

## ON SOME FRACTIONAL MAGNETIC PROBLEMS

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**ABSTRACT.** In this article, we study existence of weak solutions and the unique continuation property of the nonlocal fractional magnetic equation. First, we use a variational technique to prove existence of weak solutions for the fractional Schrödinger equation with magnetic field. Moreover, we show the doubling property main argument to unique continuation property via Carleman estimates.

### 1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

The unique continuation property (UCP) is a fundamental result in mathematics and physics, especially in the context of partial differential equations (PDEs) and the study of waves and quantum mechanics. It has significant implications in understanding the behavior of solutions to certain differential equations and is related to the notion of stability and predictability of physical systems. Essentially, UCP states that if a solution to a partial differential equation (PDE) vanishes on a sufficiently large portion of a domain, then the solution must vanish identically over the entire domain. In other words, if the solution is zero on a non-empty open set, it must be zero everywhere in the domain. In other words, UCP can be interpreted as stability and predictability conditions for physical systems. It was first introduced by Carleman in its pioneer work [5]. Due to its profound physical meaning and its implications in various fields of science, engineering and applied mathematics, UCP of differential operator has attracted many researchers and gives rise to many results concerning specially the Schrödinger operator  $H = \Delta + V(x)$ , with  $V \in L^p_{loc}$ . Jerison and Kenig has established UCP when  $p = \frac{N}{2}$ , where  $N$  is the dimension of the whole space (see [12]). Their result was later improved by Koch and Tataru [14]. Recently, UCP for the many-body Schrödinger operators has been proved by Garrigue [11] with weaker assumption on the potential than the one shown in [16]. For more results see [16]. On the other hand, the fractional Schrödinger equation for the wave function of quantum mechanical system was first introduced by Laskin to model the motion of fractional quantum mechanic particle see [17]. Since then, many researchers have found several applications of fractional differential equations. Indeed, fractional differential equations is used to describe the anomalous transport of matter, the movement of a chain or a network of particles which are linked by elastic springs (see [20]), financial processes with jump (see [6]) and references therein. For more informations on fractional magnetic operator, we refer the reader to [1] and references therein.

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Therefore, we can obviously ask what will happen if instead of the classical Schrödinger operator  $H$  we take the fractional Schrödinger operator  $H_s := (-\Delta)^s + V(x)$ ? Do we still get UCP result? This question has received many attentions from researchers. Though, almost every results were obtained by using Carleman estimates via Caffarelli-Silvestre extension see [9, 13]. We wonder if it is possible to use direct method to prove UCP for fractional magnetic Schrödinger equation.

In this paper, we first establish existence of the weak solution for fractional magnetic Schrödinger equation. Secondly, we show the doubling property first step to get the unique continuation property to the fractional magnetic Schrödinger equation given by

$$(1.1) \quad \begin{cases} (-\Delta)_{\mathcal{A}}^s u + V(x)u &= 0 & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where  $\Omega$  is a bounded subset of  $\mathbb{R}^N$ ,  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  is a sign-changing scalar potential,  $V \in L_{loc}^{\frac{N}{2}}(\Omega) \cap (L^r(\Omega) + L^\infty(\Omega))$  ( $N \geq 3, 1 \leq r < \infty$ ),  $\mathcal{A} : \mathbb{R}^N \rightarrow \mathbb{R}^N$  is the magnetic potential, and  $(-\Delta)_{\mathcal{A}}^s$  is the fractional magnetic operator which, up to normalization, defined as

$$(1.2) \quad (-\Delta)_{\mathcal{A}}^s u(x) = 2 \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - e^{i(x-y) \cdot \mathcal{A}(\frac{x+y}{2})} u(y)}{|x-y|^{N+2s}} dy, \quad \forall x \in \mathbb{R}^N,$$

here  $B_\varepsilon(x)$  denotes the ball in  $\mathbb{R}^N$  with radius  $\varepsilon > 0$  centered at  $x \in \mathbb{R}^N$ .

$(-\Delta)_{\mathcal{A}}^s u(x)$  is a magnetic version of fractional Laplacian operator given by

$$(1.3) \quad (-\Delta)^s u(x) = 2 \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \quad \forall x \in \mathbb{R}^N,$$

Motivated by above works about magnetic equation (notably [21], [13] and references therein), this paper uses variational approach to prove existence result for Schrödinger equation with magnetic field.

The rest of the paper is organized as follows: In section 2, we recall some basic proprieties of fractional magnetic Sobolev spaces, in section 3, we state and prove our existence result for magnetic equation and in section 4 we prove our result on UCP.

