ISSN: 2832-4293

Pan-American Journal of Mathematics 1 (2022), 4
https://doi.org/10.28919/cpr-pajm/1-4

THE SEQUENCE SPACE $V_{\sigma}^{I}(p)$

KHALID EBADULLAH

ABSTRACT. In this article we introduce the sequence spaces $V_{0\sigma}^{I}(p)$ and $V_{\sigma}^{I}(p)$, where $p = (p_k)$ is a sequence of positive reals and study the topology that arises on the said spaces.

1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively.
We write

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \; or \; \mathbb{C} \},$$

the space of all real or complex sequences.
Let $\ell_\infty,\ c$ and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively normed by

$$||x||_\infty = \sup_k |x_k|$$

Let v denote the space of sequences of bounded variation, that is

$$v = \{x = (x_k) : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty,\ x_{-1} = 0\}$$

v is a Banach space normed by

$$||x|| = \sum_{k=0}^{\infty} |x_k - x_{k-1}|$$

Let σ be an injection of the set of positive integers $\mathbb{N}$ into itself having no finite orbits and T be the operator defined on ℓ_∞ by $T(x_k) = (x_{\sigma(k)})$.
A positive linear functional ϕ, with $||\phi|| = 1$, is called a σ-mean or an invariant mean if $\phi(x) = \phi(Tx)$ for all $x \in \ell_\infty$.
A sequence x is said to be σ-convergent, denoted by $x \in V_\sigma$, if $\phi(x)$ takes the same value, called $\sigma - \lim x$, for all σ-means ϕ. We have

$$V_\sigma = \{x = (x_k) : \sum_{m=1}^{\infty} t_{m,k}(x) = L \text{ uniformly in } k, \text{ L} = \sigma - \lim x\},$$

where for $m \geq 0, k > 0$

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} + ... + x_{\sigma^m(k)}}{m+1}, \text{and } t_{-1,k} = 0$$

DEPARTMENT OF MATHEMATICS, MAI NEFHI COLLEGE OF SCIENCE, ERITREA
E-mail address: khalidebadullah@gmail.com.
Submitted on Jun. 26, 2022.
2020 *Mathematics Subject Classification.* 46A45.
Key words and phrases. Ideal, Filter, Invariant Mean, I-Convergent Sequence Spaces.

where $\sigma^m(k)$ denotes the mth iterate of σ at k. In particular, if σ is the translation, a σ-mean is often called a Banach limit and V_σ reduces to f, the set of almost convergent sequences. For certain kind of mappings σ, every invariant mean ϕ extends the limit functional on the space c of real convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all of whose σ-mean are equal. (cf.[1],[4],[6],[8],[9],[10],[11],[14]).

The concept of statistical convergence was first introduced by Fast [2] for real and complex sequences. Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence.
A sequence $x = (x_k)$ is said to be Statistically convergent to L if for a given $\epsilon > 0$

$$\lim_k \frac{1}{k}|\{i : |x_i - L| \geq \epsilon, i \leq k\}| = 0.$$

The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Šalát, Wilczyński[7]. Later on it was studied by Šalát, Tripathy and Ziman[12-13] and many others. Here we give some preliminaries about the notion of I-convergence.
Let X be a non empty set. Then a family of sets $I \subseteq 2^X (2^X$ denoting the power set of $X)$ is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e $A \in I, B \subseteq A \Rightarrow B \in I$.
A non-empty family of sets $F(I) \subseteq 2^X$ is said to be filter on X if and only if $\phi \notin F(I)$, for $A, B \in F(I)$ we have $A \cap B \in F(I)$ and for each $A \in F(I)$ and $A \subseteq B$ implies $B \in F(I)$.
An Ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$.
A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq$I.
A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.
For each ideal I, there is a filter $F(I)$ corresponding to I.
i.e $F(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$,where $K^c = \mathbb{N} - K$.
Definition 1.1. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in$ I. In this case we write $I - \lim x_k = L$.
The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I, \text{for some L} \in \mathbb{C}\ \}$$

Definition 1.2. A sequence $(x_k) \in \omega$ is said to be I-null if $L = 0$. In this case we write $I - \lim x_k = 0$.
Definition 1.3. A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in I$.
Definition 1.4. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists
M >0 such that $\{k \in \mathbb{N} : |x_k| > M\} \in I$.
Definition 1.5. For any set E of sequences the space of multipliers of E, denoted by $M(E)$ is given by

$$M(E) = \{a \in \omega : ax \in E \text{ for all } x \in E\}.\text{(See[15])}.$$

Definition 1.6. A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \to \mathbb{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where $K$is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$.(see[5]).

