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COMPOSITION-DIFFERENTIATION OPERATOR ON THE BERGMAN SPACE

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ABSTRACT. We investigate the properties of composition-differentiation operator D_ψ on the Bergman space of the unit disk $L_a^2(\mathbb{D})$. Specifically, we characterize the properties of the reproducing kernel for the derivatives of the Bergman space functions. Moreover, we determine the adjoint properties of D_ψ whenever ψ is self analytic map of the unit disk $\mathbb{D}$.

1. INTRODUCTION AND PRELIMINARIES

The set $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is called the unit disk of the complex plane. Let dA denote the area Lebesgue measure on $\mathbb{D}$, normalized so that the area of $\mathbb{D}$ is 1. In terms of rectangular and polar co-ordinates, we have $dA(z) = \frac{1}{\pi}dxdy = \frac{r}{\pi}drd\theta$ where $z = x + iy = re^{i\theta} \in \mathbb{D}$.
For $1 \leq p < \infty$, the classical Lebesgue spaces on the unit disk $\mathbb{D}$, denoted by $L^p(\mathbb{D})$, are defined by,

$$L^p(\mathbb{D}) := \left\{ f : \mathbb{D} \to \mathbb{C} : \|f\|_{L_a^p(\mathbb{D})} := \left(\int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{\frac{1}{p}} < \infty \right\}.$$

The Bergman space $L_a^p(\mathbb{D})$, is a subspace of $L^p(\mathbb{D})$ consisting of all analytic functions on $\mathbb{D}$. In particular,

$$L_a^p(\mathbb{D}) := L^p(\mathbb{D}) \cap \mathcal{H}(\mathbb{D}),$$

where $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions $f : \mathbb{D} \to \mathbb{C}$.
Therefore, $L_a^p(\mathbb{D})$ are Banach spaces with respect to $\|\cdot\|_p$. For $p = 2$, $L_a^2(\mathbb{D})$ is a Hilbert space with the inner product given by,

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z).$$

For $p = \infty$, we define $L^\infty(\mathbb{D})$ as the space of essentially bounded functions on the unit disk $\mathbb{D}$.
For $1 \leq p < \infty$, the Hardy space of the unit disk, $H^p(\mathbb{D})$, is defined as

$$H^p(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{H^p(\mathbb{D})} = \sup_{0<r<1} \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty \right\}.$$

If $p = 2$, $H^2(\mathbb{D})$ is a Hilbert space with inner product defined by: For each $f, g \in H^2(\mathbb{D})$,

$$\langle f, g \rangle_{H^2(\mathbb{D})} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta.$$

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We call K_w a reproducing kernel for a Hilbert space H a complex function, $K : \Omega \times \Omega \to \mathbb{C}$ such that if we put $K_w(z) = K(z, w)$, then the following two properties hold:

(i) For every $w \in \Omega$, the function K_w belongs to H and,
(ii) for all $f \in H$ and $w \in \Omega$, we have $f(w) = \langle f, K_w \rangle_H$ (the reproducing property).

The reproducing kernel K_w for the Bergman space $L^2_a(\mathbb{D})$ is given by,

$$K_w(z) = \frac{1}{(1 - \overline{w}z)^2}. \tag{1.1}$$

See [1,2,11] and references therein for more details.

Let ψ be a self analytic map on $\mathbb{D}$, then the composition operator C_ψ induced by ψ and acting on $\mathcal{H}(\mathbb{D})$ is defined by: $C_\psi f = f \circ \psi$, for all $f \in \mathcal{H}(\mathbb{D})$ while the differentiation operator D on $\mathcal{H}(\mathbb{D})$ is defined by $Df(z) = f'(z) \quad \forall f \in \mathcal{H}(\mathbb{D})$. Differentiation operator D is unbounded on the Bergman space. See [3,5,8] for more details.

Given the function $g \in L^\infty(\mathbb{D})$, we define an operator T_g acting on $\mathcal{H}(\mathbb{D})$ given by

$$T_g f = g \cdot f \quad \forall\, f \in \mathcal{H}(\mathbb{D}).$$

The operator T_g is called a Toeplitz operator.