## 2. PRELIMINARIES

In this section, we first give some basic results of fractional Sobolev spaces that will be used later. Let  $N > 2, 0 < s < 1$  be real number satisfying  $2s < N$  and the fractional critical exponent  $2_s^*$  be defined as  $2_s^* = \frac{2N}{N-2s}$ . The fractional Sobolev space  $D^s(\mathbb{R}^N)$  is defined as the closure of  $C_0^\infty(\mathbb{R}^N)$  with respect to the norm

$$\|u\|_s = \left( \|u\|_2^2 + [u]_s^2 \right)^{1/2},$$

where

$$[u]_s = \left( \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx dy \right)^{1/2}$$

is the Gagliardo seminorm and  $\|u\|_2$  the  $L^2$ -norm. The  $L^p$  of  $u$  will be denoted  $\|u\|_p$ .

Suppose that  $\mathcal{A} : \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a continuous function and denote

$$E_{\mathcal{A}}(x, y) = e^{i(x-y) \cdot \mathcal{A}(\frac{x+y}{2})}$$

Let consider the magnetic Gagliardo semi-norm defined by

$$(2.1) \quad [u]_{s, \mathcal{A}} := \left( \iint_{\mathbb{R}^{2N}} \frac{|u(x) - E_{\mathcal{A}}(x, y)u(y)|^2}{|x-y|^{N+2s}} dx dy \right)^{1/2},$$

and define  $D_{\mathcal{A}}^s(\mathbb{R}^N, \mathbb{C})$  as the closure of  $C_0^\infty(\mathbb{R}^N, \mathbb{C})$  with respect to  $[\cdot]_{s,\mathcal{A}}$ .

We also define  $\|\cdot\|_s$  and  $\|\cdot\|_{s,\mathcal{A}}$  as:

$$(2.2) \quad \|u\|_s = \left( \int_{\mathbb{R}^N} |u(x)|^2 dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}$$

and

$$(2.3) \quad \|u\|_{s,\mathcal{A}} = \left( \int_{\mathbb{R}^N} |u(x)|^2 dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x) - E_{\mathcal{A}}(x, y)u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.$$

The characterizations of magnetic Sobolev Spaces that we use are given as follows

$$(2.4) \quad H_{\mathcal{A}}^s(\Omega) = \{u \in L^2(\Omega); \|u\|_{s,\mathcal{A}} < \infty\},$$

**Assumption 2.1.** We make the following assumptions on the potential  $V$ .

$$(A_1): V \in L^r(\Omega) + L^\infty(\Omega), \text{ with } r \geq \frac{n}{2s}.$$

$(A_2)$ :  $V$  is weakly lower semicontinuous and vanishes at infinity, i.e

$$A_a = |\{x : V(x) > a\}| < \infty \text{ for all } a > 0.$$

**Lemma 2.2.** (Diamagnetic inequality) For each  $u \in D_{\mathcal{A}}^s(\mathbb{R}^N, \mathbb{C})$  it holds  $|u| \in D^s(\mathbb{R}^N)$ . More precisely,

$$[|u|]_s \leq [u]_{s,\mathcal{A}}, \quad \text{for all } u \in D_{\mathcal{A}}^s(\mathbb{R}^N, \mathbb{C}).$$

In order to define weak solution to problem (1.1), we recall the following functional space

$$(2.5) \quad X_{0,\mathcal{A}}(\Omega) = \{u \in H_{\mathcal{A}}^s(\Omega) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

equipped with the semi-norm

$$\|u\|_{X_{0,\mathcal{A}}(\Omega)} = [u]_{s,\mathcal{A}}.$$

**Lemma 2.3.** (see [1]) Let  $\Omega \subset \mathbb{R}^N$  be an bounded open. Then

$$X_{0,\mathcal{A}}(\mathbb{R}^3) \hookrightarrow H^s(\mathbb{R}^3, \mathbb{C}),$$

in other words, there exists a positive constante  $C_{n,s} > 0$  such that for all  $p \in [2, \frac{6}{3-2s}]$

$$(2.6) \quad \|u\|_p \leq C_{n,s} [u]_{s,\mathcal{A}}.$$

Furthermore, if the boundary of  $\Omega$  is Lipschitz the injection

$$X_{0,\mathcal{A}}(\Omega) \hookrightarrow L^p(\Omega, \mathbb{C})$$

is compact for any  $p \in [1, 2_s^*)$ .

The following lemma is an analogue of Lemma 2.1( [3]) in the frame of fractional magnetic laplacian.

