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ON TEMPERED (κ,ψ)-HILFER FRACTIONAL BOUNDARY VALUE PROBLEMS

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ABSTRACT. Our research is primarily focused on applying the tempered (κ,ψ)-fractional operators to investigate the existence, uniqueness, and κ-Mittag-Leffler-Ulam-Hyers stability of a specific class of boundary value problems involving implicit nonlinear fractional differential equations and tempered (κ,ψ)-Hilfer fractional derivatives. To accomplish this, we make use of the fixed point theorem of Banach and a generalization of the well-known Gronwall inequality. Additionally, we provide illustrative examples to demonstrate the practical effectiveness of our main findings.

1. INTRODUCTION

Fractional calculus extends differentiation and integration to non-integer orders, gaining attention in theoretical studies and practical applications across research domains. Its versatility has made it a crucial tool in the field. Recently, there has been a significant increase in research on fractional calculus, exploring various outcomes under different conditions and forms of fractional differential equations and inclusions. For more details on the applications of fractional calculus, the reader is directed to the books of Herrmann [10], Hilfer [11], Kilbas *et al.* [13] and Samko *et al.* [36]. Agrawal [1] introduced some generalizations of fractional integrals and derivatives and presented some of their properties. In [4,5], Benchohra *et al.* demonstrated the existence, uniqueness, and stability results for various classes of problems with different conditions with some form of extension of the well-known Hilfer fractional derivative which unifies the Riemann-Liouville and Caputo fractional derivatives.

In a recent publication [8], Diaz introduced novel definitions for the special functions κ-gamma and κ-beta. Those interested can find more information in other sources such as [7,20,21]. Sousa *et al.* presented the ψ-Hilfer fractional derivative in another work [39], highlighting important properties related to this type of fractional operator. Further insights and results based on this operator can be explored in papers like [2,37,38] and their references. Inspired by the cited papers, we have introduced a new extension of the renowned Hilfer fractional derivative [35]. This extension, called the κ-generalized ψ-Hilfer fractional derivative, enabled us to establish a generalized version of Grönwall's lemma and explore various types of Ulam stability. Additionally, we have thoroughly investigated qualitative and quantitative results for

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different classes of fractional differential problems [14, 28–34], all made possible by this new generalized fractional operator. More details can be found in [4, 5].

Tempered fractional calculus has emerged as an important class of fractional calculus operators in recent years. This class can generalize various forms of fractional calculus and possesses analytic kernels, making it an extension of fractional calculus that can describe the transition between normal and anomalous diffusion. The definitions of fractional integration with weak singular and exponential kernels were initially established by Buschman in [6], and further elaboration on this topic can be found in [3, 22, 26]. Although the Caputo tempered fractional derivative has not been extensively explored in the literature, it holds the potential to significantly contribute to this field. By studying this derivative, we aim to better understand its properties and potential applications in this unique mathematical notion, thus advancing fractional calculus. In their work cited as [19], Kucche *et al.* made substantial advancements in the field of tempered fractional integrals and derivatives. They introduced a new framework for calculating these integrals and derivatives and presented a comprehensive set of related properties and results. Continuing their research in [16], the same team extended the theory and explored tempered fractional calculus with respect to functions, introducing the tempered Hilfer-type operator.

While achieving precise solutions to differential equations proves challenging or even unattainable in several instances, our focus, aligned with nonlinear analysis and optimization, revolves around investigating approximations. It's essential to emphasize that solely stable approximations hold merit. Consequently, an array of methodologies for stability assessment, including Lyapunov and exponential stability, come into play. The stability predicament in functional equations was initially addressed by mathematician Ulam in a 1940 lecture at the University of Wisconsin. Within this context, S.M. Ulam posed the inquiry: "What are the conditions for the existence of an additive mapping in proximity to an approximately additive mapping?" [40]. The subsequent year saw Hyers tackling Ulam's conundrum for additive functions defined on Banach spaces in [12]. In 1978, Rassias [24] demonstrated the existence of unique linear mappings near approximately additive mappings, thus extending Hyers' findings. In contrast to the analysis of Lyapunov and exponential stability, Ulam-Hyers stability analysis directs its focus toward the behavior of a function under perturbations, as opposed to the stability of a dynamical system or equilibrium point. Notably, the authors of [17, 32, 35] have delved into Ulam stability concerning fractional differential problems under varying conditions. Furthermore, considerable attention has been directed towards exploring the stability of diverse functional equation types, particularly Ulam-Hyers and Ulam-Hyers-Rassias stability. This theme is pervasive in resources such as the book authored by Benchohra *et al.* [4]. Research conducted by Luo *et al.* [18] and Rus [25] has also delved into the stability of operatorial equations using the Ulam-Hyers methodology.

In [15], the authors considered the following problem:

$$\begin{cases} {}_0^C D_\delta^{\sigma,\lambda}\mathfrak{w}(\delta) = \aleph\left(\delta, \mathfrak{w}_\delta, {}_0^C D_\delta^{\sigma,\lambda}\mathfrak{w}(\delta)\right), \quad \delta \in \Xi := [0, \varkappa], \\ \mathfrak{w}(\delta) = \Lambda(\delta), \quad \delta \in [-\varrho, 0], \\ \delta_1\mathfrak{w}(0) + \delta_2\mathfrak{w}(\varkappa) = \delta_3, \end{cases}$$

where $0 < \sigma < 1$, $\lambda \geq 0$, ${}_0^C D_\delta^{\sigma,\lambda}$ is the Caputo tempered fractional derivative, $\aleph : \Xi \times C([-\varrho, 0], \mathbb{R}) \times \mathbb{R}$ is a continuous function, $\mathfrak{w} \in C([-\varrho, 0], \mathbb{R})$, $0 < \varkappa < +\infty$, $\delta_1, \delta_2, \delta_3$ are real constants, and $\varrho > 0$ is the time delay.

In [14], the authors considered the initial value problem with nonlinear implicit κ-generalized ψ-Hilfer type fractional differential equation:

$$\begin{cases} \left({}^{H}_{\kappa}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta) = \aleph\left(\delta, \mathfrak{w}(\delta), \left({}^{H}_{\kappa}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta)\right), \ \ \delta \in (\varrho_1, \varrho_2], \\ \\ \left(\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(a^+) = c_0, \end{cases}$$

where ${}^{H}_{\kappa}\mathcal{D}^{\sigma,\zeta;\psi}_{\varrho_1+}$, $\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}$ are the κ-generalized ψ-Hilfer fractional derivative of order $\sigma \in (0,\kappa)$ and type $\zeta \in [0,1]$, and κ-generalized ψ-fractional integral of order $\kappa(1-\theta)$, where $\kappa > 0$, $\aleph \in C([\varrho_1, \varrho_2] \times \mathbb{R}^2, \mathbb{R})$ and $c_0 \in \mathbb{R}$.

In order to generalize our prior results, in this paper, we establish existence and uniqueness results to the following tempered (κ,ψ)-Hilfer boundary value problem with nonlinear implicit fractional differential equation:

$$\left({}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta) = \aleph\left(\delta, \mathfrak{w}(\delta), \left({}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta)\right), \ \ \delta \in (\varrho_1, \varrho_2], \tag{1.1}$$

$$\varkappa_1\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\varrho_1{}^{+}) + \varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\varrho_2) = \varkappa_3, \tag{1.2}$$

where ${}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}$, ${}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}$ are the tempered (κ,ψ)-Hilfer fractional derivative of order $\sigma \in (0,\kappa)$, $\varepsilon \in [0,1]$ and index $\lambda \in \mathbb{R}$, and tempered (κ,ψ)-fractional integral of order $\kappa(1-\theta)$ and index λ defined in Section 2 respectively, where $\theta = \frac{1}{\kappa}(\varepsilon(\kappa-\sigma)+\sigma)$, $\kappa > 0$, $\varkappa_1, \varkappa_2, \varkappa_3 \in \mathbb{R}$, where $\varkappa_1 + \varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))} \neq 0$, $\aleph : [\varrho_1, \varrho_2] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given appropriate function specified later.

The paper is structured as follows: Section 2 starts by introducing necessary notations and reviewing preliminaries related to κ-generalized ψ-Hilfer and tempered fractional operators, as well as functions like κ-Gamma, κ-Beta, κ-Mittag-Leffler, and several auxiliary results. Additionally, the definition of the tempered (κ,ψ)-Hilfer fractional derivative and some essential theorems and lemmas are presented. In Section 3, a generalized Gronwall inequality is presented. Section 4 contains an existence and uniqueness result for the problem (1.1)-(1.2), which relies on the Banach contraction principle. Furthermore, in the Section 5, the definitions of κ-Mittag-Leffler-Ulam-Hyers stability and related remarks are provided, followed by the proof of the stability result for problem (1.1)-(1.2). The final section focuses on providing illustrative examples that effectively demonstrate the practical applicability of the main findings.

2. Preliminaries

First, we present the weighted spaces, notations, definitions, and preliminary facts which are used in this paper. Let $0 < \varrho_1 < \varrho_2 < \infty$, $\nabla = [\varrho_1, \varrho_2]$, $\sigma \in (0,\kappa)$, $\varepsilon \in [0,1]$, $\lambda \in \mathbb{R}$, $\kappa > 0$ and $\theta = \frac{1}{\kappa}(\varepsilon(\kappa-\sigma)+\sigma)$. By $C(\nabla,\mathbb{R})$ we denote the Banach space of all continuous functions from ∇ into $\mathbb{R}$ with the norm

$$\|\mathfrak{w}\|_\infty = \sup\{|\mathfrak{w}(\delta)| : \delta \in \nabla\}.$$

$AC^{\jmath}(\nabla,\mathbb{R})$, $C^{\jmath}(\nabla,\mathbb{R})$ be the spaces of continuous functions, $\jmath$-times absolutely continuous and $\jmath$-times continuously differentiable functions on ∇, respectively.
Consider the weighted Banach space

$$C_{\theta;\psi}(\nabla) = \left\{\mathfrak{w} : (\varrho_1, \varrho_2] \to \mathbb{R} : \delta \to \Psi^{\psi}_{\theta}(\delta, \varrho_1)\mathfrak{w}(\delta) \in C(\nabla,\mathbb{R})\right\},$$

where $\Psi_\theta^\psi(\delta,\varrho_1)=(\psi(\delta)-\psi(\varrho_1))^{1-\theta}$, with the norm

$$\|\mathfrak{w}\|_{C_{\theta;\psi}}=\sup_{\delta\in[\varrho_1,\varrho_2]}\left|\Psi_\theta^\psi(\delta,\varrho_1)\mathfrak{w}(\delta)\right|,$$

and

$$C_{\theta;\psi}^{\jmath}(\nabla)=\left\{\mathfrak{w}\in C^{\jmath-1}(\nabla,\mathbb{R}):\mathfrak{w}^{(\jmath)}\in C_{\theta;\psi}(\nabla)\right\},\jmath\in\mathbb{N},$$
$$C_{\theta;\psi}^{0}(\nabla)=C_{\theta;\psi}(\nabla),$$

with the norm

$$\|\mathfrak{w}\|_{C_{\theta;\psi}^{\jmath}}=\sum_{i=0}^{\jmath-1}\|\mathfrak{w}^{(i)}\|_\infty+\|\mathfrak{w}^{(\jmath)}\|_{C_{\theta;\psi}}.$$

Consider the space $X_\psi^p(\varrho_1,\varrho_2),\ (c\in\mathbb{R},\ 1\le p\le\infty)$ of those real-valued Lebesgue measurable functions $\widehat{\aleph}$ on $[\varrho_1,\varrho_2]$ for which $\|\widehat{\aleph}\|_{X_\psi^p}<\infty$, where the norm is defined by

$$\|\widehat{\aleph}\|_{X_\psi^p}=\left(\int_{\varrho_1}^{\varrho_2}\psi'(\delta)|\widehat{\aleph}(\delta)|^p d\delta\right)^{\frac{1}{p}},$$

where ψ is an increasing and positive function on $[\varrho_1,\varrho_2]$ such that ψ' is continuous on $[\varrho_1,\varrho_2]$ with $\psi(0)=0$. In particular, when $\psi(\mathfrak{w})=\mathfrak{w}$, the space $X_\psi^p(\varrho_1,\varrho_2)$ coincides with the $L_p(\varrho_1,\varrho_2)$ space.