The composition-differentiation operator denoted by D_ψ on $\mathcal{H}(\mathbb{D})$ is induced by ψ and is defined as

$$D_\psi f(z) = (f' \circ \psi)(z) \quad \forall\, f \in \mathcal{H}(\mathbb{D}) \text{ and } z \in \mathbb{D}.$$

For analytic self maps $\psi : \mathbb{D} \to \mathbb{D}$, the operator D_ψ is bounded on $L^2_a(\mathbb{D})$ (See [[10], Theorem 2.44]). Closed Graph Theorem shows that D_ψ is bounded on $L^2_a(\mathbb{D})$ whenever D_ψ takes $L^2_a(\mathbb{D})$ into itself.

Cowen and MacCluer [6] determined properties of composition operators C_ψ on the Hardy space. Ohno [10] later investigated the boundedness and compactness of the product of composition and differentiation operator operators on the Hardy space of the unit disk. Fatehi and Hammond [7] extended Ohno's results and considered a particular case when the operator D_ψ is bounded on the Hardy space of the unit disk $H^2(\mathbb{D})$ and determined the properties of this operator on $H^2(\mathbb{D})$. Nevertheless, reproducing kernel is an approach gaining popularity in the study of these operators. The purpose of this paper therefore is to determine properties of D_ψ on $L^2_a(\mathbb{D})$ when the operator D_ψ is bounded.

2. Reproducing kernel for the derivative of $L^2_a(\mathbb{D})$ functions

In this section, we determine the reproducing kernel for the derivatives of Bergman space $L^2_a(\mathbb{D})$ functions. We further establish the norm of the reproducing kernel for the derivative of the Bergman space functions together with its growth condition.

Proposition 2.1. *For a fixed point $w \in \mathbb{D}$, let K_w denote the reproducing kernel for the Bergman space $L^2_a(\mathbb{D})$. Then the following properties hold:*

(1) $\|K_w\| = \frac{1}{1-|w|^2}$.
(2) *The growth condition for the Bergman space functions is given by,*

$$|f(w)| \leq \frac{1}{\sqrt{1 - |w|^2}} \|f\|.$$

(3) *The reproducing kernel for the derivative of Bergman space functions $K^{(1)}_w$ is given by,*

$$K^{(1)}_w(z) = \frac{2z}{(1 - \overline{w}z)^3}.$$

(4) *The norm of the reproducing kernel for the derivative of the Bergman space functions is given by,*

$$\|K_w^{(1)}\| = \frac{\sqrt{2(1+5|w|^2)}}{(1-|w|^2)^2}.$$

(5) *The growth condition of the derivative of the Bergman space functions is given by,*

$$|f'(w)| \leq \sqrt{\frac{2z}{(1-\overline{w}z)^3}}\|f\|.$$

Proof. To prove 1, let K_w be the reproducing kernel of $L_a^2(\mathbb{D})$. Then letting $z = w$ in equation (1.1), we obtain

$$K_w(w) = \frac{1}{(1-|w|^2)^2}.$$

Then by the reproducing property of K_w given by,

$$\langle f, K_w \rangle = f(w) \quad \forall f \in L_a^2(\mathbb{D}),$$

we have that

$$K_w(w) = \langle K_w, K_w \rangle = \|K_w\|^2.$$

Thus

$$\|K_w\|^2 = \frac{1}{(1-|w|^2)^2},$$

and therefore

$$\|K_w\| = \frac{1}{1-|w|^2},$$

as desired.

To prove 2, we use the Cauchy Schwartz inequality.

Now for every $f \in L_a^2(\mathbb{D})$, we have

$$\begin{aligned} |f(w)| &= |\langle f, K_w \rangle| \\ &\leq \|f\|\|K_w\| \\ &= \|f\|\frac{1}{1-|w|^2}. \end{aligned}$$

It follows that

$$|f(w)| \leq \frac{1}{1-|w|^2}\|f\|, \tag{2.1}$$

where

$$\|f\| := \|f\|_{L_a^2(\mathbb{D})}.$$

Equation (2.1) is the growth condition for the Bergman space functions.

To prove 3, we need to obtain $K_w'(z)$ and then deduce $K_w^{(1)}(z)$.

Differentiating (1.1) with respect to z, we get

$$K_w'(z) = \frac{2\overline{w}}{(1-\overline{w}z)^3}. \tag{2.2}$$

We then obtain $K_w^{(1)}(z)$ by interchanging $\overline{w}$ with z in (2.2),

$$K_w^{(1)}(z) = \frac{2z}{(1-z\overline{w})^3}, \tag{2.3}$$

which is the reproducing kernel for the derivative of $L_a^2(\mathbb{D})$ functions.