**Lemma 2.4.** Assume that  $(A_1)$  holds. Then for every  $\epsilon$ , there exists  $\lambda_\epsilon$  such that

$$(2.7) \quad \int_{\Omega} V(x) |u(x)|^2 dx \leq \epsilon \|u\|_{X_{0,\mathcal{A}}(\Omega)}^2 + \lambda_\epsilon \|u\|_{L^2(\Omega)}^2.$$

*Proof.* Let  $V \in L^r(\Omega) + L^\infty(\Omega)$  i.e  $V(x) = V_1(x) + V_2(x)$  with  $V_1 \in L^r(\Omega)$  and  $V_2 \in L^\infty(\Omega)$ . we have

$$\begin{aligned} \int_{\Omega} V(x) |u(x)|^2 dx &= \int_{\Omega} V_1(x) |u(x)|^2 dx + \int_{\Omega} V_2(x) |u(x)|^2 dx \\ &= \int_{\Omega} V_1(x) |u(x)|^2 dx + \int_{\{|V_2| \leq k\}} V_2(x) |u(x)|^2 dx + \int_{\{|V_2| > k\}} V_2(x) |u(x)|^2 dx \\ (2.8) \quad &\leq (k + \|V_2\|_{\infty}) \|u\|_{L^2(\Omega)}^2 + \|V_1\|_{L^r(\{|V_2| > k\})} \|u\|_{L^{2r}(\Omega)}^2 \end{aligned}$$

where  $\frac{1}{r} + \frac{1}{t} = 1$ .

Since  $2t \in [1, 2_s^*)$  for all  $r \geq \frac{n}{2s}$ , then from Sobolev inequality we have

$$(2.9) \quad \|u\|_{L^{2t}(\Omega)}^2 \leq C_{n,s} \|u\|_{X_{0,\mathcal{A}}(\Omega)}^2$$

therefore,

$$(2.10) \quad \int_{\Omega} V(x)|u(x)|^2 dx \leq (k + \|V_2\|_{\infty}) \|u\|_{L^2(\Omega)}^2 + C_{n,s} \|V_1\|_{L^r(\{|V_2|>k\})} \|u\|_{X_{0,\mathcal{A}}(\Omega)}^2.$$

Now, choosing  $k$  big enough so that  $C_{n,s} \|V_1\|_{L^r(\{|V_2|>k\})} < \epsilon$  we get the desired result.  $\square$

### 3. EXISTENCE OF GROUND STATE SOLUTION

In this part we consider the following fractional Schrödinger problem

$$(3.1) \quad \begin{cases} (-\Delta)_{\mathcal{A}}^s u + V(x)u &= Eu & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ \|u\|_2 &= 1. \end{cases}$$

We give the following definition of weak solution to problem 1.1 that arises from the variational formulation.

**Definition 3.1.** A function  $u \in X_{0,\mathcal{A}}(\Omega)$  is a weak solution of (1.1) if  $u$  satisfies

$$(3.2) \quad \langle (-\Delta)_{\mathcal{A}}^s u, v \rangle + \int_{\Omega} V u \bar{v} dx = 0,$$

where

$$(3.3) \quad \langle (-\Delta)_{\mathcal{A}}^s u, v \rangle := \iint_{\Omega \times \Omega} \frac{(u(x) - E_{\mathcal{A}}(x, y)u(y))(\overline{v(x) - E_{\mathcal{A}}(x, y)v(y)})}{|x - y|^{N+2s}} dx dy$$

for every  $v \in X_{0,\mathcal{A}}(\Omega)$ .

**Theorem 3.2.** Assume that  $(A_1) - (A_2)$  hold. Then problem (1.1) has a weak solution in the sense of definition 3.1.

In order to formulate the variational approach of problem 1.1, we introduce the functional  $J_{\mathcal{A}} : X_{0,\mathcal{A}}(\Omega) \rightarrow \mathbb{R}$  defined as follows

$$J_{\mathcal{A}}(u) = \phi_{\mathcal{A}}(u) + \psi(u),$$

where

$$\phi_{\mathcal{A}}(u) = \frac{1}{2} \iint_{\Omega \times \Omega} \frac{|u(x) - E_{\mathcal{A}}(x, y)u(y)|^2}{|x - y|^{N+2s}} dx dy.$$

and

$$\psi(u) = \frac{1}{2} \int_{\Omega} V |u|^2 dx.$$

The followings lemmas permits to prove ours theorem

**Lemma 3.3.** Assume that  $V$  satisfy  $(A_1)$  and  $(A_2)$ , then  $J_{\mathcal{A}}(\cdot)$  is bounded from below and is weakly lower semicontinuous.