In what follows, and to keep it concise, we will take into account the following:

$$\widehat{\lambda}:=\max_{(\delta,\gamma)\in[\varrho_1,\varrho_2]\times[\varrho_1,\delta]}e^{-\lambda(\psi(\delta)-\psi(\gamma))}=\begin{cases}1, & \text{if } \lambda\ge 0,\\ e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}, & \text{if } \lambda<0.\end{cases}$$

Definition 2.1 ([8]). The κ-gamma function is defined by

$$\Gamma_\kappa(\varsigma)=\int_0^\infty \delta^{\varsigma-1}e^{-\frac{\delta^\kappa}{\kappa}}d\delta,\varsigma>0.$$

When $\kappa\to 1$ then $\Gamma_\kappa(\varsigma)\to\Gamma(\varsigma)$, we have also some useful following relations $\Gamma_\kappa(\varsigma)=\kappa^{\frac{\varsigma}{\kappa}-1}\Gamma\left(\frac{\varsigma}{\kappa}\right)$, $\Gamma_\kappa(\varsigma+\kappa)=\varsigma\Gamma_\kappa(\varsigma)$ and $\Gamma_\kappa(\kappa)=\Gamma(1)=1$. Furthermore κ-beta function is defined as follows

$$B_\kappa(\varsigma,\widehat{\varsigma})=\frac{1}{\kappa}\int_0^1\delta^{\frac{\varsigma}{\kappa}-1}(1-\delta)^{\frac{\widehat{\varsigma}}{\kappa}-1}d\delta$$

so that $B_\kappa(\varsigma,\widehat{\varsigma})=\frac{1}{\kappa}B\left(\frac{\varsigma}{\kappa},\frac{\widehat{\varsigma}}{\kappa}\right)$ and $B_\kappa(\varsigma,\widehat{\varsigma})=\frac{\Gamma_\kappa(\varsigma)\Gamma_\kappa(\widehat{\varsigma})}{\Gamma_\kappa(\varsigma+\widehat{\varsigma})}$. The Mittag-Leffler function can also be refined into the κ-Mittag-Leffler function defined as follows

$$\mathbb{E}_\kappa^{\varsigma,\widehat{\varsigma}}(\mathfrak{w})=\sum_{i=0}^\infty\frac{\mathfrak{w}^i}{\Gamma_\kappa(\varsigma i+\widehat{\varsigma})},\varsigma,\widehat{\varsigma}>0,$$

then, we can have

$$\mathbb{E}_\kappa^{\varsigma}(\mathfrak{w})=\mathbb{E}_\kappa^{\varsigma,\kappa}(\mathfrak{w})=\sum_{i=0}^\infty\frac{\mathfrak{w}^i}{\Gamma_\kappa(\varsigma i+\kappa)},\varsigma>0.$$

2.1. **Fractional Integrals.** Now, we give all the definitions to the different fractional integrals used throughout this paper.

Definition 2.2 (κ-Generalized ψ-fractional Integral [23])**.** Let $\widehat{\aleph} \in X_{\psi}^{p}(\varrho_1, \varrho_2)$ and $[\varrho_1, \varrho_2]$ be a finite or infinite interval on the real axis $\mathbb{R} = (-\infty, \infty)$, $\psi(\delta) > 0$ be an increasing function on $(\varrho_1, \varrho_2]$ and $\psi'(\delta) > 0$ be continuous on (ϱ_1, ϱ_2) and $\sigma > 0$. The generalized κ-fractional integral operators of a function $\widehat{\aleph}$ of order σ are defined by

$$\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\widehat{\aleph}(\delta) = \int_{\varrho_1}^{\delta} \bar{\Psi}_{\sigma}^{\kappa,\psi}(\delta,\gamma)\psi'(\gamma)\widehat{\aleph}(\gamma)d\gamma,$$
$$\mathcal{J}_{\varrho_2-}^{\sigma,\kappa;\psi}\widehat{\aleph}(\delta) = \int_{\delta}^{\varrho_2} \bar{\Psi}_{\sigma}^{\kappa,\psi}(\gamma,\delta)\psi'(\gamma)\widehat{\aleph}(\gamma)d\gamma,$$

with $\kappa > 0$ and $\bar{\Psi}_{\sigma}^{\kappa,\psi}(\delta,\gamma) = \dfrac{(\psi(\delta)-\psi(\gamma))^{\frac{\sigma}{\kappa}-1}}{\kappa\Gamma_{\kappa}(\sigma)}$.

Definition 2.3 (The ψ-tempered fractional Integral [19])**.** Let $\widehat{\aleph} \in X_{\psi}^{p}(\varrho_1, \varrho_2)$ and $[\varrho_1, \varrho_2]$ be a finite or infinite interval on the real axis $\mathbb{R}$, $\psi(\delta) > 0$ be an increasing function on $(\varrho_1, \varrho_2]$ and $\psi'(\delta) > 0$ be continuous on (ϱ_1, ϱ_2), $\lambda \in \mathbb{R}$ and $\sigma > 0$. The ψ-tempered fractional integral operators of a function $\widehat{\aleph}$ of order σ and index λ are defined by

$${}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma;\psi}\widehat{\aleph}(\delta) = \int_{\varrho_1}^{\delta} \frac{(\psi(\delta)-\psi(\gamma))^{\sigma-1}}{\Gamma(\sigma)} e^{-\lambda(\psi(\delta)-\psi(\gamma))}\psi'(\gamma)\widehat{\aleph}(\gamma)d\gamma,$$
$${}_{\lambda}^{T}\mathcal{J}_{\varrho_2-}^{\sigma;\psi}\widehat{\aleph}(\delta) = \int_{\delta}^{\varrho_2} \frac{(\psi(\gamma)-\psi(\delta))^{\sigma-1}}{\Gamma(\sigma)} e^{-\lambda(\psi(\gamma)-\psi(\delta))}\psi'(\gamma)\widehat{\aleph}(\gamma)d\gamma.$$

Obviously, the ψ-tempered fractional integral ${}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma;\psi}$ reduces to the ψ-Riemann-Liouville fractional integral [4,5] if $\lambda = 0$.

By incorporating Definition 2.2 and Definition 2.3, we can now present the subsequent definition of a broader fractional integral that encompasses both integrals as specific instances.

Definition 2.4 (The (κ,ψ)-tempered fractional Integral)**.** Let $\widehat{\aleph} \in X_{\psi}^{p}(\varrho_1, \varrho_2)$ and $[\varrho_1, \varrho_2]$ be a finite or infinite interval on the real axis $\mathbb{R}$, $\psi(\delta) > 0$ be an increasing function on $(\varrho_1, \varrho_2]$ and $\psi'(\delta) > 0$ be continuous on (ϱ_1, ϱ_2), $\lambda \in \mathbb{R}$, $\kappa > 0$ and $\sigma > 0$. The (κ,ψ)-tempered fractional integral operators of a function $\widehat{\aleph}$ of order σ and index λ are defined by

$$\begin{aligned}
{}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\widehat{\aleph}(\delta) &= e^{-\lambda\psi(\delta)}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}\left(\widehat{\aleph}(\delta)e^{\lambda\psi(\delta)}\right)\\
&= \int_{\varrho_1}^{\delta} \bar{\Psi}_{\sigma}^{\kappa,\psi}(\delta,\gamma)e^{-\lambda(\psi(\delta)-\psi(\gamma))}\psi'(\gamma)\widehat{\aleph}(\gamma)d\gamma,\\
{}_{\lambda}^{T}\mathcal{J}_{\varrho_2-}^{\sigma,\kappa;\psi}\widehat{\aleph}(\delta) &= e^{\lambda\psi(\delta)}\mathcal{J}_{\varrho_2-}^{\sigma,\kappa;\psi}\left(\widehat{\aleph}(\delta)e^{-\lambda\psi(\delta)}\right)\\
&= \int_{\delta}^{\varrho_2} \bar{\Psi}_{\sigma}^{\kappa,\psi}(\gamma,\delta)e^{-\lambda(\psi(\gamma)-\psi(\delta))}\psi'(\gamma)\widehat{\aleph}(\gamma)d\gamma,
\end{aligned}$$

with $\bar{\Psi}_{\sigma}^{\kappa,\psi}(\delta,\gamma) = \dfrac{(\psi(\delta)-\psi(\gamma))^{\frac{\sigma}{\kappa}-1}}{\kappa\Gamma_{\kappa}(\sigma)}$. Now, the (κ,ψ)-tempered fractional integral ${}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma,\kappa;\psi}$ reduces to the ψ-tempered fractional integral ${}_{\lambda}^{T}\mathcal{J}_{\varrho_1+}^{\sigma;\psi}$ if $\kappa = 1$.

2.2. **Fractional derivatives.** In this section, we present the definitions of various fractional derivatives that are utilized.

Definition 2.5 (κ-Generalized ψ-Hilfer Derivative [4,5])**.** Let $\jmath - 1 < \dfrac{\sigma}{\kappa} \leq \jmath$ with $\jmath \in \mathbb{N}$, $\nabla = [\varrho_1, \varrho_2]$ an interval such that $-\infty \leq \varrho_1 < \varrho_2 \leq \infty$ and $\widehat{\aleph}, \psi \in C^{\jmath}([\varrho_1, \varrho_2], \mathbb{R})$ two functions such that ψ is increasing

and $\psi'(\delta) \neq 0$, for all $\delta \in \nabla$. The κ-generalized ψ-Hilfer fractional derivatives ${}^{H}_{\kappa}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_1+}(\cdot)$ and ${}^{H}_{\kappa}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_2-}(\cdot)$ of a function $\widehat{\aleph}$ of order σ and type $0 \leq \varepsilon \leq 1$, with $\kappa > 0$ are defined by

$$\begin{aligned}
{}^{H}_{\kappa}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_1+}\widehat{\aleph}(\delta) &= \left(\mathcal{J}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_1+}\left(\frac{1}{\psi'(\delta)}\frac{d}{d\delta}\right)^{\jmath}\left(\kappa^{\jmath}\mathcal{J}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_1+}\widehat{\aleph}\right)\right)(\delta)\\
&= \left(\mathcal{J}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_1+}\delta^{\jmath}_{\psi}\left(\kappa^{\jmath}\mathcal{J}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_1+}\widehat{\aleph}\right)\right)(\delta)
\end{aligned}$$

and

$$\begin{aligned}
{}^{H}_{\kappa}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_2-}\widehat{\aleph}(\delta) &= \left(\mathcal{J}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_2-}\left(-\frac{1}{\psi'(\delta)}\frac{d}{d\delta}\right)^{\jmath}\left(\kappa^{\jmath}\mathcal{J}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_2-}\widehat{\aleph}\right)\right)(\delta)\\
&= \left(\mathcal{J}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_2-}(-1)^{\jmath}\delta^{\jmath}_{\psi}\left(\kappa^{\jmath}\mathcal{J}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_2-}\widehat{\aleph}\right)\right)(\delta),
\end{aligned}$$

where $\delta^{\jmath}_{\psi} = \left(\frac{1}{\psi'(\delta)}\frac{d}{d\delta}\right)^{\jmath}$.