Definition 1.7. A convergence field of I-covergence is a set

$$F(I) = \{x = (x_k) \in \ell_\infty : \text{there exists } I - \lim x \in \mathbb{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of ℓ_∞ with respect to the supremum norm, $F(I) = \ell_\infty \cap c^I$(See[5,12,13]).
Define a function $\hbar : F(I) \to \mathbb{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to \mathbb{R}$ is a Lipschitz function.(see[5,12,13]).

Recently Khan and Ebadullah[3] introduced the following classes of sequence spaces.
Let $x = (x_k) \in \ell_\infty$,

$$V^I_{0\sigma}(m, \epsilon) = \{(x_k) \in \ell_\infty : (\forall m)(\exists \epsilon > 0)\{k \in \mathbb{N} : |t_{m,k}(x)| \geq \epsilon\} \in I\}$$

$$V^I_\sigma(m, \epsilon) = \{(x_k) \in \ell_\infty : (\forall m)(\exists \epsilon > 0)\{k \in \mathbb{N} : |t_{m,k}(x) - L| \geq \epsilon\} \in I, \text{for some L} \in \mathbb{C} \}$$

2. Main Results

In this article we introduce the following classes of sequence spaces.
Let $x = (x_k) \in \ell_\infty$ and $p = (p_k)$ a sequence of positive reals

$$V^I_{0\sigma}(p) = \{(x_k) \in \ell_\infty : (\forall m)(\exists \epsilon > 0)\{k \in \mathbb{N} : |t_{m,k}(x)|^{p_k} \geq \epsilon\} \in I\}$$

$$V^I_\sigma(p) = \{(x_k) \in \ell_\infty : (\forall m)(\exists \epsilon > 0)\{k \in \mathbb{N} : |t_{m,k}(x) - L|^{p_k} \geq \epsilon\} \in I, \text{for some L} \in \mathbb{C} \}$$

Theorem 2.1. $V^I_\sigma(p)$ and $V^I_{0\sigma}(p)$ are linear spaces.

Proof. Let $(x_k), (y_k) \in V^I_\sigma(p)$ and let α, β be scalars. Then for a given $\epsilon > 0$.
we have

$$\{k \in \mathbb{N} : |t_{m,k}(x) - L_1|^{p_k} \geq \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \text{C}\} \in I$$
$$\{k \in \mathbb{N} : |t_{m,k}(y) - L_2|^{p_k} \geq \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \text{C}\} \in I$$

where

$$M_1 = D.max\{1, \sup_k |\alpha|^{p_k}\}$$
$$M_2 = D.max\{1, \sup_k |\beta|^{p_k}\}$$

and

$$D = max\{1, 2^{H-1}\} \text{ where } H = \sup_k p_k \geq 0.$$

Let

$$A_1 = \{k \in \mathbb{N} : |t_{m,k}(x) - L_1|^{p_k} < \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \text{C}\} \in F(I)$$
$$A_2 = \{k \in \mathbb{N} : |t_{m,k}(x)(y) - L_2|^{p_k} < \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \text{C}\} \in F(I)$$

be such that $A^c_1,\ A^c_2 \in I$.
Then

$$A_3 = \{k \in \mathbb{N} : |(\alpha t_{m,k}(x) + \beta t_{m,k}(y) - (\alpha L_1 + \beta L_2)|^{p_k}) < \epsilon\}$$
$$\supseteq \{k \in \mathbb{N} : |\alpha|^{p_k} |t_{m,k}(x) - L_1|^{p_k} < \frac{\epsilon}{2M_1} |\alpha|^{p_k}.D\}$$
$$\cap \{k \in \mathbb{N} : |\beta|^{p_k} |t_{m,k}(y) - L_2|^{p_k} < \frac{\epsilon}{2M_2} |\beta|^{p_k}.D\}$$

Thus $A^c_3 = A^c_1 \cap A^c_2 \in I$.
Hence $(\alpha t_{m,k}(x) + \beta t_{m,k}(y)) \in V^I_\sigma(p)$.