*Proof.* Let  $u \in X_{0,\mathcal{A}}(\Omega)$ . First, we show that if  $V \in L^r(\Omega)$  then  $J_{\mathcal{A}}(\cdot)$  is nonnegative whenever  $\|V\|_r \leq C_{n,s}^{-1}$ . Indeed, let us consider  $\phi_{\mathcal{A}}(u)$ . We have by Hölder and Sobolev inequalities

$$(3.4) \quad \begin{aligned} \int_{\Omega} \frac{V(x)}{\|V_1\|_r} |u(x)|^2 dx &\leq \left\| \frac{V_1}{\|V_1\|_r} \right\|_r \|u^2\|_{\frac{r}{r-1}} \\ &\leq \|u\|_{\frac{2r}{r-1}}^2 \\ \|V_1\|_r^{-1} \psi(u) &\leq C_{n,s} [u]_{s,\mathcal{A}}^2. \end{aligned}$$

It follows from (3.4) that

$$\begin{aligned} J_{\mathcal{A}}(u) &= \phi_{\mathcal{A}}(u) + \psi(u) \\ &\geq (1 - C_{n,s}\|V_1\|_r)\phi_{\mathcal{A}}(u) \geq 0. \end{aligned}$$

Now, consider  $V$  as in  $(A_1)$  i.e.  $V = V_1 + V_2$ , where  $V_1 \in L^r(\Omega)$  and  $V_2 \in L^\infty(\Omega)$ , with  $r \geq \frac{n}{2s}$ . Let  $h(x) = (V_1(x) - \lambda)^-$ , where  $f^-(x) = \min(f(x), 0)$ . Then,  $h$  satisfies  $\|h\|_r \leq C_{n,s}$ . Indeed, by (2.6)

$$\begin{aligned} \|h\|_r &\leq C_{n,s} \iint_{\Omega \times \Omega} \frac{|h(x) - h(y)|^2}{|x - y|^{N+2s}} dx dy \\ &\leq C_{n,s} \left[ \iint_{\{V_1(x) > \lambda, V_1(y) > \lambda\}} \frac{|V_1(x) - V_1(y)|^2}{|x - y|^{N+2s}} dx dy + 2 \iint_{\{V_1(x) > \lambda, V_1(y) \leq \lambda\}} \frac{|V_1(x) - \lambda|^2}{|x - y|^{N+2s}} dx dy \right] \\ (3.5) \quad &\leq C_{n,s} (I_1(\lambda) + I_2(\lambda)) \end{aligned}$$

Since,  $V$  is vanishing at infinity so is  $V_1$  therefore, by the dominated convergence theorem for  $\lambda$  great enough the integrals  $I_1(\lambda)$  and  $I_2(\lambda)$  could be choose small such that  $I_1(\lambda) + I_2(\lambda) \leq 1$ . That shows the desired result.

From what precedes, we have

$$\begin{aligned} J_{\mathcal{A}}(u) &= \frac{1}{2} \iint_{\Omega \times \Omega} \frac{|(u(x) - E_{\mathcal{A}}(x, y)u(y))|^2}{|x - y|^{N+2s}} dx dy + \frac{1}{2} \int_{\Omega} V(x)|u(x)|^2 dx \\ &= \phi_{\mathcal{A}}(u) + \frac{1}{2} \int_{\Omega} (V_1(x) - \lambda) |u(x)|^2 dx + \frac{1}{2} \int_{\Omega} V_2(x)|u(x)|^2 dx + \lambda \\ &\geq \phi_{\mathcal{A}}(u) + \frac{1}{2} \int_{\Omega} h(x)|u(x)|^2 dx + \|V_2\|_{\infty} \|u\|_2^2 + \lambda \\ &\geq \phi_{\mathcal{A}}(u) - \frac{1}{2} \|h\|_r \|V_1\|_{\frac{2r}{r-1}}^2 - \|V_2\|_{\infty} \|u\|_2^2 + \lambda \\ &\geq \frac{1}{2} \phi_{\mathcal{A}}(u) + \lambda - \|V_2\|_{\infty} \|u\|_2^2 \\ (3.6) \quad &\geq \left( \frac{1}{2} + \lambda - C_{n,s} \|V_2\|_{\infty} \right) [u]_{s, \mathcal{A}}^2 \geq 0 \end{aligned}$$

(since  $\lambda$  is supposed to be big enough). And therefore, the functional  $J_{\mathcal{A}}(\cdot)$  is bounded from below and coercive. Moreover,  $J_{\mathcal{A}}(\cdot)$  is weakly lower semicontinuous since,  $V$  is assumed to be weakly lower semicontinuous and so is the Gagliardo seminorm  $[\cdot]_{s, \mathcal{A}}$  by applying Fatou lemma.  $\square$

**Lemma 3.4.** Let  $u \in X_{0, \mathcal{A}}(\Omega)$ ,  $u$  is a minimizer of  $J_{\mathcal{A}}$  if and only if  $u$  is weak solution to problem.

