in measure. \square

Theorem 4.3. Let $\{\bar{e}_h\}$ be a sequence of \mathbb{U} . If $\hat{e} = (\hat{e}_1, \hat{e}_2, \dots) = \bar{e}_1 \vee \bar{e}_2 \vee \dots$ and $\mu(\hat{e})$ is finite, then \exists a monotone-increasing sequence $\{\bar{q}_j\}$ which converges to \hat{e} in measure, where the i th component of \bar{q}_j is the i th component of one of the \bar{e}_h for $i, j \in \mathbb{N}$.

Proof. In the proof of the previous theorem, it was established that the sequence $\{\bar{g}_j\}$ converges to \hat{e} in measure. Let

$$\bar{q}_1 = \bar{g}_1, \bar{q}_2 = \bar{g}_1 \vee \bar{g}_2, \dots, \bar{q}_j = \bar{g}_1 \vee \bar{g}_2 \vee \dots \vee \bar{g}_j; j \in \mathbb{N}$$

Clearly $\{\bar{q}_j\}$ is a monotone-increasing sequence which converges to \hat{e} in measure. \square

5. CONVERGENCE IN THE $L\sigma$ -RING S

In this section we show that under certain conditions, if $\{a_n\}$ is a sequence in the σ -ring S and $\{a_n\}$ converges to L in \mathbb{U} , then $L \in S$.

The following theorem is the analogue of the result in lines 14 and 15 of [4].

Theorem 5.1. If S is an $l\sigma$ -ring of elements of the lattice (\mathbb{U}, \leq) , $\bar{a}_i \in S$, $\mu(\bar{a}_i) < \infty$ for each $i \in \mathbb{N}$, and $\mu(\bigvee_{i=1}^{\infty} \bar{a}_i) < \infty$, then $\bigwedge_{i=1}^{\infty} \bar{a}_i \in S$.

Proof. Let $\bar{u} = (u_1, u_2, \dots) = \bigvee_{i=1}^{\infty} \bar{a}_i$ and $\bar{l} = (l_1, l_2, \dots) = \bigwedge_{i=1}^{\infty} \bar{a}_i$. We establish that $l \in S$ by showing that

$$(5.1) \quad \bigvee_{i=1}^{\infty} (\bar{u} - \bar{a}_i) = \bar{u} - \bar{l} \in S$$

and hence that $\bar{l} = \bar{u} - (\bar{u} - \bar{l}) \in S$, since the k th coordinate of \bar{l} is less than or equal to the k th coordinate of \bar{u} , for each $k \in \mathbb{N}$ and both coordinates must be finite.

We prove (5.1) by considering the k th coordinate of the tuple $\{\bar{u} - \bar{a}_n\}$, where k is an arbitrary natural number. There are two cases to consider.

Case1 $l_k \in \{a_{nk} \mid n = 1, 2, \dots\}$ so that $l_k = a_{n_0k}$, say.

$$u_k - l_k = u_k - a_{n_0k} \geq u_k - a_{nk}, \forall n \in \mathbb{N}$$

Case2 $l_k \notin \{a_{nk} \mid n = 1, 2, \dots\}$

There exists a subsequence $\{a_{i_{k_n}k}\}; n = 1, 2, \dots$ of the sequence $\{a_{nk}\}; n = 1, 2, \dots$ converging to l_k from above. Also $a_{nk} \geq l_k$ for $n = 1, 2, \dots$ and for each $k \in \{1, 2, \dots\}$.

By considering all different relative sizes of u_k, a_{nk} and $l_k, k \in \mathbb{N}$, it is easy to see that

$$u_k - a_{nk} \leq u_k - l_k, \forall n \in \mathbb{N} \text{ and } \forall k \in \mathbb{N}$$

Given $\epsilon > 0, \exists N \in \mathbb{N} \ni 0 \leq a_{i_{k_n}k} - l_k < \epsilon, \forall n > N$.