To prove 4, we use equation (2.3) which gives $K_w^{(1)}$.

Now by the reproducing property of $K_w^{(1)}$, we have that

$$\langle f, K_w^{(1)} \rangle = f'(w)$$

and therefore

$$\langle K_w^{(1)}, K_w^{(1)}\rangle = (K_w^{(1)})'(w) = \|K_w^{(1)}\|^2.$$

But

$$K_w^{(1)}(w) = \frac{2w}{(1-\overline{w}w)^3}.$$

We then get the derivative of the reproducing kernel as follows,

$$\begin{aligned}(K_w^{(1)})'(w) &= \frac{2(1-\overline{w}w)^3 - 2w[3(1-\overline{w}w)^2(-\overline{w})]}{(1-w\overline{w})^6}\\ &= \frac{2(1-\overline{w}w)^3 + 6\overline{w}w(1-\overline{w}w)^2}{(1-\overline{w}w)^6}\\ &= \frac{2(1+5|w|^2)}{(1-|w|^2)^4}.\end{aligned}$$

It then follows that

$$\|K_w^{(1)}\|^2 = \frac{2(1+5|w|^2)}{(1-|w|^2)^4}.$$

Thus

$$\|K_w^{(1)}\| = \frac{\sqrt{2(1+5|w|^2)}}{(1-|w|^2)^2}.$$

To prove 5, we again employ Cauchy-Schwartz inequality.
By Cauchy-Schwartz inequality, we have that for every $f \in L_a^2(\mathbb{D})$,

$$\begin{aligned}f'(w) = \langle f, K_w^{(1)}\rangle &= |\langle f, K_w^{(1)}\rangle|\\ &\leq \|f\|\|K_w^{(1)}\|\\ &= \|f\|\frac{\sqrt{2(1+5|w|^2)}}{(1-|w|^2)^2}.\end{aligned}$$

Hence

$$|f'(w)| \leq \frac{\sqrt{2(1+5|w|^2)}}{(1-|w|^2)^2}\|f\|,$$

as desired.
This completes the proof. □

3. Properties of composition-differentiation operator on the Bergman space.

In this section, we determine certain properties of composition-differentiation operator on the Bergman space. In particular, we prove a compactness property of the composition-differentiation operator D_ψ that $\|D_\psi\| \geq \sqrt{2}$ whenever $\psi(0) \neq 0$. We further determine the adjoint of the composition-differentiation operator by employing the use of nonconstant linear fractional self maps. However, we first prove the following proposition.

Proposition 3.1. *Let ψ be an analytic self map in $\mathbb{D}$, then for any $w \in \mathbb{D}$, the following properties hold in the Bergman space $L_a^2(\mathbb{D})$:*
1. $\langle f, K_w^{(1)}\rangle = f'(w)$.
2. $C_\psi^*(K_w) = K_{\psi(w)}$ where C_ψ^* denotes the adjoint of C_ψ on $L_a^2(\mathbb{D})$.
3. C_ψ is invertible if and only if ψ is an automorphism. In that case, $C_\psi^{-1} = C_{\psi^{-1}}$.
4. $T_\psi^*(K_w) = \overline{\psi(w)}K_w$
5. $D_\psi^*(K_w) = K_{\psi(w)}^{(1)}$
6. $D_\psi D_{\psi^{-1}} = D_{\psi^{-1}}D_\psi$ with $D_\psi^* D_{\psi^{-1}}^* K_w = D_{\psi^{-1}}^* D_\psi^* K_w = (K_w^{(1)})^{(1)}$. In particular, D_ψ is not invertible.