*Proof.* Let us check that the minimizer of  $J_{\mathcal{A}}$  satisfies the problem (1.1). We have

$$\begin{aligned} 0 &= \frac{d}{dt} J_{\mathcal{A}}(u + tw) \Big|_{t=0} \\ &= \underbrace{\left[ \frac{1}{2} \iint_{\Omega \times \Omega} \frac{d}{dt} \frac{|(u(x) - E_{\mathcal{A}}(x, y)u(y)) + t(w(x) - E_{\mathcal{A}}(x, y)w(y))|^2}{|x - y|^{N+2s}} dx dy \right]}_{I_1} \\ (3.7) \quad &+ \underbrace{\left[ \frac{1}{2} \int_{\Omega} \frac{d}{dt} V(x)(u(x) + tw(x))^2 dx \right]}_{I_2} \Big|_{t=0}. \end{aligned}$$

Note that

$$(3.8) \quad \begin{aligned} I_1 &= \frac{1}{2} \iint_{\Omega \times \Omega} \frac{(w(x) - E_{\mathcal{A}}(x, y)w(y))(\overline{(u(x) - E_{\mathcal{A}}(x, y)u(y))})}{|x - y|^{N+2s}} dx dy \\ &+ \frac{1}{2} \iint_{\Omega \times \Omega} \frac{(\overline{(w(x) - E_{\mathcal{A}}(x, y)w(y))})(u(x) - E_{\mathcal{A}}(x, y)u(y))}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

By changing the role of  $u$  and  $w$  in the first integral we obtain:

$$(3.9) \quad \begin{aligned} I_1 &= \frac{1}{2} \iint_{\Omega \times \Omega} \frac{(\overline{(w(x) - E_{\mathcal{A}}(x, y)w(y))})(u(x) - E_{\mathcal{A}}(x, y)u(y))}{|x - y|^{N+2s}} dx dy \\ &+ \frac{1}{2} \iint_{\Omega \times \Omega} \frac{(\overline{(w(x) - E_{\mathcal{A}}(x, y)w(y))})(u(x) - E_{\mathcal{A}}(x, y)u(y))}{|x - y|^{N+2s}} dx dy \\ &= \iint_{\Omega \times \Omega} \frac{((u(x) - E_{\mathcal{A}}(x, y)u(y)))(\overline{(w(x) - E_{\mathcal{A}}(x, y)w(y))})}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

The same ideas on  $I_2$  give

$$I_2 = \int_{\Omega} V(x)u(x)\overline{w(x)} dx.$$

It follows that

$$(3.10) \quad \begin{aligned} 0 &= \frac{d}{dt} J_{\mathcal{A}}(u + tw) \Big|_{t=0} \\ &= \iint_{\Omega \times \Omega} \frac{(u(x) - E_{\mathcal{A}}(x, y)u(y))(\overline{(w(x) - E_{\mathcal{A}}(x, y)w(y))})}{|x - y|^{N+2s}} dx dy \\ &+ \int_{\Omega} V(x)u(x)\overline{w(x)} dx. \end{aligned}$$

Therefore,  $u$  satisfies problem 1.1.

Now, let see the proof of the converse (that every weak solution is the minimiser of  $J_{\mathcal{A}}$ ). Let  $u \in X_{0,\mathcal{A}}(\Omega)$  be a weak solution to problem then in the sense of Definition 3.1,  $u$  satisfy

$$(3.11) \quad \iint_{\Omega \times \Omega} \frac{(u(x) - E_{\mathcal{A}}(x, y)u(y))(\overline{(v(x) - E_{\mathcal{A}}(x, y)v(y))})}{|x - y|^{N+2s}} dx dy + \int_{\Omega} V u \overline{v} dx = 0$$

for every  $v \in X_{0,\mathcal{A}}(\Omega)$ , which proves that  $J'_{\mathcal{A}}(u) = 0$ . So, since  $J_{\mathcal{A}}$  attains its minimum, then  $u$  minimizes  $J_{\mathcal{A}}$ .  $\square$

The proof of Theorem 3.2 follows from Lemmas (3.3) and (3.4).

#### 4. AN UCP RESULT

In this part, we prove a weak ucp property of the fractional magnetic operator  $(-\Delta)_{\mathcal{A}}^s$ .