Definition 2.6 (The tempered ψ-Hilfer Derivative [16])**.** Let $\jmath - 1 < \sigma \leq \jmath$ with $\jmath \in \mathbb{N}$, $\lambda \in \mathbb{R}$, $\nabla = [\varrho_1, \varrho_2]$ an interval such that $-\infty \leq \varrho_1 < \varrho_2 \leq \infty$ and $\widehat{\aleph}, \psi \in C^{\jmath}([\varrho_1, \varrho_2], \mathbb{R})$ two functions such that ψ is increasing and $\psi'(\delta) \neq 0$, for all $\delta \in \nabla$. The tempered ψ-Hilfer fractional derivatives ${}^{TH}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}(\cdot)$ and ${}^{TH}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_2-}(\cdot)$ of a function $\widehat{\aleph}$ of order σ, index λ and type $0 \leq \varepsilon \leq 1$, are defined by

$$\begin{aligned}
{}^{TH}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\widehat{\aleph}(\delta) &= \left({}^{T}_{\lambda}\mathcal{J}^{\varepsilon(\jmath-\sigma);\psi}_{\varrho_1+}\left(\frac{1}{\psi'(\delta)}\frac{d}{d\delta}+\lambda\right)^{\jmath}\left({}^{T}_{\lambda}\mathcal{J}^{(1-\varepsilon)(\jmath-\sigma);\psi}_{\varrho_1+}\widehat{\aleph}\right)\right)(\delta)\\
&= \left({}^{T}_{\lambda}\mathcal{J}^{\varepsilon(\jmath-\sigma);\psi}_{\varrho_1+}\mho^{\jmath}_{\psi}\left({}^{T}_{\lambda}\mathcal{J}^{(1-\varepsilon)(\jmath-\sigma);\psi}_{\varrho_1+}\widehat{\aleph}\right)\right)(\delta)\\
&= e^{-\lambda\psi(\delta)} \times {}^{H}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_1+}\left(\widehat{\aleph}(\delta)e^{\lambda\psi(\delta)}\right)
\end{aligned}$$

and

$$\begin{aligned}
{}^{TH}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_2-}\widehat{\aleph}(\delta) &= \left({}^{T}_{\lambda}\mathcal{J}^{\varepsilon(\jmath-\sigma);\psi}_{\varrho_2-}\left(-\frac{1}{\psi'(\delta)}\frac{d}{d\delta}+\lambda\right)^{\jmath}\left({}^{T}_{\lambda}\mathcal{J}^{(1-\varepsilon)(\jmath-\sigma);\psi}_{\varrho_2-}\widehat{\aleph}\right)\right)(\delta)\\
&= \left({}^{T}_{\lambda}\mathcal{J}^{\varepsilon(\jmath-\sigma);\psi}_{\varrho_2-}(-1)^{\jmath}\mho^{\jmath}_{\psi}\left({}^{T}_{\lambda}\mathcal{J}^{(1-\varepsilon)(\jmath-\sigma);\psi}_{\varrho_2-}\widehat{\aleph}\right)\right)(\delta)\\
&= e^{\lambda\psi(\delta)} \times {}^{H}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_2-}\left(\widehat{\aleph}(\delta)e^{-\lambda\psi(\delta)}\right),
\end{aligned}$$

where $\mho^{\jmath}_{\psi} = \left(\frac{1}{\psi'(\delta)}\frac{d}{d\delta}+\lambda\right)^{\jmath}$, ${}^{H}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_1+}(\cdot)$ and ${}^{H}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_2-}(\cdot)$ are the left-sided and right-sided ψ-Hilfer fractional derivatives, defined in [39].

By incorporating Definition 2.5 and Definition 2.6, we will now give the following definition of a more generalized fractional derivative that encompasses both tempered ψ-Hilfer derivative and κ-generalized ψ-Hilfer derivative as specific cases.

Definition 2.7 (The tempered (κ,ψ)-Hilfer Derivative [27])**.** Let $\jmath - 1 < \frac{\sigma}{\kappa} \leq \jmath$ with $\jmath \in \mathbb{N}$, $\lambda \in \mathbb{R}$, $\kappa > 0$, $\nabla = [\varrho_1, \varrho_2]$ an interval such that $-\infty \leq \varrho_1 < \varrho_2 \leq \infty$ and $\widehat{\aleph}, \psi \in C^{\jmath}([\varrho_1, \varrho_2], \mathbb{R})$ two functions such that ψ is increasing and $\psi'(\delta) \neq 0$, for all $\delta \in \nabla$. The tempered (κ,ψ)-Hilfer fractional derivatives (left-sided and right-sided) ${}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}(\cdot)$ and ${}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_2-}(\cdot)$ of a function $\widehat{\aleph}$ of order σ, index λ and type $0 \leq \varepsilon \leq 1$, are defined by

$$\begin{aligned}
{}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\widehat{\aleph}(\delta) &= \left({}^{T}_{\lambda}\mathcal{J}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_1+}\left(\frac{1}{\psi'(\delta)}\frac{d}{d\delta}+\lambda\right)^{\jmath}\left(\kappa^{\jmath}\,{}^{T}_{\lambda}\mathcal{J}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_1+}\widehat{\aleph}\right)\right)(\delta)\\
&= \left({}^{T}_{\lambda}\mathcal{J}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_1+}\mho^{\jmath}_{\psi}\left(\kappa^{\jmath}\,{}^{T}_{\lambda}\mathcal{J}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_1+}\widehat{\aleph}\right)\right)(\delta)\\
&= e^{-\lambda\psi(\delta)} \times {}^{H}_{\kappa}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_1+}\left(\widehat{\aleph}(\delta)e^{\lambda\psi(\delta)}\right)
\end{aligned}$$

and

$$\begin{aligned}
{}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_2-}\widehat{\aleph}(\delta) &= \left({}^{T}_{\lambda}\mathcal{J}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_2-}\left(-\frac{1}{\psi'(\delta)}\frac{d}{d\delta}+\lambda\right)^{\jmath}\left(\kappa^{\jmath}\,{}^{T}_{\lambda}\mathcal{J}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_2-}\widehat{\aleph}\right)\right)(\delta)\\
&= \left({}^{T}_{\lambda}\mathcal{J}^{\varepsilon(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_2-}(-1)^{\jmath}\mho^{\jmath}_{\psi}\left(\kappa^{\jmath}\,{}^{T}_{\lambda}\mathcal{J}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_2-}\widehat{\aleph}\right)\right)(\delta)\\
&= e^{\lambda\psi(\delta)}\times{}^{H}_{\kappa}\mathcal{D}^{\sigma,\varepsilon;\psi}_{\varrho_2-}\left(\widehat{\aleph}(\delta)e^{-\lambda\psi(\delta)}\right),
\end{aligned}$$

where $\mho^{\jmath}_{\psi}=\left(\dfrac{1}{\psi'(\delta)}\dfrac{d}{d\delta}+\lambda\right)^{\jmath}$. The tempered (κ,ψ)-Hilfer fractional derivative ${}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}$ reduces to the tempered ψ-Hilfer fractional derivative ${}^{TH}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}$ if $\kappa=1$.

2.3. **Necessary properties of fractional operators.** Subsequently, employing Definition 2.4, particularly highlighting the notion that we have the ability to write

$$ {}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\widehat{\aleph}(\delta)=e^{-\lambda\psi(\delta)}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\left(\widehat{\aleph}(\delta)e^{\lambda\psi(\delta)}\right), \tag{2.1}$$

and by following the same steps of Theorem 2.2 and Theorem 2.3 from [35], we can deduce the following theorems.

Theorem 2.8 ([27]). *Let $\widehat{\aleph}:[\varrho_1,\varrho_2]\to\mathbb{R}$ be an integrable function, and take $\sigma>0$, $\lambda\in\mathbb{R}$ and $\kappa>0$. Then ${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\widehat{\aleph}$ exists for all $\delta\in[\varrho_1,\varrho_2]$.*

Theorem 2.9 ([27]). *Let $\widehat{\aleph}\in X^{p}_{\psi}(\varrho_1,\varrho_2)$ and take $\sigma>0$, $\lambda\in\mathbb{R}$ and $\kappa>0$. Then ${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\widehat{\aleph}\in C([\varrho_1,\varrho_2],\mathbb{R})$.*

Lemma 2.10 ([27]). *Let $\sigma>0$, $\varepsilon>0$, $\lambda\in\mathbb{R}$ and $\kappa>0$. Then, we have the following semigroup property given by*

$${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\,{}^{T}_{\lambda}\mathcal{J}^{\varepsilon,\kappa;\psi}_{\varrho_1+}\aleph(\delta)={}^{T}_{\lambda}\mathcal{J}^{\sigma+\varepsilon,\kappa;\psi}_{\varrho_1+}\aleph(\delta)={}^{T}_{\lambda}\mathcal{J}^{\varepsilon,\kappa;\psi}_{\varrho_1+}\,{}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\aleph(\delta)$$

and

$${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_2-}\,{}^{T}_{\lambda}\mathcal{J}^{\varepsilon,\kappa;\psi}_{\varrho_2-}\aleph(\delta)={}^{T}_{\lambda}\mathcal{J}^{\sigma+\varepsilon,\kappa;\psi}_{\varrho_2-}\aleph(\delta)={}^{T}_{\lambda}\mathcal{J}^{\varepsilon,\kappa;\psi}_{\varrho_2-}\,{}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_2-}\aleph(\delta).$$

Lemma 2.11 ([35]). *Let $\sigma,\varepsilon>0$, and $\kappa>0$. Then, we have*

$$\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\bar{\Psi}^{\kappa,\psi}_{\varepsilon}(\delta,\varrho_1)=\bar{\Psi}^{\kappa,\psi}_{\sigma+\varepsilon}(\delta,\varrho_1)$$

and

$$\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_2-}\bar{\Psi}^{\kappa,\psi}_{\varepsilon}(\varrho_2,\delta)=\bar{\Psi}^{\kappa,\psi}_{\sigma+\varepsilon}(\varrho_2,\delta).$$

Lemma 2.12 ([27]). *Let $\sigma,\varepsilon>0$, $\lambda\in\mathbb{R}$ and $\kappa>0$. Then, we have*

$${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\bar{\Psi}^{\kappa,\psi}_{\varepsilon}(\delta,\varrho_1)=e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\,\bar{\Psi}^{\kappa,\psi}_{\sigma+\varepsilon}(\delta,\varrho_1)$$

and

$${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_2-}e^{\lambda(\psi(\delta)-\psi(\varrho_1))}\bar{\Psi}^{\kappa,\psi}_{\varepsilon}(\varrho_2,\delta)=e^{\lambda(\psi(\delta)-\psi(\varrho_1))}\,\bar{\Psi}^{\kappa,\psi}_{\sigma+\varepsilon}(\varrho_2,\delta).$$