Proof. To prove 1, for every $f \in L_a^2(\mathbb{D})$, we have

$$\begin{aligned} f(w) &= \langle f, K_w \rangle \\ &= \int_{\mathbb{D}} f(z)\overline{K_w(z)}dA(z) \\ &= \int_{\mathbb{D}} \frac{f(z)}{(1-\overline{z}w)^2}dA(z). \end{aligned} \tag{3.1}$$

Differentiating equation (3.1) with respect to w we obtain

$$\begin{aligned} f'(w) &= \frac{\partial}{\partial w}\int_{\mathbb{D}} \frac{f(z)}{(1-\overline{z}w)^2}dA(z) \\ &= \int_{\mathbb{D}} \frac{\partial}{\partial w}\frac{f(z)}{(1-\overline{z}w)^2}dA(z) \\ &= \int_{\mathbb{D}} f(z)\frac{2\overline{z}}{(1-\overline{z}w)^3}dA(z) \\ &= \int_{\mathbb{D}} f(z)\overline{K_z^{(1)}(w)}dA(z) \\ &= \langle f, K_w^{(1)} \rangle, \end{aligned}$$

as desired. For 2, for every $f \in L_a^2(\mathbb{D})$ and $w \in \mathbb{D}$, we have by the definition of C_ψ and the reproducing property of K_w,

$$\begin{aligned} \langle f, C_\psi^* K_w \rangle &= \langle C_\psi f, K_w \rangle \\ &= \langle f \circ \psi, K_w \rangle \\ &= f(\psi(w)) \\ &= \langle f, K_{\psi(w)} \rangle. \end{aligned}$$

So $C_\psi^* K_w = K_{\psi(w)}$ as desired.
For 3, if ψ is an automorphism of $\mathbb{D}$, then by assertion 2, we have

$$\begin{aligned} \langle C_\psi C_{\psi^{-1}} f, K_w \rangle &= \langle C_{\psi^{-1}} f, C_\psi^* K_w \rangle \\ &= \langle C_{\psi^{-1}} f, K_{\psi(w)} \rangle \\ &= \langle f, K_{\psi^{-1}(\psi(w))} \rangle \\ &= \langle f, K_w \rangle. \end{aligned}$$

Thus $C_\psi C_{\psi^{-1}} = I$. Similarly, it can be shown that $C_{\psi^{-1}} C_\psi = I$.
This implies that C_ψ is invertible and $C_\psi^{-1} = C_{\psi^{-1}}$, as desired.
For 4, by the inner product property, we have

$$\begin{aligned} T_\psi^* K_w(z) &= \langle T_\psi^* K_w, K_z \rangle \\ &= \overline{\langle T_\psi K_z, K_w \rangle} \\ &= \overline{T_\psi K_z(w)} \\ &= \overline{\psi(w) K_z(w)} \\ &= \overline{\psi(w)} \langle K_w, K_z \rangle \\ &= \overline{\psi(w)} K_w(z). \end{aligned}$$

Since z was arbitrary, it follows that $T^*_\psi K_w = \overline{\psi(w)} K_w$, as desired.
For 5, let $f \in L^2_a(\mathbb{D})$, then by assertion 1, we have

$$\begin{aligned}\langle f, D^*_\psi K_w\rangle &= \langle D_\psi f, K_w\rangle \\ &= D_\psi f(w) \\ &= (f' \circ \psi)(w) \\ &= f'(\psi(w)) \\ &= \langle f, K^{(1)}_{\psi(w)}\rangle.\end{aligned}$$

So $D^*_\psi K_w = K^{(1)}_{\psi(w)}$.
For 6, by assertion 4, we have that

$$\begin{aligned}\langle D_\psi D_{\psi^{-1}} f, K_w\rangle &= \langle D_{\psi^{-1}} f, D^*_\psi K_w\rangle \\ &= \langle D_{\psi^{-1}} f, K^{(1)}_{\psi(w)}\rangle \\ &= \langle f, (K^{(1)}_{\psi^{-1}(\psi(w))})^{(1)}\rangle \\ &= \langle f, (K^{(1)}_w)^{(1)}\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle D_{\psi^{-1}} D_\psi f, K_w\rangle &= \langle D_\psi f, D^*_{\psi^{-1}} K_w\rangle \\ &= \langle D_\psi f, K^{(1)}_{\psi^{-1}(w)}\rangle \\ &= \langle f, (K^{(1)}_{\psi^{-1}(\psi(w))})^{(1)}\rangle \\ &= \langle f, (K^{(1)}_w)^{(1)}\rangle,\end{aligned}$$

where $(K^{(1)}_w)^{(1)}$ is the second derivative of the reproducing kernel K_w.
Therefore $D_\psi D_{\psi^{-1}} = D_{\psi^{-1}} D_\psi$ with $D^*_\psi D^*_{\psi^{-1}} K_w = D^*_{\psi^{-1}} D^*_\psi K_w = (K^{(1)}_w)^{(1)}$.
In particular, D_ψ is not invertible. □

In the next results, we approximate the lower bound of the composition-differentiation operator D_ψ on the Bergman space $L^2_a(\mathbb{D})$.