**Definition 4.1.** A function  $u \in L^2(\Omega)$  has a zero of infinite order at  $x_0 \in \Omega$  if for each  $n \in \mathbb{N}$ , there exists a constant  $C_1 > 0$  such that

$$(4.1) \quad \int_{B(x_0, R)} |u|^2 \leq C_1 R^n.$$

**Definition 4.2.** A family of functions enjoys the unique continuation property for short U.C.P., if no function besides possibly the zero function vanishes in a set of positive measure of  $\Omega$ .

**Definition 4.3.** A family of functions has the strong unique continuation property for short S.U.C.P., if no function besides possibly the zero function has a zero of infinite order.

**Definition 4.4.** A family of functions enjoys the weak unique continuation property for short W.U.C.P. , if no function besides possibly the zero function vanishes in an open subset of  $\Omega$ .

Now, we state the main result of this part.

**Theorem 4.5.** Let  $V \in L^{\frac{N}{2}}_{loc}(\Omega)$  and suppose that  $u \in X_{0,\mathcal{A}}(\Omega)$  be a solution of

$$(4.2) \quad (-\Delta)_{\mathcal{A}}^s u + V(x)u = 0.$$

If  $u = 0$  on a set of positive measure  $E$ , then  $u$  has a zero of infinite order.

Now, we will need the following inverse Poincaré's Inequality.

**Lemma 4.6.** Let  $r > 0$ ,  $B_r$  and  $B_{2r}$  be two concentric balls contained in  $\Omega$ . Assume  $V \in L^{\frac{N}{2}}_{loc}(\Omega)$ . If  $u$  is a solution of Problem 1.1 then, we have

$$(4.3) \quad \iint_{B_r \times B_r} \frac{|u(x) - E_{\mathcal{A}}(x, y)u(y)|^2}{|x - y|^{N+2s}} dx dy \leq \frac{C_2}{r^2} \int_{B_{2r}} |u|^2.$$

*Proof.* Let  $\varphi \in C_0^\infty(\Omega)$  be a real-valued function with  $\text{supp } \varphi \subset B_{2r}$ ,  $\varphi \equiv 1$  for  $x \in B_r$  and  $|(-\Delta)_{\mathcal{A}}^s \varphi| \leq \frac{2}{r}$ . Let  $u \in X_{0,\mathcal{A}}(\Omega)$  satisfying 4.2, taking  $v = u\varphi^2$  as test function, we obtain

$$(4.4) \quad \iint_{B_{2r} \times B_{2r}} \frac{(u(x) - E_{\mathcal{A}}(x, y)u(y))(\overline{(\varphi^2 u)(x) - E_{\mathcal{A}}(x, y)(\varphi^2 u)(y)})}{|x - y|^{N+2s}} dx dy + \int_{B_{2r}} V u(\overline{\varphi^2 u}) dx = 0.$$

Using the following identity

$$(4.5) \quad (a - b)(c\bar{a} - d\bar{b}) = c|a - b|^2 + \bar{b}(a - b)(c - d),$$

where  $a, b, c, d$  are complex numbers; the first term of (4.4) can be written as

$$(4.6) \quad \iint_{B_{2r} \times B_{2r}} \frac{|u(x) - E_{\mathcal{A}}(x, y)u(y)|^2 \varphi^2(x)}{|x - y|^{N+2s}} dx dy + \iint_{B_{2r} \times B_{2r}} \frac{\overline{E_{\mathcal{A}}(x, y)u(y)}(u(x) - E_{\mathcal{A}}(x, y)u(y))(\varphi^2(x) - \varphi^2(y))}{|x - y|^{N+2s}} dx dy$$

and then (4.4) becomes

$$(4.7) \quad \iint_{B_{2r} \times B_{2r}} \frac{|u(x) - E_{\mathcal{A}}(x, y)u(y)|^2 \varphi^2(x)}{|x - y|^{N+2s}} dx dy = - \iint_{B_{2r} \times B_{2r}} \frac{\overline{E_{\mathcal{A}}(x, y)u(y)}(u(x) - E_{\mathcal{A}}(x, y)u(y))(\varphi^2(x) - \varphi^2(y))}{|x - y|^{N+2s}} dx dy - \int_{B_{2r}} V u(\overline{\varphi^2 u}) dx.$$