Theorem 2.13 ([27]). *Let $0<\varrho_1<\varrho_2<\infty,\sigma>0,0\le\theta<1$, $\lambda\in\mathbb{R}$, $\kappa>0$ and $\mathfrak{w}\in C_{\theta;\psi}(\nabla)$. If $\dfrac{\sigma}{\kappa}>1-\theta$, then*

$$\left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\varrho_1)=\lim_{\delta\to\varrho_1{}^{+}}\left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta)=0.$$

Lemma 2.14 ([27]). *Let $\delta>\varrho_1,\sigma>0,0\le\varepsilon\le1,\lambda\in\mathbb{R},\kappa>0$. Then for $0<\theta<1;\theta=\frac{1}{\kappa}(\varepsilon(\kappa-\sigma)+\sigma)$, we have*

$$\left[{}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\left(\Psi^{\psi}_{\theta}(\gamma,\varrho_1)\right)^{-1}e^{-\lambda(\psi(\gamma)-\psi(\varrho_1))}\right](\delta)=0.$$

Theorem 2.15 ([27]). *If* $\aleph \in C^{\jmath}_{\theta;\psi}[\varrho_1,\varrho_2], \jmath-1<\dfrac{\sigma}{\kappa}<\jmath, 0\leq\varepsilon\leq 1, \lambda\in\mathbb{R}$, *where* $\jmath\in\mathbb{N}$ *and* $\kappa>0$, *then*

$$\begin{aligned}&\left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\ {}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\aleph\right)(\delta)\\&=\aleph(\delta)-e^{-\lambda\psi(\delta)}\sum_{i=1}^{\jmath}\frac{(\psi(\delta)-\psi(\varrho_1))^{\theta-i}}{\kappa^{i-\jmath}\Gamma_{\kappa}(\kappa(\theta-i+1))}\left\{\delta^{\jmath-i}_{\psi}\left(\mathcal{J}^{(1-\varepsilon)(\kappa\jmath-\sigma),\kappa;\psi}_{\varrho_1+}\aleph(\varrho_1)e^{\lambda\psi(\varrho_1)}\right)\right\},\end{aligned}$$

where

$$\theta=\frac{1}{\kappa}\left(\varepsilon(\kappa\jmath-\sigma)+\sigma\right).$$

In particular, if $\jmath=1$, *we have*

$$\begin{aligned}&\left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\ {}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\aleph\right)(\delta)\\&=\aleph(\delta)-e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\frac{(\psi(\delta)-\psi(\varrho_1))^{\theta-1}}{\Gamma_{\kappa}(\varepsilon(\kappa-\sigma)+\sigma)}\ {}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\aleph(\varrho_1).\end{aligned}$$

Lemma 2.16 ([27]). *Let* $\sigma>0, 0\leq\varepsilon\leq 1, \lambda\in\mathbb{R}$, *and* $\mathfrak{w}\in C^{1}_{\theta;\psi}(\nabla)$, *where* $\kappa>0$, *then for* $\delta\in(\varrho_1,\varrho_2]$, *we have*

$$\left({}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\ {}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta)=\mathfrak{w}(\delta).$$

Lemma 2.17. *Let* $\sigma,\kappa>0$ *and* $\lambda\in\mathbb{R}$. *Then, we have*

$$\begin{aligned}&{}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)\\&=e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\left[\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)-1\right],\end{aligned}$$

and

$${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)\leq\widehat{\lambda}\left[\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)-1\right].$$

Proof. By using relation (2.1), we may write the following

$$\begin{aligned}&{}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)\\&=e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)&=\int_{\varrho_1}^{\delta}\psi'(\gamma)\bar{\Psi}^{\kappa,\psi}_{\sigma}(\delta,\gamma)\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\gamma)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)d\gamma\\&=\int_{\varrho_1}^{\delta}\psi'(\gamma)\bar{\Psi}^{\kappa,\psi}_{\sigma}(\delta,\gamma)\sum_{i=0}^{\infty}\frac{(\psi(\gamma)-\psi(\varrho_1))^{\frac{\sigma i}{\kappa}}}{\Gamma_{\kappa}(\sigma i+\kappa)}d\gamma.\end{aligned}$$

With $\mu=\psi(\gamma)-\psi(\varrho_1)$, we get

$$\begin{aligned}&\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)\\&=\frac{1}{\kappa\Gamma_{\kappa}(\sigma)}\sum_{i=0}^{\infty}\frac{1}{\Gamma_{\kappa}(\sigma i+\kappa)}\int_{0}^{\psi(\delta)-\psi(\varrho_1)}\frac{\mu^{\frac{\sigma i}{\kappa}}}{(\psi(\delta)-\psi(\varrho_1)-\mu)^{1-\frac{\sigma}{\kappa}}}d\mu\\&=\frac{1}{\kappa\Gamma_{\kappa}(\sigma)}\sum_{i=0}^{\infty}\frac{(\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}-1}}{\Gamma_{\kappa}(\sigma i+\kappa)}\int_{0}^{\psi(\delta)-\psi(\varrho_1)}\mu^{\frac{\sigma i}{\kappa}}\left(1-\frac{\mu}{\psi(\delta)-\psi(\varrho_1)}\right)^{\frac{\sigma}{\kappa}-1}d\mu.\end{aligned}$$

Making the change of variables $\nabla = \dfrac{\mu}{\psi(\delta)-\psi(\varrho_1)}$ and using the definition of κ-beta function, we have

$$\begin{aligned}
&\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)\\
&=\frac{1}{\kappa\Gamma_{\kappa}(\sigma)}\sum_{i=0}^{\infty}\frac{(\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}(i+1)}}{\Gamma_{\kappa}(\sigma i+\kappa)}\int_0^1\nabla^{\frac{\sigma i}{\kappa}}(1-\nabla)^{\frac{\sigma}{\kappa}-1}\,d\nabla\\
&=\frac{1}{\Gamma_{\kappa}(\sigma)}\sum_{i=0}^{\infty}\frac{(\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}(i+1)}\,\Gamma_{\kappa}(\sigma i+\kappa)\Gamma_{\kappa}(\sigma)}{\Gamma_{\kappa}(\sigma i+\kappa)\Gamma_{\kappa}(\sigma(i+1)+\kappa)}\\
&=\sum_{i=0}^{\infty}\frac{(\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}(i+1)}}{\Gamma_{\kappa}(\sigma(i+1)+\kappa)}\\
&=\sum_{j=0}^{\infty}\frac{(\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma j}{\kappa}}}{\Gamma_{\kappa}(\sigma j+\kappa)}-1.
\end{aligned}$$

Thus, we have

$$\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)=\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)-1.$$

Consequently,

$$\begin{aligned}
&{}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)\\
&\quad=e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\left[\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)-1\right].
\end{aligned}$$

The second result can be proved by following the same steps. □

3. The Gronwall Inequality

In this section, we present a generalized Gronwall inequality that will play a crucial role in our Ulam stability results. The proof of this result incorporates the properties of the functions κ-gamma, κ-beta, and κ-Mittag-Leffler.

Theorem 3.1. *Let $\mathfrak{w}$, $\mathfrak{z}$ be two integrable functions and $\mathfrak{x}$ continuous, with domain $[\varrho_1,\varrho_2]$. Let $\psi\in C^1[\varrho_1,\varrho_2]$ an increasing function such that $\psi'(\delta)\neq 0$, $\delta\in[\varrho_1,\varrho_2]$, $\sigma>0$, $\kappa>0$ and $\lambda\in\mathbb{R}$. Assume that:*

(1) *$\mathfrak{w}$ and $\mathfrak{z}$ are nonnegative;*
(2) *$\mathfrak{x}$ is nonnegative and nondecreasing.*

If

$$\mathfrak{w}(\delta)\leq\mathfrak{z}(\delta)+\mathfrak{x}(\delta)\,\Gamma_{\kappa}(\sigma)\int_{\varrho_1}^{\delta}\psi'(\gamma)\,e^{-\lambda(\psi(\delta)-\psi(\gamma))}\bar{\Psi}^{\kappa,\psi}_{\sigma}(\delta,\gamma)\mathfrak{w}(\gamma)\,d\gamma,$$

then

$$\mathfrak{w}(\delta)\leq\mathfrak{z}(\delta)+\int_{\varrho_1}^{\delta}\sum_{i=1}^{\infty}\left[\widehat{\lambda}\mathfrak{x}(\delta)\,\Gamma_{\kappa}(\sigma)\right]^{i}\psi'(\gamma)\,\bar{\Psi}^{\kappa,\psi}_{i\sigma}(\delta,\gamma)\mathfrak{z}(\gamma)\,d\gamma, \tag{3.1}$$

for all $\delta\in[\varrho_1,\varrho_2]$, where

$$\widehat{\lambda}:=\max_{(\delta,\gamma)\in[\varrho_1,\varrho_2]\times[\varrho_1,\delta]}e^{-\lambda(\psi(\delta)-\psi(\gamma))}=\begin{cases}1, & \text{if } \lambda\geq 0,\\ e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}, & \text{if } \lambda<0.\end{cases}$$

And if $\mathfrak{z}$ is a nondecreasing function on $[\varrho_1,\varrho_2]$, then we have

$$\mathfrak{w}(\delta)\leq\mathfrak{z}(\delta)\,\mathbb{E}^{\sigma}_{\kappa}\left(\widehat{\lambda}\mathfrak{x}(\delta)\,\Gamma_{\kappa}(\sigma)\,(\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right).$$

Proof. Let

$$\Upsilon v(\delta) = \widehat{\lambda}_{\mathfrak{x}}(\delta)\Gamma_{\kappa}(\sigma)\int_{\varrho_1}^{\delta}\psi'(\xi)\,\bar{\Psi}_{\sigma}^{\kappa,\psi}(\delta,\xi)v(\xi)\,d\xi, \tag{3.2}$$

for all $\delta \in [\varrho_1, \varrho_2]$ and locally integral function v. Then,

$$\mathfrak{w}(\delta) \leq \mathfrak{z}(\delta) + \Upsilon\mathfrak{w}(\delta).$$

Iterating for $\jmath \in \mathbb{N}$, we can write

$$\mathfrak{w}(\delta) \leq \sum_{i=0}^{\jmath-1}\Upsilon^{i}\mathfrak{z}(\delta) + \Upsilon^{\jmath}\mathfrak{w}(\delta).$$

Thus, by mathematical induction, and if $\mathfrak{w}$ is a nonnegative function, we prove the following relation:

$$\Upsilon^{\jmath}\mathfrak{w}(\delta) \leq \int_{\varrho_1}^{\delta}\left[\widehat{\lambda}_{\mathfrak{x}}(\delta)\,\Gamma_{\kappa}(\sigma)\right]^{\jmath}\psi'(\xi)\,\bar{\Psi}_{\jmath\sigma}^{\kappa,\psi}(\delta,\xi)\mathfrak{w}(\xi)\,d\xi. \tag{3.3}$$