Hence $V_{\sigma}^{I}(p)$ is a linear space.
Similarly the result can be proved for $V_{0\sigma}^{I}(p)$.

Theorem 2.2. Let $(p_k) \in \ell_\infty$. Then the spaces $V_{\sigma}^{I}(p)$ is a normed linear space, normed by

$$||x_k||_* = \sup_k |t_{m,k}(x)|^{\frac{p_k}{M}}. \tag{2.1}$$

where $M = max\{1, \sup_k p_k\}$.

Proof. Let $x = (x_k),\ y = (y_k) \in V_{\sigma}^{I}(p)$.
(1) Clearly, $||x||_* = 0$ if and only if $x = 0$.
(2) $||x||_* = ||-x||_*$ is obvious.
(3) Since $\frac{p_k}{M} \leq 1$ and $M > 1$, using Minkowski's inequality we have

$$\sup_k |t_{m,k}(x) + t_{m,k}(y)|^{\frac{p_k}{M}} \leq \sup_k |t_{m,k}(x)|^{\frac{p_k}{M}} + \sup_k |t_{m,k}(y)|^{\frac{p_k}{M}}$$

(4) Now for any complex λ we have (λ_k) such that $\lambda_k \to \lambda$, $(k \to \infty)$.
Let $(x_k) \in V_{\sigma}^{I}(p)$ such that $|t_{m,k}(x) - L|^{p_k} \geq \epsilon$.
Therefore, $||t_{m,k}(x) - L||_* = \sup_k |t_{m,k}(x) - L|^{\frac{p_k}{M}} \leq \sup_k |t_{m,k}(x)|^{\frac{p_k}{M}} + \sup_k |L|^{\frac{p_k}{M}}$.
Hence $||\lambda_k t_{m,k}(x) - \lambda L||_* \leq ||\lambda_k t_{m,k}(x)||_* + ||\lambda L||_* = \lambda_k ||t_{m,k}(x)||_* + \lambda ||L||_*$ as $(k \to \infty)$.

Theorem 2.3. $V_{\sigma}^{I}(p)$ is a closed subspace of $\ell_\infty(p)$.

Proof. Let $(x_k^{(n)})$ be a cauchy sequence in $V_{\sigma}^{I}(p)$ such that $x^{(n)} \to x$.
We show that $x \in V_{\sigma}^{I}(p)$.
Since $(x_k^{(n)}) \in V_{\sigma}^{I}(p)$, then there exists a_n such that

$$\{k \in \mathbb{N} : |t_{m,k}(x^{(n)}) - a_n|^{p_k} \geq \epsilon\} \in I$$

We need to show that
(1)(a_n) converges to a.
(2)If $U = \{k \in \mathbb{N} : |x_k - a|^{p_k} < \epsilon\}$, then $U^c \in I$.

(1) Since $(x_k^{(n)})$ is a cauchy sequence in $V_{\sigma}^{I}(p)$ then for a given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sup_k |t_{m,k}(x_k^{(n)}) - t_{m,k}(x_k^{(i)})|^{p_k} < \frac{\epsilon}{3}, \text{for all n,i} \geq k_0$$

For a given $\epsilon > 0$, we have

$$B_{ni} = \{k \in \mathbb{N} : |t_{m,k}(x_k^{(n)}) - t_{m,k}(x_k^{(i)})|^{p_k} < \frac{\epsilon}{3}M\}$$

$$B_i = \{k \in \mathbb{N} : |t_{m,k}(x_k^{(i)}) - a_i|^{p_k} < \frac{\epsilon}{3}M\}$$

$$B_n = \{k \in \mathbb{N} : |t_{m,k}(x_k^{(n)}) - a_n|^{p_k} < \frac{\epsilon}{3}M\}$$

Then $B_{ni}^c, B_i^c, B_n^c \in I$.
Let $B^c = B_{ni}^c \cap B_i^c \cap B_n^c$,
where

$$B = \{k \in \mathbb{N} : |a_i - a_n|^{p_k} < \epsilon\} \in F(I).$$

Then $B^c \in I$.
We choose $k_0 \in B^c$, then for each $n, i \geq k_0$, we have

$$\{k \in \mathbb{N} : |a_i - a_n|^{p_k} < \epsilon\} \supseteq \{k \in \mathbb{N} : |t_{m,k}(x_k^{(i)}) - a_i|^{p_k} < \frac{\epsilon}{3}\}$$
$$\cap\{k \in \mathbb{N} : |t_{m,k}(x_k^{(n)}) - t_{m,k}(x_k^{(i)})|^{p_k} < \frac{\epsilon}{3}\}$$
$$\cap\{k \in \mathbb{N} : |t_{m,k}(x_k^{(n)}) - a_n|^{p_k} < \frac{\epsilon}{3}\}$$