Passing to absolute value we have

$$\begin{aligned}
 & \iint_{B_{2r} \times B_{2r}} \frac{|u(x) - E_A(x, y)u(y)|^2 \varphi^2(x)}{|x - y|^{N+2s}} dx dy \\
 \leq & \iint_{B_{2r} \times B_{2r}} \frac{|E_A(x, y)u(y)(u(x) - E_A(x, y)u(y))(\varphi^2(x) - \varphi^2(y))|}{|x - y|^{N+2s}} dx dy \\
 & + \int_{B_{2r}} |V||u|^2 \varphi^2 dx \\
 \leq & \iint_{B_{2r} \times B_{2r}} \frac{|E_A(x, y)u(y)||u(x) - E_A(x, y)u(y)||\varphi^2(x) - \varphi^2(y)|}{|x - y|^{N+2s}} dx dy \\
 & + \int_{B_{2r}} |V||u|^2 \varphi^2 dx \\
 \leq & \epsilon \iint_{B_{2r} \times B_{2r}} \frac{|u(x) - E_A(x, y)u(y)|^2 (\varphi(x) + \varphi(y))^2}{|x - y|^{N+2s}} dx dy + \\
 & + \frac{1}{\epsilon} \iint_{B_{2r} \times B_{2r}} \frac{|u(y)|^2 |\varphi(x) - \varphi(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{B_{2r}} |V||u|^2 \varphi^2 dx \\
 (4.8) \quad & \leq I_1 + I_2 + I_3,
 \end{aligned}$$

where

$$\begin{aligned}
 I_1 &:= \epsilon \iint_{B_{2r} \times B_{2r}} \frac{|u(x) - E_A(x, y)u(y)|^2 (\varphi(x) + \varphi(y))^2}{|x - y|^{N+2s}} dx dy \\
 I_2 &:= \frac{1}{\epsilon} \iint_{B_{2r} \times B_{2r}} \frac{|u(y)|^2 |\varphi(x) - \varphi(y)|^2}{|x - y|^{N+2s}} dx dy \\
 I_3 &:= \int_{B_{2r}} |V||u|^2 \varphi^2 dx.
 \end{aligned}$$

Since,  $\varphi \leq 1$  on  $B_{2r}$  we have

$$(4.9) \quad I_1 \leq 4\epsilon \iint_{B_{2r} \times B_{2r}} \frac{|u(x) - E_A(x, y)u(y)|^2}{|x - y|^{N+2s}} dx dy.$$

As for  $I_2$ , we have

$$\begin{aligned}
 I_2 &= \frac{1}{\epsilon} \iint_{B_{2r} \times B_{2r}} \frac{|u(y)|^2 |\varphi(x) - \varphi(y)|^2}{|x - y|^{N+2s}} dx dy \\
 &= \frac{1}{\epsilon} \int_{B_{2r}} |u(y)|^2 \left[ \int_{B_{2r}} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{N+2s}} dx \right] dy \\
 &= \frac{1}{\epsilon} \int_{B_{2r}} |u(y)|^2 G_s \varphi(y) dy \\
 (4.10) \quad &\leq \frac{C_1}{\epsilon r^2} \int_{B_{2r}} |u(y)|^2 dy,
 \end{aligned}$$

where

$$G_s \varphi(y) = \int_{B_{2r}} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{N+2s}} dx.$$

From (2.4), we have

$$(4.11) \quad \int_{B_{2r}} V \varphi^2 |u|^2 dx \leq \epsilon \iint_{B_{2r} \times B_{2r}} \frac{|u(x) - E_A(x, y)u(y)|^2}{|x - y|^{N+2s}} dx dy + \lambda(\epsilon) \int_{B_{2r}} |u|^2.$$

Then, putting together (4.9)-(4.11), (4.8) becomes

$$\iint_{B_{2r} \times B_{2r}} \frac{|u(x) - E_A(x, y)u(y)|^2 \varphi^2(x)}{|x - y|^{N+2s}} dx dy \leq 5\epsilon \iint_{B_{2r} \times B_{2r}} \frac{|u(x) - E_A(x, y)u(y)|^2}{|x - y|^{N+2s}} dx dy + \left( \frac{C_1}{\epsilon r^2} + \lambda(\epsilon) \right) \int_{B_{2r}} |u|^2$$

Now, choosing  $\epsilon$  such that  $0 < \epsilon < \frac{1}{5}$  and  $\lambda(\epsilon)r^2 < C$  it follows that

$$\iint_{B_{2r} \times B_{2r}} \frac{|u(x) - E_A(x, y)u(y)|^2 \varphi^2(x)}{|x - y|^{N+2s}} dx dy \leq \frac{C}{r^2} \int_{B_{2r}} |u|^2.$$

Hence,

$$\iint_{B_r \times B_r} \frac{|u(x) - E_A(x, y)u(y)|^2}{|x - y|^{N+2s}} dx dy \leq \frac{C}{r^2} \int_{B_{2r}} |u|^2.$$

□

**Proof of Theorem:** We denote  $E$  a set of positive measure,  $E^c$  the complement of  $E$ ,  $B_r(x_0)$  the ball of radius  $r$  centered at  $x_0$  and  $|S|$  the Lebesgue's measure of a set  $S$ .