Proposition 3.2. *Let ψ be an analytic self map on $\mathbb{D}$. Then $\|D_\psi\| \geq \sqrt{2}$. Moreover if $\psi(0) \neq 0$, then $\|D_\psi\| > \sqrt{2}$.*

Proof. From assertion 5 of Proposition 3.1, we also have that

$$D^*_\psi(K_w) = K^{(1)}_\psi(w).$$

Recall,

$$\|K_w\| = \frac{1}{1-|w|^2},$$

and

$$\|K^{(1)}_w\| = \sqrt{\frac{2(1+5|w|^2)}{(1-|w|^2)^4}}.$$

Now we define $\|D^*_\psi(K_w)\|$ given by

$$\|D^*_\psi(K_w)\| = \sqrt{\frac{2(1+5|w|^2)}{(1-|w|^2)^4}}. \tag{3.2}$$

Dividing equation (3.2) by $\|K_w\|^2$ we obtain

$$\frac{\|D_\psi^*(K_w)\|^2}{\|K_w\|^2} = \frac{(1-|w|^2)^2 2(1+5|\psi(w)|^2)}{(1-|\psi(w)|^2)^4}, \tag{3.3}$$

Implying that

$$\frac{\|D_\psi^*(K_w)\|}{\|K_w\|} = \frac{(1-|w|^2\sqrt{(2+10|\psi(w)|^2)}}{(1-|\psi(w)|^2)^2}.$$

It follows that, if D_ψ is bounded on L_a^2 then,

$$\|D_\psi\| \geq \sup_{w\in\mathbb{D}} \sqrt{\frac{(1-|w|^2)^2(2+10|\psi(w)|^2)}{(1-|\psi(w)|^2)^4}} \geq \sup_{w\in\mathbb{D}} \sqrt{\frac{(1-|w|^2)^2}{(1-|\psi(w)|^2)^4}}, \tag{3.4}$$

since $\|D_\psi^*\| = \|D_\psi\|$. Letting $w=0$ in equation (3.4), we obtain,

$$\|D_\psi\| \geq \sqrt{\frac{(2+10|\psi(0)|^2)}{(1-|\psi(0)|^2)^4}}$$

This shows that $\|D_\psi\| \geq \sqrt{2}$, for all ψ and that whenever $\psi(0)\neq 0$ then $\|D_\psi\| > \sqrt{2}$. □

Consider $\varphi(z) = \frac{az+b}{cz+d}$ be a nonconstant linear fractional self-map of $\mathbb{D}$ where $a,b,c,d\in\mathbb{C}$, then the map

$$\sigma(z) = \frac{\overline{a}z-\overline{c}}{-\overline{b}z+\overline{d}},$$

also takes $\mathbb{D}$ to $\mathbb{D}$.

These two maps φ and σ have been used mostly to investigate adjoint properties of composition operators. See [4,5,9] for more details.

In our next result, we also employ the maps φ and σ to determine the adjoint properties of the composition-differentiation operators D_ψ. Moreover, we deduce some consequences from the adjoint relations obtained.

Theorem 3.3. *For the two maps φ and σ,*

$$D_\varphi^* T^*_{K^{(1)}_{\sigma(0)}} = T_{K^{(1)}_{\varphi(0)}} D_\sigma. \tag{3.5}$$

Proof. Let $\varphi(z) = \frac{az+b}{cz+d}$. Then

$$\varphi(0) = \frac{b}{d}.$$

We also have that

$$\sigma(z) = \frac{\overline{a}z-\overline{c}}{-\overline{b}z+\overline{d}}$$

which implies that

$$\sigma(0) = -\frac{\overline{c}}{\overline{d}}.$$

Reproducing kernel of the derivatives of the Bergman space functions was obtained in chapter 3 as

$$K_w^{(1)}(z) = \frac{2\overline{z}}{(1-\overline{w}z)^3} \tag{3.6}$$

That is, for all $f\in L_a^2(\mathbb{D})$, $f'(z) = \langle f, K_w^{(1)}\rangle$.