Then (a_n) is a cauchy sequence of scalars in $\mathbb{C}$, so there exists a scalar $a \in \mathbb{C}$ such that $(a_n) \to a$, as $n \to \infty$.

(2) Let $0 < \delta < 1$ be given.Then we show that if

$$U = \{k \in \mathbb{N} : |t_{m,k}(x) - a|^{p_k} < \delta\},$$

then $U^c \in I$.
Since $t_{m,k}(x^{(n)}) \to t_{m,k}(x)$, then there exists $q_0 \in \mathbb{N}$ such that

$$P = \{k \in \mathbb{N} : |t_{m,k}(x^{(q_0)} - t_{m,k}(x)|^{p_k} < \frac{\delta}{3}\} \quad [2.2]$$

which implies that $P^c \in I$
The number q_0 can be so choosen that together with [2.2], we have

$$Q = \{k \in \mathbb{N} : |a_{q_0} - a|^{p_k} < \frac{\delta}{3}\}$$

such that $Q^c \in I$
Since $\{k \in \mathbb{N} : |t_{m,k}(x_k^{(q_0)}) - a_{q_0}|^{p_k} \geq \delta\} \in I$.
Then we have a subset S of $\mathbb{N}$ such that $S^c \in I$, where

$$S = \{k \in \mathbb{N} : |t_{m,k}(x_k^{(q_0)}) - a_{q_0}|^{p_k} < \frac{\delta}{3}\}.$$

Let $U^c = P^c \cap Q^c \cap S^c$, where $U = \{k \in \mathbb{N} : |t_{m,k}(x) - a|^{p_k} < \delta\}$.
Therefore for each $k \in U^c$, we have

$$\{k \in \mathbb{N} : |t_{m,k}(x) - a|^{p_k} < \delta\} \supseteq \{k \in \mathbb{N} : |t_{m,k}(x^{(q_0)} - t_{m,k}(x)|^{p_k} < \frac{\delta}{3}\}$$
$$\cap\{k \in \mathbb{N} : |t_{m,k}(x_k^{(q_0)}) - a_{q_0}|^{p_k} < \frac{\delta}{3}\}$$
$$\cap\{k \in \mathbb{N} : |a_{q_0} - a|^{p_k} < \frac{\delta}{3}\}$$

Then the result follows.
Since the inclusion $V_\sigma^I(p) \subset l_\infty(p)$ are strict so in view of Theorem 2.3 we have the following result.

Theorem 2.4. The space $V_\sigma^I(p)$ is nowhere dense subset of $\ell_\infty(p)$.

Theorem 2.5. The space $V_\sigma^I(p)$ is not seperable.

Proof. Let $M = \{m_1 < m_2 < m_3 < ....\}$ be an infinite subset of $\mathbb{N}$ such that $M \in I$.
Let

$$p_k = \begin{cases} 1, \text{if k} \in M, \\ 2, otherwise. \end{cases}$$

Let $P_0 = \{(x_k) : t_{m,k}(x) = 0$ or 1, for $k = m_j,\ j \in \mathbb{N}\ and\ t_{m,k}(x) = 0$, otherwise$\}$.
Since M is infinite, so P_0 is uncountable.
Consider the class of open balls $B_1 = \{B(z, \frac{1}{2}) : z \in P_0\}$.
Let C_1 be an open cover of $V_\sigma^I(p)$ containing B_1.
Since B_1 is uncountable, so C_1 cannot be reduced to a countable subcover for $V_\sigma^I(p)$.
Thus $V_\sigma^I(p)$ is not seperable.

Theorem 2.6. The function $\hbar : V_\sigma^I(p) \to \mathbb{R}$ is the Lipschitz function and is uniformly continuous.