We know that relation (3.3) is true for $\jmath = 1$. Suppose that the formula is true for some $\jmath = i \in \mathbb{N}$, then the induction hypothesis implies

$$\begin{aligned}
\Upsilon^{i+1}\mathfrak{w}(\delta) &= \Upsilon\left(\Upsilon^{i}\mathfrak{w}(\delta)\right)\\
&\leq \Upsilon\left(\int_{\varrho_1}^{\delta}\left[\widehat{\lambda}_{\mathfrak{x}}(\delta)\,\Gamma_{\kappa}(\sigma)\right]^{i}\psi'(\xi)\,\bar{\Psi}_{i\sigma}^{\kappa,\psi}(\delta,\xi)\mathfrak{w}(\xi)\,d\xi\right)\\
&= \widehat{\lambda}_{\mathfrak{x}}(\delta)\Gamma_{\kappa}(\sigma)\\
&\quad\times\int_{\varrho_1}^{\delta}\psi'(\xi)\,\bar{\Psi}_{\sigma}^{\kappa,\psi}(\delta,\xi)\left(\int_{\varrho_1}^{\xi}\left[\widehat{\lambda}_{\mathfrak{x}}(\xi)\,\Gamma_{\kappa}(\sigma)\right]^{i}\psi'(\gamma)\,\mathfrak{w}(\gamma)\,\bar{\Psi}_{i\sigma}^{\kappa,\psi}(\delta,\gamma)d\gamma\right)d\xi.
\end{aligned}$$

Since $\mathfrak{x}$ is a nondecreasing function, that is $\mathfrak{x}(\xi) \leq \mathfrak{x}(\delta)$, for all $\xi \leq \delta$, then we obtain

$$\Upsilon^{i+1}\mathfrak{w}(\delta) \leq \left[\widehat{\lambda}_{\mathfrak{x}}(\delta)\,\Gamma_{\kappa}(\sigma)\right]^{i+1}\int_{\varrho_1}^{\delta}\int_{\varrho_1}^{\xi}\psi'(\xi)\,\bar{\Psi}_{\sigma}^{\kappa,\psi}(\delta,\xi)\psi'(\gamma)\,\bar{\Psi}_{i\sigma}^{\kappa,\psi}(\delta,\gamma)\mathfrak{w}(\gamma)\,d\gamma d\xi. \tag{3.4}$$

From Equation (3.4) and by Dirichlet's formula, we can have

$$\Upsilon^{i+1}\mathfrak{w}(\delta) \leq \left[\widehat{\lambda}_{\mathfrak{x}}(\delta)\,\Gamma_{\kappa}(\sigma)\right]^{i+1}\int_{\varrho_1}^{\delta}\psi'(\xi)\,\mathfrak{w}(\xi)\int_{\xi}^{\delta}\bar{\Psi}_{\sigma}^{\kappa,\psi}(\delta,\gamma)\psi'(\gamma)\,\bar{\Psi}_{i\sigma}^{\kappa,\psi}(\gamma,\xi)d\gamma d\xi. \tag{3.5}$$

On other hand, we have

$$\begin{aligned}
&\int_{\xi}^{\delta}\psi'(\gamma)\left[\psi(\delta)-\psi(\gamma)\right]^{\frac{\sigma}{\kappa}-1}\left[\psi(\gamma)-\psi(\xi)\right]^{\frac{i\sigma}{\kappa}-1}d\gamma\\
&\quad=\int_{\xi}^{\delta}\psi'(\gamma)\left[\psi(\delta)-\psi(\xi)\right]^{\frac{\sigma}{\kappa}-1}\left[1-\frac{\psi(\gamma)-\psi(\xi)}{\psi(\delta)-\psi(\xi)}\right]^{\frac{\sigma}{\kappa}-1}\left[\psi(\gamma)-\psi(\xi)\right]^{\frac{i\sigma}{\kappa}-1}d\gamma.
\end{aligned}$$

With a change of variables $\mu = \dfrac{\psi(\gamma)-\psi(\xi)}{\psi(\delta)-\psi(\xi)}$ and using the definition of κ-beta function and the relation with gamma function $B_\kappa(\varsigma,\widehat{\varsigma}) = \dfrac{\Gamma_\kappa(\varsigma)\Gamma_\kappa(\widehat{\varsigma})}{\Gamma_\kappa(\varsigma+\widehat{\varsigma})}$, we have

$$\begin{aligned}
&\int_\xi^\delta \psi'(\gamma)\left[\psi(\delta)-\psi(\gamma)\right]^{\frac{\sigma}{\kappa}-1}\left[\psi(\gamma)-\psi(\xi)\right]^{\frac{i\sigma}{\kappa}-1}d\gamma \\
&= \left[\psi(\delta)-\psi(\xi)\right]^{\frac{i\sigma+\sigma}{\kappa}-1}\int_0^1 \left[1-\mu\right]^{\frac{\sigma}{\kappa}-1}\mu^{\frac{i\sigma}{\kappa}-1}d\mu \\
&= \kappa\left[\psi(\delta)-\psi(\xi)\right]^{\frac{i\sigma+\sigma}{\kappa}-1}\frac{\Gamma_\kappa(\sigma)\Gamma_\kappa(i\sigma)}{\Gamma_\kappa(\sigma+i\sigma)}. \qquad (3.6)
\end{aligned}$$

By replacing Equation (3.6) in Equation (3.5), we get

$$\Upsilon^{i+1}\mathfrak{w}(\delta) \le \int_{\varrho_1}^\delta \left[\widehat{\lambda}_{\mathfrak{x}}(\delta)\Gamma_\kappa(\sigma)\right]^{i+1}\psi'(\xi)\,\mathfrak{w}(\xi)\,\bar{\Psi}^{\kappa,\psi}_{(i+1)\sigma}(\delta,\xi)d\xi.$$

Let us now prove that $\Upsilon^\jmath\mathfrak{w}(\delta)\to 0$ as $\jmath\to\infty$. Since $\mathfrak{x}$ is a continuous function on $[\varrho_1,\varrho_2]$, there exist a constant $F>0$ such that $\widehat{\lambda}_{\mathfrak{x}}(\delta)\le F$ for all $\delta\in[\varrho_1,\varrho_2]$. Then, we obtain

$$\Upsilon^\jmath\mathfrak{w}(\delta) \le \int_{\varrho_1}^\delta \left[F\Gamma_\kappa(\sigma)\right]^\jmath\psi'(\xi)\,\mathfrak{w}(\xi)\,\bar{\Psi}^{\kappa,\psi}_{\jmath\sigma}(\delta,\xi)d\xi.$$

Consider the series

$$\sum_{\jmath=1}^\infty \frac{\left[F\Gamma_\kappa(\sigma)\right]^\jmath}{\Gamma_\kappa(\sigma\jmath)}.$$

Using the property of the generalized κ-gamma, we have

$$\sum_{\jmath=1}^\infty \frac{\left[F\Gamma_\kappa(\sigma)\right]^\jmath}{\Gamma_\kappa(\sigma\jmath)} = \sum_{\jmath=1}^\infty \frac{\left[F\kappa^{\frac{\sigma}{\kappa}-1}\Gamma\left(\frac{\sigma}{\kappa}\right)\right]^\jmath}{\kappa^{\frac{\sigma\jmath}{\kappa}-1}\Gamma\left(\frac{\sigma\jmath}{\kappa}\right)} = \sum_{\jmath=1}^\infty \frac{\kappa\left[F\kappa^{-1}\Gamma\left(\frac{\sigma}{\kappa}\right)\right]^\jmath}{\Gamma\left(\frac{\sigma\jmath}{\kappa}\right)},$$

By using Stirling approximation and the root test, we can show that the series converges. Therefore, we conclude that

$$\mathfrak{w}(\delta) \le \sum_{i=0}^\infty \Upsilon^i\mathfrak{z}(\delta) \le \mathfrak{z}(\delta) + \int_{\varrho_1}^\delta \sum_{i=1}^\infty \left[\widehat{\lambda}_{\mathfrak{x}}(\delta)\Gamma_\kappa(\sigma)\right]^i\psi'(\xi)\,\bar{\Psi}^{\kappa,\psi}_{i\sigma}(\delta,\xi)\mathfrak{z}(\xi)\,d\xi.$$

Now, since $\mathfrak{z}$ is nondecreasing, so, for all $\xi\in[\varrho_1,\delta]$, we have $\mathfrak{z}(\xi)\le\mathfrak{z}(\delta)$ and we can write

$$\begin{aligned}
\mathfrak{w}(\delta) &\le \mathfrak{z}(\delta) + \int_{\varrho_1}^\delta \sum_{i=1}^\infty \left[\widehat{\lambda}_{\mathfrak{x}}(\delta)\Gamma_\kappa(\sigma)\right]^i\psi'(\xi)\,\bar{\Psi}^{\kappa,\psi}_{i\sigma}(\delta,\xi)\mathfrak{z}(\xi)\,d\xi \\
&\le \mathfrak{z}(\delta)\left[1+\int_{\varrho_1}^\delta \sum_{i=1}^\infty \left[\widehat{\lambda}_{\mathfrak{x}}(\delta)\Gamma_\kappa(\sigma)\right]^i\psi'(\xi)\,\bar{\Psi}^{\kappa,\psi}_{i\sigma}(\delta,\xi)d\xi\right] \\
&= \mathfrak{z}(\delta)\left[1+\sum_{i=1}^\infty \frac{\left[\widehat{\lambda}_{\mathfrak{x}}(\delta)\Gamma_\kappa(\sigma)\right]^i}{\sigma i\Gamma_\kappa(\sigma i)}\left[\psi(\delta)-\psi(\varrho_1)\right]^{\frac{\sigma i}{\kappa}}\right],
\end{aligned}$$

and by using the properties of κ-gamma function and the definition of κ-Mittag-Leffler function in Definition 2.1, we have

$$\begin{aligned}
\mathfrak{w}(\delta) &\le \mathfrak{z}(\delta)\left[1+\sum_{i=1}^\infty \kappa\left[\widehat{\lambda}_{\mathfrak{x}}(\delta)\Gamma_\kappa(\sigma)\right]^i\bar{\Psi}^{\kappa,\psi}_{i\sigma+\kappa}(\delta,\varrho_1)\right] \\
&= \mathfrak{z}(\delta)\,\mathbb{E}^\sigma_\kappa\left(\widehat{\lambda}_{\mathfrak{x}}(\delta)\Gamma_\kappa(\sigma)\left(\psi(\delta)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}\right).
\end{aligned}$$

□

4. Existence of Solutions

We consider the following fractional differential equation

$$\left({}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta)=\varpi(\delta),\ \ \delta\in(\varrho_1,\varrho_2], \tag{4.1}$$

where $0<\sigma<\kappa, 0\leq\varepsilon\leq 1, \lambda\in\mathbb{R}$ with the condition

$$\varkappa_1\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\varrho_1{}^{+})+\varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\varrho_2)=\varkappa_3, \tag{4.2}$$

where $\theta=\frac{\varepsilon(\kappa-\sigma)+\sigma}{\kappa}$, $\varpi(\cdot)\in C(\nabla,\mathbb{R})$, $\varkappa_1,\varkappa_2,\varkappa_3\in\mathbb{R}$, $\varkappa_1+\varkappa_2e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\neq 0$.

The following theorem shows that the problem (4.1)-(4.2) have a unique solution.