Now

$$(u_k - l_k) - (u_k - a_{i_{k_n}k}) \leq a_{i_{k_n}k} - l_k, \forall n > N$$

Therefore

$$0 \leq (u_k - l_k) - (u_k - a_{i_{k_n}k}) < \epsilon, \forall n > N$$

Hence $\{u_k - a_{i_{k_n}k}\}; n = 1, 2, \dots$ is a subsequence of the sequence $\{u_k - a_{nk}\}; n = 1, 2, \dots$ which converges to $(u_k - l_k)$ from below.

From Case 1 and Case 2, we see that $\bigvee_{i=1}^{\infty} (\bar{u} - \bar{a}_i) = \bar{u} - \bar{l} \in S$.

We have therefore established (5.1). \square

Theorem 5.2. Let S be an σ -ring of elements of (\mathbb{U}, \leq) and let $\{\bar{a}_n\}$ be a sequence of elements of $S \ni \mu(\bar{a}_i) < \infty$ for each $i \in \mathbb{N}$, and $\mu(\bigvee_{i=1}^{\infty} \bar{a}_i) < \infty$. If $\{\bar{a}_n\}$ converges in measure to $L \in U$, then $L \in S$.

Proof. Assume that $\{\bar{a}_n\}$ converges in measure to $L \in U$.

We first establish that if for each $k \in \mathbb{N}$, either:

(1) $a_{nk} \geq L_k, \forall n \in \mathbb{N}$ and $a_{1k} \geq a_{2k} \geq \dots$ or

(2) $a_{nk} \leq L_k, \forall n \in \mathbb{N}$ and $a_{1k} \leq a_{2k} \leq \dots$

then $L \in S$, where $L = (L_1, L_2, \dots)$.

Consider the sequence $\{\bar{b}_n\}$, where

$$\bar{b}_1 = \bar{a}_1 \vee \bar{a}_2 \vee \bar{a}_3 \dots$$

$$\bar{b}_2 = \bar{a}_2 \vee \bar{a}_3 \vee \bar{a}_4 \dots$$

$$\bar{b}_3 = \bar{a}_3 \vee \bar{a}_4 \vee \bar{a}_5 \dots$$

\vdots

Also, let $B = \bar{b}_1 \wedge \bar{b}_2 \wedge \bar{b}_3 \dots \in S$, from the previous theorem.

It is clear that $L = B \in S$.

We next establish that for any sequence $\{a_n\}$ of elements of S that satisfies the given conditions, L is also in S .

There exists a subsequence $\{\bar{a}_{i_{n1}}\}; n = 1, 2, \dots$ of the sequence $\{\bar{a}_n\}; n = 1, 2, \dots \ni$ the subsequence $\{\bar{a}_{i_{n1}}\}; n = 1, 2, \dots$ is either monotone increasing or monotone decreasing. Also, there exists a subsequence $\{\bar{a}_{i_{n2}}\}; n = 1, 2, \dots$ of the subsequence $\{\bar{a}_{i_{n1}}\}; n = 1, 2, \dots \ni$ the subsequence $\{\bar{a}_{i_{n2}}\}; n = 1, 2, \dots$ is either monotone increasing or monotone decreasing.

In general, there exists a subsequence $\{\bar{a}_{i_{nm}}\}; n = 1, 2, \dots$ of the subsequence $\{\bar{a}_{i_{n(m-1)}}\}; n = 1, 2, \dots$ for each $m \in \{2, 3, \dots\} \ni$ the sequence $\{\bar{a}_{i_{nm}m}\}; n = 1, 2, \dots$ is either monotone increasing or monotone decreasing. We can conclude that there exists a subsequence $\{\bar{b}_n\}; n = 1, 2, \dots$ of $\{\bar{a}_n\}; n = 1, 2, \dots \ni$ for each $k \in \mathbb{N}$, the sequence $\{\bar{b}_{nk}\}; n = 1, 2, \dots$ is either monotone increasing or monotone decreasing. Now, $\{\bar{b}_n\}; n = 1, 2, \dots$ will converge in measure to L and will also satisfy the condition at the beginning of this proof. Hence $L \in S$. \square

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