Proof. Let $x, y \in V_\sigma^I(p)$ and $x \neq y$.Then the sets

$$A_x = \{k \in \mathbb{N} : |t_{m,k}(x) - \hbar(x)|^{p_k} \geq ||x-y||_*\} \in I,$$

$$A_y = \{k \in \mathbb{N} : |t_{m,k}(y) - \hbar(y)|^{p_k} \geq ||x-y||_*\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |t_{m,k}(x) - \hbar(x)|^{p_k} < ||x-y||_*\} \in F(I),$$

$$B_y = \{k \in \mathbb{N} : |t_{m,k}(y) - \hbar(y)|^{p_k} < ||x-y||_*\} \in F(I).$$

Hence also

$$B = B_x \cap B_y \in F(I),$$

so that $B \neq \phi$.
Now taking k in B,
$|\hbar(x) - \hbar(y)|^{p_k}$

$$\leq |\hbar(x) - t_{m,k}(x)|^{p_k} + |t_{m,k}(x) - t_{m,k}(y)|^{p_k} + |t_{m,k}(y) - \hbar(y)|^{p_k}$$

$\leq 3||x-y||_*$.
Thus $\hbar$ is a Lipschitz function.

Theorem 2.7. $c_0^I(p) \subset V_{0\sigma}^I(p) \subset \ell_\infty^I(p)$.

Proof. Let $(x_k) \in c_0^I(p)$.
Then we have $\{k \in \mathbb{N} : |x_k|^{p_k} \geq \epsilon\} \in I$
Since $c_0 \subset V_{0\sigma}$.
$(x_k) \in V_{0\sigma}^I(p)$ implies $\{k \in \mathbb{N} : |t_{m,k}(x)|^{p_k} \geq \epsilon\} \in I$.
Now let

$$A_1 = \{k \in \mathbb{N} : |x_k|^{p_k} < \epsilon\} \in F(I).$$

$$A_2 = \{k \in \mathbb{N} : |t_{m,k}(x)|^{p_k} < \epsilon\} \in F(I).$$

be such that $A_1^c, A_2^c \in I$.
As $\ell_\infty(p) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\}$, taking supremum over k we get $A_1^c \subset A_2^c$.
Hence $c_0^I(p) \subset V_{0\sigma}^I(p) \subset \ell_\infty^I(p)$.

Theorem 2.8. $c^I(p) \subset V_\sigma^I(p) \subset \ell_\infty^I(p)$.

Proof. Let $(x_k) \in c^I(p)$. Then we have $\{k \in \mathbb{N} : |x_k - L|^{p_k} \geq \epsilon\} \in I$.
Since $c \subset V_\sigma \subset \ell_\infty$

$(x_k) \in V_\sigma^I(p)$ implies $\{k \in \mathbb{N} : |t_{m,k}(x) - L|^{p_k} \geq \epsilon\} \in I$.
Now let

$$B_1 = \{k \in \mathbb{N} : |t_k - L|^{p_k} < \epsilon\} \in F(I).$$

$$B_2 = \{k \in \mathbb{N} : |t_{m,k}(x) - L|^{p_k} < \epsilon\} \in F(I).$$

be such that $B_1^c, B_2^c \in I$.
As $\ell_\infty(p) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\}$, taking supremum over k we get $B_1^c \subset B_2^c$.
Hence $c^I(p) \subset V_\sigma^I(p) \subset \ell_\infty^I(p)$.

Theorem 2.9. If $H = \sup_k p_k < \infty$, then we have $\ell_\infty^I \subset M(V_\sigma^I(p))$, where the inclusion may be proper.

Proof. Let $a \in \ell_\infty^I$.This implies that $\sup_k |a_k| < 1 + K$. for some $K > 0$ and all k.
Therefore $x \in V_\sigma^I(p)$ implies

$$\sup_k(|a_k t_{m,k}(x)|^{p_k}) \leq (1+K)^H \sup_k(|t_{m,k}(x)|^{p_k}) < \infty.$$

which gives $\ell_\infty^I \subset M(V_\sigma^I(p))$.
To show that the inclusion may be proper, consider the case when $p_k = \frac{1}{k}$ for all k. Take $a_k = k$ for all k.Therefore $x \in V_\sigma^I(p)$ implies

$$\sup_k(|a_k t_{m,k}(x)|^{p_k}) \leq \sup_k(|k|^{\frac{1}{k}}) \sup_k(|t_{m,k}(x)|^{p_k}) < \infty.$$

Thus in this case $a = (a_k) \in M(V_\sigma^I(p))$ while $a \notin \ell_\infty^I$.