Letting $w=\varphi(0)$ in equation (3.6), we obtain

$$K^{(1)}_{\varphi(0)}(z) = \frac{2z}{(1-(\overline{\frac{b}{d}})z)^3} \tag{3.7}$$

$$= \frac{2z}{(1-\frac{\overline{b}}{\overline{d}}z)^3}. \tag{3.8}$$

Multiplying equation (4.7) by $\overline{d}^3$, we obtain

$$K^{(1)}_{\varphi(0)}(z) = \frac{2\overline{d}^3 z}{(\overline{d} - \overline{b}z)^3}. \tag{3.9}$$

Also, letting $w = \sigma(0)$ in (3.6), we get

$$K^{(1)}_{\sigma(0)}(z) = \frac{2z}{(1 - (-\frac{\overline{c}}{\overline{d}})z)^3}. \tag{3.10}$$

Multiplying the R.H.S of (3.10) by d^3, we obtain

$$K^{(1)}_{\sigma(0)}(z) = \frac{2d^3 z}{(d + cz)^3}.$$

Moreover,

$$\begin{aligned} T_{K^{(1)}_{\varphi(0)}} D_\sigma(K_w)(z) &= K^{(1)}_{\psi(0)} D_\sigma(K_w)(z) \\ &= K^{(1)}_{\varphi(0)} K'_w(\sigma(z)). \end{aligned}$$

But from equation (3.6),

$$K^{(1)}_w(z) = \frac{2\overline{w}}{(1 - \overline{w}z)^3}.$$

It then follows that,

$$\begin{aligned} K'_w(\sigma(z)) &= \frac{2\overline{w}}{(1 - \overline{w}\sigma(z))^3} \\ &= \frac{2\overline{w}}{\left(1 - \overline{w}\left(\frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}\right)\right)^3}. \end{aligned}$$

Therefore,

$$\begin{aligned} T_{K^{(1)}_{\varphi(0)}} D_\sigma(K_w)(z) &= \frac{2\overline{d}^3 z}{(\overline{d} - \overline{b}z)^3} . \frac{2\overline{w}}{\left(1 - \overline{w}\left(\frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}\right)\right)^3} \\ &= \frac{2\overline{d}^3 z . 2\overline{w}}{\left(\overline{d} - \overline{b}z - \overline{w}(\overline{a}z - \overline{c})\right)^3} \\ &= \frac{4\overline{d^3 w} z}{\left(\overline{d} - \overline{b}z - \overline{aw}z + \overline{cw}\right)^3}. \end{aligned}$$

Hence

$$T_{K^{(1)}_{\varphi(0)}} D_\sigma(K_w)(z) = \frac{4\overline{d^3 w} z}{\left(\overline{d} - \overline{b}z - \overline{aw}z + \overline{cw}\right)^3}. \tag{3.11}$$

Simplifying the denominator of the R.H.S of (3.11), we obtain

$$\left((\overline{cw} + \overline{d}) - (\overline{aw} + \overline{b})z\right)^3 = \left(\overline{cw} + \overline{d}\right)^3 \left(1 - \left(\frac{\overline{aw} + \overline{b}}{\overline{cw} + \overline{d}}\right) z\right)^3.$$

Therefore equation (3.11) can be written as

$$T_{K^{(1)}_{\varphi(0)}} D_\sigma(K_w)(z) = \frac{4\overline{d^3 w}}{(\overline{cw} + \overline{d})^3} . \frac{z}{\left(1 - \left(\frac{\overline{aw} + \overline{b}}{\overline{cw} + \overline{d}}\right) z\right)^3},$$

for any $w \in \mathbb{D}$.
Next, we compute $D_\varphi^* T^*_{K^{(1)}_{\sigma(0)}}(K_w)$.
From the adjoint property of T given in Proposition 3.1 assertion 4, we have that

$$\begin{aligned} D_\varphi^* T^*_{K^{(1)}_{\sigma(0)}}(K_w) &= D_\varphi^* \overline{K^{(1)}_{\sigma(0)}} K_w \\ &= K^{(1)}_{\sigma(0)} D_\varphi^* K_w. \end{aligned}$$