Assume that  $u \in H_{\mathcal{A}}^s(\gamma, \Omega)$  vanishes on the set  $E$ . Almost every point of  $E$  are a point of density of  $E$  i.e if  $x_0$  is such a point then

$$(4.12) \quad \lim_{r \rightarrow 0} \frac{|E \cap B_r(x_0)|}{|B_r(x_0)|} = 1.$$

In other words, if  $x_0$  is a density point of  $E$ , given some  $\epsilon > 0$ , there is an  $r_0 = r_0(\epsilon)$  such that for any  $r \leq r_0$  we have

$$(4.13) \quad \frac{|E^c \cap B_r(x_0)|}{|B_r(x_0)|} < \epsilon.$$

Taking  $r_0$  smaller, if necessary, we can assume  $B_{r_0}(x_0) \subset \Omega$ .

Since  $u = 0$  on  $E$ , the Hölder inequality yield

$$(4.14) \quad \begin{aligned} \int_{B_r} |u|^2 &= \int_{B_r \cap E^c} |u|^2 \\ &\leq \left( \int_{B_r \cap E^c} |u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}} |B_r \cap E^c|^{\frac{2}{N}} \\ &\leq \epsilon^{\frac{2}{N}} |B_r|^{\frac{2}{N}} \left( \int_{B_r \cap E^c} |u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}}. \end{aligned}$$

According to 4.14, Sobolev inequalities and Diamagnetic inequality, we have

$$\begin{aligned} \int_{B_r} |u|^2 &\leq C\epsilon^{\frac{2}{N}} (r^N)^{\frac{2}{N}} \left( \int_{B_r} |u|^2 + \iint_{B_r \times B_r} \frac{|u(x) - E_A(x, y)u(y)|^2}{|x - y|^{N+2s}} dx dy \right) \\ &\leq C\epsilon^{\frac{2}{N}} r_0^2 \int_{B_r} |u|^2 + C\epsilon^{\frac{2}{N}} r^2 \iint_{B_r \times B_r} \frac{|u(x) - E_A(x, y)u(y)|^2}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

And we obtain

$$(4.15) \quad \int_{B_r} |u|^2 \leq C\epsilon^{\frac{2}{N}} r^2 \iint_{B_r \times B_r} \frac{|u(x) - E_A(x, y)u(y)|^2}{|x - y|^{N+2s}} dx dy.$$

From inverse Poincaré's Inequality we have for any  $r \leq r_0$

$$(4.16) \quad \int_{B_r} |u|^2 \leq C\epsilon^{\frac{2}{N}} \int_{B_{2r}} |u|^2.$$

Now, we introduce the following function

$$(4.17) \quad f(r) = \int_{B_r} |u|^2$$

and fix  $n \in \mathbb{N}$ , choose  $\epsilon > 0$  such that  $C\epsilon^{\frac{2}{N}} = 2^{-n}$ . Since  $r_0$  depends on  $\epsilon$  then  $r_0$  will also depends on  $n$  and from (4.16) we have

$$(4.18) \quad f(r) \leq 2^{-n} f(2r) \quad \text{for } r \leq r_0.$$

By iteration of (4.18) we get

$$(4.19) \quad f(r') \leq 2^{-kn} f(2^k r') \quad \text{for } r' \leq 2^{-(k-1)} r_0.$$

Let fix  $r$  and choose  $k$  such that  $2^{-k} r_0 \leq r \leq 2^{-(k-1)} r_0$ . Then according to 4.19, it follows that

$$(4.20) \quad f(r) \leq 2^{-kn} f(2^k r).$$

In addition, since the function  $f$  is increasing, we have

$$(4.21) \quad f(2^k r) \leq f(2r_0).$$

We use (4.20) and (4.21) to get

$$(4.22) \quad f(r) \leq 2^{-kn} f(2r_0).$$

Since  $2^{-k} \leq \frac{r}{r_0}$ , we have

$$(4.23) \quad f(r) \leq \left(\frac{r}{r_0}\right)^n f(2r_0).$$

It follows that  $x_0$  is zero of infinite order of  $u$ .

Notes and Comments. First, we may go further and try to show existence of ground state solution. In other word, problem (1.1) constrained with  $\|u\|_2 = 1$  has a minimizer under mild assumption on  $V$ . To this aim we could use Lions compactness prinpinpe see [19].

Secondly, many authors get UCP for fractional operator by using Caffarelli-Silvestre extension( [4]). Could we prove this result without using Caffarelli-Silvestre extension( [4])? We will try to elucidate it in forthcoming paper.

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