Theorem 2.10. Let (p_k) and (q_k) be two sequences of positive real numbers. Then $V_\sigma^I(p) \supseteq V_\sigma^I(q)$ if and only if $\lim_{k \in K} \inf \frac{p_k}{q_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.

Proof. Let $\lim_{k \in K} \inf \frac{p_k}{q_k} > 0$ and $(x_k) \in V_\sigma^I(q)$.
Then there exists $\beta > 0$ such that $p_k > \beta q_k$, for all sufficiently large $k \in K$.
Since $(x_k) \in V_\sigma^I(q)$ for a given $\epsilon > 0$, we have

$$B_0 = \{k \in \mathbb{N} : |t_{m,k}(x) - L|^{q_k} \geq \epsilon\} \in I$$

Let $G_0 = K^c \cup B_0$ Then $G_0 \in I$.
Then for all sufficiently large $k \in G_0$,

$$\{k \in \mathbb{N} : |t_{m,k}(x) - L|^{p_k}) \geq \epsilon\} \subseteq \{k \in \mathbb{N} : |t_{m,k}(x) - L|^{\beta q_k}) \geq \epsilon\} \in I.$$

Therefore $(x_k) \in V_\sigma^I(p)$.
The converse part of the result follows obviously.

Theorem 2.11. Let (p_k) and (q_k) be two sequences of positive real numbers. Then $V_\sigma^I(q) \supseteq V_\sigma^I(p)$ if and only if $\lim_{k \in K} \inf \frac{q_k}{p_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.

Proof. The proof follows similarly as the proof of Theorem 2.10.

Theorem 2.12. Let (p_k) and (q_k) be two sequences of positive real numbers. Then $V_\sigma^I(p) = V_\sigma^I(q)$ if and only if $\lim_{k \in K} \inf \frac{p_k}{q_k} > 0$, and $\lim_{k \in K} \inf \frac{q_k}{p_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K^c \in I$.

Proof. On combining Theorem 2.10 and 2.11 we get the required result.

REFERENCES

[1] Z.U. Ahmad, M. Mursaleen, An application of Banach limits, Proc. Amer. Math. Soc. 103 (1983),244-246.

[2] H. Fast, Sur la convergence statistique, Colloq.Math. 2 (1951), 241-244.

[3] V.A. Khan, K. Ebadullah, On some new I-convergent sequence spaces, Math. Aeterna, 3 (2013), 151-159.

[4] V.A. Khan, K. Ebadullah, On a new difference sequence space of invariant means defined by Orlicz functions, Bull. Allahabad Math. Soc. 26 (2011), 259-272.

[5] V.A. Khan, K. Ebadullah, On some I-Convergent sequence spaces defined by a modulus function, Theory Appl. Math. Comp. Sci. 1 (2011), 22-30.

[6] J.P. King, Almost summable sequences, Proc.Amer. Math. Soc. 17 (1966), 1219-1225.

[7] P. Kostyrko, T. Šalát, W. Wilczyński, I-convergence, Real analysis exchange, 26 (2000), 669-686.

[8] G.G. Lorentz, A contribution to the theory of divergent series, Acta Math. 80 (1948), 167-190.

[9] M. Mursaleen, Matrix transformation between some new sequence spaces, Houston J. Math. 9 (1983), 505-509.

[10] M. Mursaleen, On some new invariant matrix methods of summability, Quart. J. Math. Oxford, 34 (1983), 77-86.

[11] R.A. Raimi, Invariant means and invariant matrix methods of summability, Duke J. Math. 30 (1963), 81-94.

[12] T. Šalát, B.C. Tripathy, M. Ziman, On some properties of I-convergence, Tatra Mt. Math. Publ. 28 (2004), 279-286.

[13] T. Šalát, B.C. Tripathy, M. Ziman, On I-convergence field, Ital. J. Pure Appl. Math. 17 (2005), 45-54.

[14] P. Schafer, Infinite matrices and Invariant means, Proc. Amer. Math. Soc. 36 (1972), 104-110.

[15] S. Simons, The spaces $l_(p_\nu)$ and $m_(p_\nu)$, Proc. London Math. Soc. 15 (1965), 422-436.