But by assertion 5 of Proposition 3.1, we have that

$$D_\varphi^* K_w = K^{(1)}_{\varphi(w)}$$

This implies that,

$$\begin{aligned} \overline{K^{(1)}_{\sigma(0)}}(w) D_\varphi^*(K_w) &= \overline{K^{(1)}_{\sigma(0)}(w)} K^{(1)}_{\varphi(w)} \\ &= \frac{2\overline{d}^3 w}{(\overline{cw}+\overline{d})^3} . \frac{2z}{\left(1-\overline{\varphi(w)}z\right)^3} \\ &= \frac{4\overline{d^3 w}}{\left(\overline{cw}+\overline{d}z\right)^3} . \frac{z}{\left(1-\left(\frac{\overline{aw}+\overline{b}}{\overline{cw}+\overline{d}}\right)z\right)^3} \end{aligned}$$

Therefore, $D_\varphi^* T^*_{K^{(1)}_{\sigma(0)}}$ and $T_{K^{(1)}_{\varphi(0)}} D_\sigma$ both agree on span of reproducing functions and therefore constitutes a dense subset of $L_a^2(\mathbb{D})$.
Thus both the operators are identical on $L_a^2(\mathbb{D})$. □

Remark 3.4. The relation obtained in Theorem 3.3 above is similar to that obtained by Fatehi and Hammond [7] in the setting of the Hardy space $H^2(\mathbb{D})$. Moreover, as noted in [7], it has a resemblance to Cowen's adjoint formula for composition operators, which is written as $C_\varphi^* T^*_{K^{(1)}_{\sigma(0)}} = T_{K^{(1)}_{\varphi(0)}} C_\sigma$ (See[8, Theorem 2]).

Now, let us focus on special cases obtained by taking $b, c = 0$ and $r = \frac{a}{d}$ where $a, d \in \mathbb{R}$. Then $\sigma(z) = \frac{a}{d}z$ and $\varphi(z) = \sigma(z) = \frac{a}{d}z$.
Take $\rho(z) = rz$ for some real number $r = \frac{a}{d}$.
Then clearly, $\rho(z) = \sigma(z) = \varphi(z)$ which implies that $\rho(0) = 0$ and so $K^{(1)}_{\rho(0)} = z$.
For ease of notation, we let

$$T_z = T_{K^{(1)}_{\varphi(0)}} = T_{K^{(1)}_{\sigma(0)}}.$$

and

$$D_\rho = D_\varphi = D_\sigma.$$

Then the relation in equation (3.5) reduces to

$$D_\rho^* T_z^* = T_z D_\rho. \tag{3.12}$$

We can then deduce the following consequences:

Corollary 3.5. *Let T_z and D_ρ be as defined above. Then*

(1) $T_z^* T_z = I$
(2) $D_\rho^* = T_z D_\rho T_z$
(3) $D_\rho T_z$ *is self adjoint*
(4) $D_\rho D_\rho^* = (D_\rho T_z)^2$

Proof. To prove 1, for $f \in L_a^2(\mathbb{D})$, we have that

$$\begin{aligned} \langle T_z^* T_z f, f \rangle &= \langle T_z f, T_z f \rangle \\ &= \|T_z f\|^2 \\ &= \|f\|^2 \\ &= \langle f, f \rangle. \end{aligned}$$

So $T_z^* T_z = I$, as desired.
To prove 2, from assertion 1, we have that $T_z^* T_z = I$.
Therefore $D_\rho^* = D_\rho^* T_z^* T_z$.
But from (3.12), $D_\rho^* T_z^* = T_z D_\rho$.
We then have that, $D_\rho^* T_z^* T_z = T_z D_\rho T_z$,
implying that $D_\rho^* = T_z D_\rho T_z$, as desired.
To prove (3), let $(D_\rho T_z)^*$ be the adjoint of $D_\rho T_z$, then

$$\begin{aligned} (D_\rho T_z)^* &= T_z^* D_\rho^* \\ &= T_z^* (T_z D_\rho T_z) \\ &= T_z^* T_z D_\rho T_z. \end{aligned}$$

Since $T_z^* T_z = I$, it then follows that $T_z^* T_z D_z T_z = D_\rho T_z$,
as desired.
To prove (4), from assertion (2), $D_\rho^* = T_z D_\rho T_z$.
It then follows that,

$$\begin{aligned} D_\rho D_\rho^* &= D_\rho (T_z D_\rho T_z) \\ &= D_\rho T_z D_\rho T_z \\ &= (D_\rho T_z)^2. \end{aligned}$$

This completes the proof. □

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