Theorem 4.1. *Let $0<\sigma<\kappa, 0\leq\varepsilon\leq 1, \lambda\in\mathbb{R}$, $\kappa>0$, $\varpi(\cdot)\in C(\nabla,\mathbb{R})$. The problem (4.1)-(4.2) has a unique solution given by:*

$$\mathfrak{w}(\delta)=\frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}}{\Psi^{\psi}_{\theta}(\delta,\varrho_1)\Gamma_{\kappa}(\kappa\theta)}\left[\frac{\varkappa_3-\varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\varrho_2)}{\varkappa_1+\varkappa_2e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}}\right]+\left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\delta). \tag{4.3}$$

Proof. Assume $\mathfrak{w}$ satisfies (4.1)-(4.2). By applying the fractional integral operator ${}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}(\cdot)$ on both sides of the fractional equation (4.1) and using Theorem 2.15, we obtain

$$\mathfrak{w}(\delta)=\frac{{}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}(\varrho_1)}{\Psi^{\psi}_{\theta}(\delta,\varrho_1)\Gamma_{\kappa}(\kappa\theta)}e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}+\left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\delta). \tag{4.4}$$

Applying $\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}(\cdot)$ on both sides of (4.4), using Lemma 2.10, Lemma 2.12 and taking $\delta=\varrho_2$, we have

$$\begin{aligned}&\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\varrho_2)\\ &=\ e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}{}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}(\varrho_1)+\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\varrho_2).\end{aligned} \tag{4.5}$$

Multiplying both sides of (4.5) by $\varkappa_2$, we get

$$\begin{aligned}&\varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\varrho_2)\\ &=\ \varkappa_2e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}{}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}(\varrho_1)+\varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\varrho_2).\end{aligned}$$

Using condition (4.2), we obtain

$$\varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\varrho_2)=\varkappa_3-\varkappa_1\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\varrho_1{}^{+}).$$

Thus,

$$\begin{aligned}\varkappa_3-\varkappa_1\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\varrho_1{}^{+})=\varkappa_2e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}{}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}(\varrho_1)\\ +\varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\varrho_2).\end{aligned}$$

Then,

$$\begin{aligned}\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\varrho_1{}^{+})&=\frac{\varkappa_3}{\varkappa_1+\varkappa_2e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}}\\ &\quad-\frac{\varkappa_2}{\varkappa_1+\varkappa_2e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}}\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\varrho_2).\end{aligned} \tag{4.6}$$

Substituting (4.6) into (4.4), we obtain (4.3).

Reciprocally, applying ${}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}$ on both sides of (4.3) and using Lemma 2.12 and Lemma 2.10, we get

$$\begin{aligned}\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta) = &\left[\frac{\varkappa_3-\varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\varrho_2)}{\varkappa_1+\varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}}\right]e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\\ &+\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\delta),\ \delta\in(\varrho_1,\varrho_2].\end{aligned}\tag{4.7}$$

Next, taking the limit $\delta\to{\varrho_1}^{+}$ of (4.7) and using Theorem 2.13, with $\kappa(1-\theta)<\kappa(1-\theta)+\sigma$, we obtain

$$\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)({\varrho_1}^{+}) = \frac{\varkappa_3-\varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\varrho_2)}{\varkappa_1+\varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}}.\tag{4.8}$$

Now, taking $\delta=\varrho_2$ in (4.7), to get

$$\begin{aligned}\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\varrho_2) = &\left[\frac{\varkappa_3-\varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\varrho_2)}{\varkappa_1+\varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}}\right]e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\\ &+\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\varrho_2).\end{aligned}\tag{4.9}$$

From (4.8) and (4.9), we obtain

$$\begin{aligned}&\varkappa_1\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)({\varrho_1}^{+})+\varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)(\varrho_2)\\ &=\left[\frac{\varkappa_2\varkappa_3-\varkappa_2^2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\varrho_2)}{\varkappa_1+\varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}}\right]\\ &\quad\times e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}+\varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\varrho_2)\\ &\quad+\frac{\varkappa_1\varkappa_3-\varkappa_1\varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\varrho_2)}{\varkappa_1+\varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}}\\ &=\varkappa_3,\end{aligned}$$

which proves that (4.2) is satisfied. Apply ${}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}(\cdot)$ on both sides of (4.3). Then, from Lemma 2.14 and Lemma 2.16 we obtain equation (4.1). □

Lemma 4.2. *Let $\theta=\dfrac{\varepsilon(\kappa-\sigma)+\sigma}{\kappa}$ where $0<\sigma<\kappa$ and $0\le\varepsilon\le 1,\lambda\in\mathbb{R}$, let $\aleph:\nabla\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Then, the problem (1.1)-(1.2) is equivalent to the following integral equation:*

$$\mathfrak{w}(\delta)=\frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}}{\Psi^{\psi}_{\theta}(\delta,\varrho_1)\Gamma_{\kappa}(\kappa\theta)}\left[\frac{\varkappa_3-\varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\varrho_2)}{\varkappa_1+\varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}}\right]+\left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\varpi\right)(\delta),\tag{4.10}$$

where ϖ be a function satisfying the functional equation

$$\varpi(\delta)=\aleph(\delta,\mathfrak{w}(\delta),\varpi(\delta)).$$

The following hypotheses will be used in the sequel :

(*Ax1*) The function $\aleph:\nabla\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous.

(*Ax2*) There exist constants $\zeta_1>0$ and $0<\zeta_2<1$ such that

$$|\aleph(\delta,\mathfrak{w}_1,\mathfrak{z}_1)-\aleph(\delta,\mathfrak{w}_2,\mathfrak{z}_2)|\le\zeta_1|\mathfrak{w}_1-\mathfrak{w}_2|+\zeta_2|\mathfrak{z}_1-\mathfrak{z}_2|$$

for any $\mathfrak{w}_1,\mathfrak{w}_2,\mathfrak{z}_1,\mathfrak{z}_2\in\mathbb{R}$ and $\delta\in\nabla$.

We are now in a position to state and prove our existence result for the problem (1.1)-(1.2) based on based on Banach's fixed point theorem [9].

Theorem 4.3. *Assume* (Ax1)-(Ax2) *hold. If*

$$\mathcal{L} = \frac{\zeta_1 \left(\psi(\varrho_2) - \psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}}{1-\zeta_2} \left[\frac{|\varkappa_2|\widehat{\lambda}^2}{\Gamma_\kappa(\kappa+\sigma)\left|\varkappa_1 + \varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right|} + \frac{\widehat{\lambda}\Gamma_\kappa(\kappa\theta)}{\Gamma_\kappa(\sigma+\kappa\theta)} \right] < 1, \tag{4.11}$$

then the problem (1.1)-(1.2) *has a unique solution in* $C_{\theta;\psi}(\nabla)$.

Proof. Transform problem (1.1)-(1.2) into a fixed point problem by considering the operator $\mathcal{T} : C_{\theta;\psi}(\nabla) \to C_{\theta;\psi}(\nabla)$ by

$$\left(\mathcal{T}\mathfrak{w}\right)(\delta) = \frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\delta,\varrho_1)\Gamma_\kappa(\kappa\theta)} \left[\frac{\varkappa_3 - \varkappa_2 \left({}^T_\lambda \mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+} \varpi\right)(\varrho_2)}{\varkappa_1 + \varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}} \right] + \left({}^T_\lambda \mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+} \varpi\right)(\delta), \tag{4.12}$$

where ϖ be a function satisfying the functional equation

$$\varpi(\delta) = \aleph(\delta, \mathfrak{w}(\delta), \varpi(\delta)).$$

By Theorem 2.9, we have $\mathcal{T}\mathfrak{w} \in C_{\theta;\psi}(\nabla)$. We show that the operator $\mathcal{T}$ has a unique fixed point in $C_{\theta;\psi}(\nabla)$.

Let $\mathfrak{w}, \mathfrak{z} \in C_{\theta;\psi}(\nabla)$. Then for any for $\delta \in \nabla$, we have

$$\begin{aligned} |\mathcal{T}\mathfrak{w}(\delta) - \mathcal{T}\mathfrak{z}(\delta)| \leq{}& \frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\delta,\varrho_1)\Gamma_\kappa(\kappa\theta)} \left[\frac{|\varkappa_2| \left({}^T_\lambda \mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+} |\varpi_1(\gamma) - \varpi_2(\gamma)|\right)(\varrho_2)}{\left|\varkappa_1 + \varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right|} \right] \\ &+ \left({}^T_\lambda \mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1^+} |\varpi_1(\gamma) - \varpi_2(\gamma)|\right)(\delta), \end{aligned}$$

where ϖ_1 and ϖ_1 be functions satisfying the functional equations

$$\begin{aligned} \varpi_1(\delta) &= \aleph(\delta, \mathfrak{w}(\delta), \varpi_1(\delta)), \\ \varpi_2(\delta) &= \aleph(\delta, \mathfrak{z}(\delta), \varpi_2(\delta)). \end{aligned}$$

By (*Ax2*), we have

$$\begin{aligned} |\varpi_1(\delta) - \varpi_2(\delta)| &= |\aleph(\delta, \mathfrak{w}(\delta), \varpi_1(\delta)) - \aleph(\delta, \mathfrak{z}(\delta), \varpi_2(\delta))| \\ &\leq \zeta_1 |\mathfrak{w}(\delta) - \mathfrak{z}(\delta)| + \zeta_2 |\varpi_1(\delta) - \varpi_2(\delta)|. \end{aligned}$$

Then,

$$|\varpi_1(\delta) - \varpi_2(\delta)| \leq \frac{\zeta_1}{1-\zeta_2} |\mathfrak{w}(\delta) - \mathfrak{z}(\delta)|.$$

Therefore, for each $\delta \in \nabla$ we get

$$\begin{aligned} |\mathcal{T}\mathfrak{w}(\delta) - \mathcal{T}\mathfrak{z}(\delta)| \leq{}& \frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}}{\Psi_\theta^\psi(\delta,\varrho_1)\Gamma_\kappa(\kappa\theta)} \left[\frac{\zeta_1|\varkappa_2| \left({}^T_\lambda \mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+} |\mathfrak{w}(\gamma) - \mathfrak{z}(\gamma)|\right)(\varrho_2)}{(1-\zeta_2)\left|\varkappa_1 + \varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right|} \right] \\ &+ \frac{\zeta_1}{1-\zeta_2} \left({}^T_\lambda \mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1^+} |\mathfrak{w}(\gamma) - \mathfrak{z}(\gamma)|\right)(\delta). \end{aligned}$$

Thus,

$$
\begin{aligned}
&|\mathcal{T}\mathfrak{w}(\delta)-\mathcal{T}\mathfrak{z}(\delta)| \\
&\leq \left[\frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\zeta_1|\varkappa_2|\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\left(\psi(\gamma)-\psi(\varrho_1)\right)^{\theta-1}\right)(\varrho_2)}{\Psi^{\psi}_{\theta}(\delta,\varrho_1)\Gamma_{\kappa}(\kappa\theta)(1-\zeta_2)\left|\varkappa_1+\varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right|}\right. \\
&\quad \left.+\frac{\zeta_1}{1-\zeta_2}\left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\left(\psi(\gamma)-\psi(\varrho_1)\right)^{\theta-1}\right)(\delta)\right]\|\mathfrak{w}-\mathfrak{z}\|_{C_{\theta;\psi}} \\
&\leq \left[\frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\zeta_1|\varkappa_2|\left(\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}e^{-\lambda(\psi(\delta)-\psi(\gamma))}\left(\psi(\gamma)-\psi(\varrho_1)\right)^{\theta-1}\right)(\varrho_2)}{\Psi^{\psi}_{\theta}(\delta,\varrho_1)\Gamma_{\kappa}(\kappa\theta)(1-\zeta_2)\left|\varkappa_1+\varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right|}\right. \\
&\quad \left.+\frac{\zeta_1}{1-\zeta_2}\left(\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}e^{-\lambda(\psi(\delta)-\psi(\gamma))}\left(\psi(\gamma)-\psi(\varrho_1)\right)^{\theta-1}\right)(\delta)\right]\|\mathfrak{w}-\mathfrak{z}\|_{C_{\theta;\psi}}.
\end{aligned}
$$

By Lemma 2.11, we have

$$
\begin{aligned}
|\mathcal{T}\mathfrak{w}(\delta)-\mathcal{T}\mathfrak{z}(\delta)| &\leq \left[\frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}\zeta_1|\varkappa_2|\widehat{\lambda}\left(\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}\left(\psi(\gamma)-\psi(\varrho_1)\right)^{\theta-1}\right)(\varrho_2)}{\Psi^{\psi}_{\theta}(\delta,\varrho_1)\Gamma_{\kappa}(\kappa\theta)(1-\zeta_2)\left|\varkappa_1+\varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right|}\right. \\
&\quad \left.+\frac{\zeta_1\widehat{\lambda}}{1-\zeta_2}\left(\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\left(\psi(\gamma)-\psi(\varrho_1)\right)^{\theta-1}\right)(\delta)\right]\|\mathfrak{w}-\mathfrak{z}\|_{C_{\theta;\psi}} \\
&\leq \left[\frac{\zeta_1|\varkappa_2|\widehat{\lambda}^2\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}}{\Psi^{\psi}_{\theta}(\delta,\varrho_1)\Gamma_{\kappa}(\kappa+\sigma)(1-\zeta_2)\left|\varkappa_1+\varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right|}\right. \\
&\quad \left.+\frac{\zeta_1\widehat{\lambda}\Gamma_{\kappa}(\kappa\theta)\left(\psi(\delta)-\psi(\varrho_1)\right)^{\frac{\sigma+\kappa\theta}{\kappa}-1}}{\Gamma_{\kappa}(\sigma+\kappa\theta)(1-\zeta_2)}\right]\|\mathfrak{w}-\mathfrak{z}\|_{C_{\theta;\psi}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left|\Psi^{\psi}_{\theta}(\delta,\varrho_1)\left(\mathcal{T}\mathfrak{w}(\delta)-\mathcal{T}\mathfrak{z}(\delta)\right)\right| \\
&\leq \left[\frac{\zeta_1|\varkappa_2|\widehat{\lambda}^2\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}}{\Gamma_{\kappa}(\kappa+\sigma)(1-\zeta_2)\left|\varkappa_1+\varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right|}\right. \\
&\quad \left.+\frac{\zeta_1\widehat{\lambda}\Gamma_{\kappa}(\kappa\theta)\left(\psi(\delta)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}}{\Gamma_{\kappa}(\sigma+\kappa\theta)(1-\zeta_2)}\right]\|\mathfrak{w}-\mathfrak{z}\|_{C_{\theta;\psi}} \\
&\leq \frac{\zeta_1\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}}{1-\zeta_2}\|\mathfrak{w}-\mathfrak{z}\|_{C_{\theta;\psi}} \\
&\quad \times\left[\frac{|\varkappa_2|\widehat{\lambda}^2}{\Gamma_{\kappa}(\kappa+\sigma)\left|\varkappa_1+\varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right|}+\frac{\widehat{\lambda}\Gamma_{\kappa}(\kappa\theta)}{\Gamma_{\kappa}(\sigma+\kappa\theta)}\right].
\end{aligned}
$$

Thus,

$$
\|\mathcal{T}\mathfrak{w}-\mathcal{T}\mathfrak{z}\|_{C_{\theta;\psi}} \leq \mathcal{L}\|\mathfrak{w}-\mathfrak{z}\|_{C_{\theta;\psi}}.
$$

By (4.11), the operator $\mathcal{T}$ is a contraction on $C_{\theta;\psi}(\nabla)$. Hence, by Banach's contraction principle, $\mathcal{T}$ has a unique fixed point $\mathfrak{w}\in C_{\theta;\psi}(\nabla)$, which is a solution to our problem (1.1)-(1.2). □

5. κ-MITTAG-LEFFLER-ULAM-HYERS STABILITY

In this Section, we consider the κ-Mittag-Leffler-Ulam-Hyers stability for our problem (1.1)-(1.2). Let $\mathfrak{w} \in C_{\theta;\psi}(\nabla)$, $\epsilon > 0$. We consider the following inequality:

$$\begin{aligned} &\left|\left({}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta) - \aleph\left(\delta, \mathfrak{w}(\delta), \left({}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta)\right)\right| \\ &\leq \epsilon\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right),\ \delta \in (\varrho_1, \varrho_2]. \end{aligned} \tag{5.1}$$

Definition 5.1 ([32]). Problem (1.1)-(1.2) is said to be κ-Mittag-Leffler-Ulam-Hyers stable with respect to $\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)$ if there exists $a_{\mathbb{E}^{\sigma}_{\kappa}} > 0$ where for each $\epsilon > 0$ and for each solution $\mathfrak{w} \in C_{\theta;\psi}(\nabla)$ of inequality (5.1) there exists a solution $\mathfrak{z} \in C_{\theta;\psi}(\nabla)$ of (1.1)-(1.2) with

$$|\mathfrak{w}(\delta) - \mathfrak{z}(\delta)| \leq a_{\mathbb{E}^{\sigma}_{\kappa}}\epsilon\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right), \qquad \delta \in \nabla.$$

Definition 5.2 ([32]). Problem (1.1)-(1.2) is generalized κ-Mittag-Leffler-Ulam-Hyers stable with respect to $\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)$ if there exists $v : C([0,\infty),[0,\infty))$ with $v(0) = 0$ such that for each $\epsilon > 0$ and for each solution $\mathfrak{w} \in C_{\theta;\psi}(\nabla)$ of inequality (5.1) there exists a solution $\mathfrak{z} \in C_{\theta;\psi}(\nabla)$ of (1.1)-(1.2) with

$$|\mathfrak{w}(\delta) - \mathfrak{z}(\delta)| \leq v(\epsilon)\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right), \qquad \delta \in \nabla.$$

Remark 5.3. Its clear that : Definition 5.1 $\Longrightarrow$ Definition 5.2.

Remark 5.4. A function $\mathfrak{w} \in C_{\theta;\psi}(\nabla)$ is a solution of inequality (5.1) if and only if there exist $\wp \in C_{\theta,\kappa;\psi}(\nabla)$ such that

(1) $|\wp(\delta)| \leq \epsilon\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)$, $\delta \in (\varrho_1, \varrho_2]$,

(2) $\left({}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta) = \aleph\left(\delta, \mathfrak{w}(\delta), \left({}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta)\right) + \wp(\delta)$, $\delta \in (\varrho_1, \varrho_2]$.

Theorem 5.5. *Assume that* (Ax1), (Ax2) *and (4.11) hold. Then,* (1.1)-(1.2) *is* κ*-Mittag-Leffler-Ulam-Hyers stable with respect to* $\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)$ *and consequently generalized* κ*-Mittag-Leffler-Ulam-Hyers stable.*

Proof. Let $\mathfrak{w} \in C_{\theta;\psi}(\nabla)$ be a solution if inequality (5.1), and let us assume that $\mathfrak{z}$ is the unique solution of the problem

$$\begin{cases} \left({}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\mathfrak{z}\right)(\delta) = \aleph\left(\delta, \mathfrak{z}(\delta), \left({}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\mathfrak{z}\right)(\delta)\right);\ \delta \in (\varrho_1, \varrho_2], \\ \varkappa_1\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{z}\right)({\varrho_1}^{+}) + \varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{z}\right)(\varrho_2) = \varkappa_3, \\ \left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{z}\right)({\varrho_1}^{+}) = \left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}\right)({\varrho_1}^{+}). \end{cases}$$

By Lemma 4.2, we obtain for each $\delta \in (\varrho_1, \varrho_2]$

$$\mathfrak{w}(\delta) = \frac{e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}}{\Psi^{\psi}_{\theta}(\delta,\varrho_1)\Gamma_{\kappa}(\kappa\theta)}\left[\frac{\varkappa_3 - \varkappa_2\left({}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta)+\sigma,\kappa;\psi}_{\varrho_1+}w\right)(\varrho_2)}{\varkappa_1 + \varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}}\right] + \left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}w\right)(\delta),$$

where $w \in C_{\theta;\psi}(\nabla)$, be a function satisfying the functional equation

$$w(\delta) = \aleph(\delta, \mathfrak{z}(\delta), w(\delta)).$$

Since $\mathfrak{w}$ is a solution of the inequality (5.1), by Remark 5.4, we have

$$\left({}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta) = \aleph\left(\delta, \mathfrak{w}(\delta), \left({}^{TH}_{\kappa}\mathcal{D}^{\sigma,\varepsilon,\lambda;\psi}_{\varrho_1+}\mathfrak{w}\right)(\delta)\right) + \wp(\delta), \quad \delta \in (\varrho_1, \varrho_2]. \tag{5.2}$$

Clearly, the solution of (5.2) is given by

$$\mathfrak{w}(\delta)=\frac{{}^{T}_{\lambda}\mathcal{J}^{\kappa(1-\theta),\kappa;\psi}_{\varrho_1+}\mathfrak{w}(\varrho_1)}{\Psi^{\psi}_{\theta}(\delta,\varrho_1)\Gamma_{\kappa}(\kappa\theta)}e^{-\lambda(\psi(\delta)-\psi(\varrho_1))}+\left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}(\varpi+\wp)\right)(\delta).$$

where ϖ be a function satisfying the functional equation

$$\varpi(\delta)=\aleph(\delta,\mathfrak{w}(\delta),\varpi(\delta)).$$

Hence, for each $\delta\in(\varrho_1,\varrho_2]$, we have

$$\begin{aligned}|\mathfrak{w}(\delta)-\mathfrak{z}(\delta)|&\leq\left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}|\varpi(\gamma)-w(\gamma)|\right)(\delta)+\left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\wp\right)(\delta)\\&\leq\epsilon\,{}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)+\frac{\zeta_1}{1-\zeta_2}\left({}^{T}_{\lambda}\mathcal{J}^{\sigma,\kappa;\psi}_{\varrho_1+}|\mathfrak{w}(\gamma)-\mathfrak{z}(\gamma)|\right)(\delta).\end{aligned}$$

Using Lemma 2.12 and Lemma 2.17, we get

$$\begin{aligned}|\mathfrak{w}(\delta)-\mathfrak{z}(\delta)|&\leq\epsilon\widehat{\lambda}\left[\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)-1\right]\\&\quad+\frac{\eta_1}{1-\eta_2}\int_{\varrho_1}^{\delta}\psi'(\gamma)\,e^{-\lambda(\psi(\delta)-\psi(\gamma))}\bar{\Psi}^{\kappa,\psi}_{\sigma}(\delta,\gamma)|\mathfrak{w}(\gamma)-\mathfrak{z}(\gamma)|d\gamma.\end{aligned}$$

By applying Theorem 3.1, we obtain

$$\begin{aligned}|\mathfrak{w}(\delta)-\mathfrak{z}(\delta)|&\leq\epsilon\widehat{\lambda}\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)\\&\quad+\int_{\varrho_1}^{\delta}\sum_{i=1}^{\infty}\left(\frac{\eta_1\widehat{\lambda}}{1-\eta_2}\right)^{i}\psi'(\gamma)\,\bar{\Psi}^{\kappa,\psi}_{i\sigma}(\delta,\gamma)\epsilon\widehat{\lambda}\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\gamma)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)d\gamma\\&\leq\epsilon\widehat{\lambda}\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)\mathbb{E}^{\sigma}_{\kappa}\left[\frac{\eta_1\widehat{\lambda}}{1-\eta_2}\left(\psi(\delta)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}\right]\\&\leq\epsilon\widehat{\lambda}\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)\mathbb{E}^{\sigma}_{\kappa}\left[\frac{\eta_1\widehat{\lambda}}{1-\eta_2}\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}\right].\end{aligned}$$

Then for each $\delta\in(\varrho_1,\varrho_2]$, we have

$$|\mathfrak{w}(\delta)-\mathfrak{z}(\delta)|\leq a_{\mathbb{E}^{\sigma}_{\kappa}}\epsilon\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right),$$

where

$$a_{\mathbb{E}^{\sigma}_{\kappa}}=\widehat{\lambda}\mathbb{E}^{\sigma}_{\kappa}\left[\frac{\eta_1\widehat{\lambda}}{1-\eta_2}\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}\right].$$

Hence, the problem (1.1)-(1.2) is κ-Mittag-Leffler-Ulam-Hyers stable with respect to $\mathbb{E}^{\sigma}_{\kappa}\left((\psi(\delta)-\psi(\varrho_1))^{\frac{\sigma}{\kappa}}\right)$. If we set $v(\epsilon)=a_{\mathbb{E}^{\sigma}_{\kappa}}\epsilon$, then the problem (1.1)-(1.2) is also generalized κ-Mittag-Leffler-Ulam-Hyers stable. □

6. Examples

In this segment, we are going to provide practical examples that showcase the fulfillment of the conditions outlined in the theorems of the existence and stability results. We will initially present the general case of our problem (1.1)-(1.2).

Example 6.1. By taking $\varepsilon=\sigma=\frac{1}{2}$, $\lambda=3$, $\kappa=\frac{3}{2}$, $\psi(\delta)=\delta^3$, $\varrho_1=1$, $\varrho_2=\pi$, $\varkappa_1=\varkappa_2=1$ and $\varkappa_3=e$, from the problem (1.1)-(1.2), we obtain the following boundary value problem with (κ,ψ)-Hilfer nonlinear implicit fractional differential equation:

$$\left({}^{TH}_{\frac{3}{2}}\mathcal{D}^{\frac{1}{2},\frac{1}{2},3;\psi}_{1+}\mathfrak{w}\right)(\delta)=\aleph\left(\delta,\mathfrak{w}(\delta),\left({}^{TH}_{\frac{3}{2}}\mathcal{D}^{\frac{1}{2},\frac{1}{2},3;\psi}_{1+}\mathfrak{w}\right)(\delta)\right),\quad\delta\in(1,\pi],\tag{6.1}$$

$$\left({}^{T}_{\lambda}\mathcal{J}^{\frac{2}{9},\frac{2}{3};\psi}_{1+}\mathfrak{w}\right)(1^{+})+\left({}^{T}_{\lambda}\mathcal{J}^{\frac{2}{9},\frac{2}{3};\psi}_{1+}\mathfrak{w}\right)(\pi)=e, \tag{6.2}$$

where $\nabla=[1,\pi]$, $\theta=\frac{1}{\kappa}(\varepsilon(\kappa-\sigma)+\sigma)=\frac{2}{3}$ and

$$\aleph(\delta,\mathfrak{w},\mathfrak{z})=\frac{\sqrt{\delta^3-1}(|\cos(\delta)|+\mathfrak{w}+\mathfrak{z})}{513e^{-\delta+2\pi}},\ \delta\in\nabla,\ \mathfrak{w},\mathfrak{z}\in\mathbb{R}.$$

We have

$$C_{\theta;\psi}(\nabla)=C_{\frac{2}{3};\psi}(\nabla)=\left\{\mathfrak{w}:(1,\pi]\to\mathbb{R}:(\sqrt[3]{\delta^3-1})\mathfrak{w}\in C(\nabla,\mathbb{R})\right\}.$$

It is clear that the function $\aleph$ is continuous on ∇. Then, the condition (*Ax1*) is satisfied.
For each $\mathfrak{w},\bar{\mathfrak{w}},\mathfrak{z},\bar{\mathfrak{z}}\in\mathbb{R}$ and $\delta\in\nabla$, we have

$$|\aleph(\delta,\mathfrak{w},\bar{\mathfrak{w}})-\aleph(\delta,\mathfrak{z},\bar{\mathfrak{z}})|\leq\frac{\sqrt{\delta^3-1}}{513e^{-\delta+2\pi}}\left(|\mathfrak{w}-\bar{\mathfrak{w}}|+|\mathfrak{z}-\bar{\mathfrak{z}}|\right),\ \delta\in\nabla,$$

and so the condition (*Ax2*) is satisfied with $\zeta_1=\zeta_2=\frac{\sqrt{\pi^3-1}}{513e^{\pi}}$. Also, the condition (4.11) of Theorem 4.3 is satisfied. Indeed, we have

$$\begin{aligned}\mathcal{L}&=\frac{\zeta_1\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}}{1-\zeta_2}\left[\frac{|\varkappa_2|\widehat{\lambda}^2}{\Gamma_\kappa(\kappa+\sigma)\left|\varkappa_1+\varkappa_2e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right|}+\frac{\widehat{\lambda}\Gamma_\kappa(\kappa\theta)}{\Gamma_\kappa(\sigma+\kappa\theta)}\right]\\&=\frac{\sqrt{\pi^3-1}\sqrt[3]{\pi^3-1}}{513e^{\pi}-\sqrt{\pi^3-1}}\left[\frac{1}{\sqrt[3]{\frac{3}{2}}\Gamma\left(\frac{4}{3}\right)\left(1+e^{-3(\pi^3-1)}\right)}+\sqrt[3]{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)\right]\\&\approx 0.00310176086069368\\&<1.\end{aligned}$$

Then the problem (6.1)-(6.2) has a unique solution in $C_{\frac{2}{3};\psi}([1,\pi])$ and is κ-Mittag-Leffler-Ulam-Hyers stable with respect to $\mathbb{E}^{\frac{1}{3}}_{\frac{3}{2}}\left(\sqrt[3]{\delta^3-1}\right)$.

Example 6.2. Taking $\varepsilon\to 0$, $\sigma=\frac{1}{2}$, $\lambda=0$, $\kappa=1$, $\psi(\delta)=\delta$, $\varrho_1=1$, $\varrho_2=e$, $\varkappa_1=1$, $\varkappa_2=0$ and $\varkappa_3=\pi$, we get a particular case of problem (1.1)-(1.2) using the Riemann–Liouville fractional derivative, given by

$$\left({}^{TH}_{1}\mathcal{D}^{\frac{1}{2},0,0;\psi}_{1+}\mathfrak{w}\right)(\delta)=\left({}^{RL}\mathbb{D}^{\frac{1}{2}}_{1^{+}}\mathfrak{w}\right)(\delta)=\aleph\left(\delta,\mathfrak{w}(\delta),\left({}^{RL}\mathbb{D}^{\frac{1}{2}}_{1^{+}}\mathfrak{w}\right)(\delta)\right),\ \ \delta\in(1,e], \tag{6.3}$$

$$\left({}^{T}_{0}\mathcal{J}^{\frac{1}{2},1;\psi}_{1+}\mathfrak{w}\right)(1^{+})=\pi, \tag{6.4}$$

where $\nabla=[1,e]$, $\theta=\frac{1}{\kappa}(\varepsilon(\kappa-\sigma)+\sigma)=\frac{1}{2}$,

$$\aleph(\delta,\mathfrak{w},\mathfrak{z})=\frac{3e^2+\mathfrak{w}\sqrt{\ln\delta}+\mathfrak{z}}{333e^{\delta}},\ \delta\in\nabla,\ \mathfrak{w},\mathfrak{z}\in\mathbb{R}.$$

We have

$$C_{\theta;\psi}(\nabla)=C_{\frac{1}{2};\psi}(\nabla)=\left\{\mathfrak{w}:(1,e]\to\mathbb{R}:(\sqrt{\delta-1})\mathfrak{w}\in C(\nabla,\mathbb{R})\right\}.$$

Clearly, the continuous function $\aleph$ is continuous. Hence, the condition (*Ax1*) is satisfied.
For each $\mathfrak{w},\bar{\mathfrak{w}},\mathfrak{z},\bar{\mathfrak{z}}\in\mathbb{R}$ and $\delta\in\nabla$, we have

$$|\aleph(\delta,\mathfrak{w},\bar{\mathfrak{w}})-\aleph(\delta,\mathfrak{z},\bar{\mathfrak{z}})|\leq\frac{\sqrt{\ln\delta}}{333e^{\delta}}|\mathfrak{w}-\bar{\mathfrak{w}}|+\frac{1}{333e^{\delta}}|\mathfrak{z}-\bar{\mathfrak{z}}|,\ \delta\in\nabla,$$

and so the condition $(Ax2)$ is satisfied with $\zeta_1 = \zeta_2 = \dfrac{1}{333e}$.
Also, we have

$$\begin{aligned}\mathcal{L} &= \frac{\zeta_1 \left(\psi(\varrho_2) - \psi(\varrho_1)\right)^{\frac{\sigma}{\kappa}}}{1-\zeta_2} \left[\frac{|\varkappa_2|\widehat{\lambda}^2}{\Gamma_\kappa(\kappa+\sigma)\left|\varkappa_1 + \varkappa_2 e^{-\lambda(\psi(\varrho_2)-\psi(\varrho_1))}\right|} + \frac{\widehat{\lambda}\Gamma_\kappa(\kappa\theta)}{\Gamma_\kappa(\sigma+\kappa\theta)}\right] \\ &= \frac{\sqrt{\pi(e-1)}}{333e-1} \\ &\approx 0.00256958784 \\ &< 1.\end{aligned}$$

Since the conditions of Theorem 4.3 and Theorem 5.5 are satisfied, then the problem (6.1)-(6.2) has a unique solution in $C_{\frac{1}{2};\psi}([1,e])$ and is Mittag-Leffler-Ulam-Hyers stable with respect to $\mathbb{E}_1^{\frac{1}{2}}\left(\sqrt{\delta-1}\right)$.

7. Conclusion

In this research, we introduced a new derivative operator called the tempered (κ,ψ)-Hilfer fractional derivative. Alongside this introduction, we conducted a thorough investigation into the essential properties of this operator, considering the unique characteristics of the functions κ-Gamma and κ-Beta. Moreover, we demonstrated the practical significance of our definitions by establishing the existence and uniqueness of the solutions for two tempered (κ,ψ)-Hilfer problems. These problems encompassed nonlinear implicit fractional differential equations accompanied by boundary conditions. Our approach to proving existence and uniqueness relied on the application of the Banach contraction principle. Furthermore, we formulated and validated a generalized Gronwall inequality, which played a crucial role in demonstrating the κ-Mittag-Leffler-Ulam-Hyers stability. To exemplify the applicability of our key results and to show that the requirements of our theorems can be verified, we presented several specific examples. These examples effectively highlighted the versatility and broadening effect of our proposed operator across various cases. Notably, this newly introduced operator acts as an extension, encompassing previously established fractional derivatives like the ψ-Hilfer fractional derivative already present in the literature. This broader framework significantly enriches the ongoing advancement of fractional calculus, paving the way for promising avenues of future exploration in this ever-evolving and dynamic field.

Competing interests. The authors declare no competing interests.

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