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FINITE LOGARITHMIC ORDER MEROMORPHIC SOLUTIONS OF COMPLEX LINEAR DELAY-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this article, we study the growth of meromorphic solutions of linear delay-differential equation of the form

$$\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z)f^{(j)}(z+c_i) = F(z),$$

where $A_{ij}(z)$ $(i = 0, 1, \ldots, n, j = 0, 1, \ldots, m, n, m \in \mathbb{N})$ and $F(z)$ are meromorphic of finite logarithmic order, $c_i (i = 0, \ldots, n)$ are distinct non-zero complex constants. We extend those results obtained recently by Chen and Zheng, Bellaama and Belaïdi to the logarithmic lower order.

1. INTRODUCTION AND MAIN RESULTS

Throughout this article, we assume the readers are familiar with the fundamental results and standard notations of the Nevanlinna distribution theory of meromorphic functions such as $m(r, f)$, $N(r, f)$, $M(r, f)$, $T(r, f)$, which can be found in [13, 15, 25]. The concepts of logarithmic order and logarithmic type of entire or meromorphic functions were introduced by Chern, [9, 10]. Since then, many authors used them in order to generalize previous results obtained on the growth of solutions of linear difference equations and linear differential equations in which the coefficients are entire or meromorphic functions in the complex plane $\mathbb{C}$ of positive order different to zero, see for example [1, 6, 11, 14, 19, 21, 22], their new results were on the logarithmic order, the logarithmic lower order and the logarithmic exponent of convergence, where they considered the case when the coefficients are of zero order see, for example, [2–4, 7, 12, 17, 18, 23]. In this article, we also use these concepts to investigate the lower logarithmic order of solutions to more general homogeneous and non homogeneous linear delay-differential equations, where we generalize those results obtained in [5, 8]. We start by stating some important definitions.

Definition 1.1 ([3, 10])**.** The logarithmic order and the logarithmic lower order of a meromorphic function f are defined by

$$\rho_{\log}(f) = \limsup_{r\longrightarrow+\infty} \frac{\log T(r, f)}{\log\log r}, \quad \mu_{\log}(f) = \liminf_{r\longrightarrow+\infty} \frac{\log T(r, f)}{\log\log r}.$$

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where $T(r,f)$ denotes the Nevanlinna characteristic of the function f. If f is an entire function, then

$$\rho_{\log}(f)=\limsup_{r\longrightarrow+\infty}\frac{\log\log M(r,f)}{\log\log r}=\limsup_{r\longrightarrow+\infty}\frac{\log T(r,f)}{\log\log r},$$

$$\mu_{\log}(f)=\liminf_{r\longrightarrow+\infty}\frac{\log\log M(r,f)}{\log\log r}=\liminf_{r\longrightarrow+\infty}\frac{\log T(r,f)}{\log\log r},$$

where $M(r,f)$ denotes the maximum modulus of f in the circle $|z|=r$.

It is clear that, the logarithmic order of any non-constant rational function f is one, and thus, any transcendental meromorphic function in the plane has logarithmic order no less than one. Moreover, any meromorphic function with finite logarithmic order in the plane is of order zero.

Definition 1.2 ([3,7]). The logarithmic type and the logarithmic lower type of a meromorphic function f are defined by

$$\tau_{\log}(f)=\limsup_{r\longrightarrow+\infty}\frac{T(r,f)}{(\log r)^{\rho_{\log}(f)}},\quad \underline{\tau}_{\log}(f)=\liminf_{r\longrightarrow+\infty}\frac{T(r,f)}{(\log r)^{\mu_{\log}(f)}}.$$

If f is an entire function, then

$$\tau_{\log,M}(f)=\limsup_{r\longrightarrow+\infty}\frac{\log M(r,f)}{(\log r)^{\rho_{\log}(f)}},$$

$$\underline{\tau}_{\log,M}(f)=\liminf_{r\longrightarrow+\infty}\frac{\log M(r,f)}{(\log r)^{\mu_{\log}(f)}}.$$

It is clear that, the logarithmic type of any non-constant polynomial P equals its degree $\deg P$, that any non-constant rational function is of finite logarithmic type, and that any transcendental meromorphic function whose logarithmic order equals one in the plane must be of infinite logarithmic type.

Definition 1.3 ([10]). Let f be a meromorphic function. Then, the logarithmic exponent of convergence of poles of f is defined by

$$\lambda_{\log}\left(\frac{1}{f}\right)=\limsup_{r\longrightarrow+\infty}\frac{\log n(r,f)}{\log\log r}=\limsup_{r\longrightarrow+\infty}\frac{\log N(r,f)}{\log\log r}-1,$$

where $n(r,f)$ denotes the number of poles and $N(r,f)$ is the counting function of poles of f in the disc $|z|\leq r$.

Definition 1.4 ([25]). Let $a\in\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$, the deficiency of a with respect to a meromorphic function f is given by

$$\delta(a,f)=\liminf_{r\to+\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)}=1-\limsup_{r\to+\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)}.$$

Recently, the research on the properties of meromorphic solutions of complex delay-differential equations has become a subject of great interest from the viewpoint of Nevanlinna theory and its difference analogues. In [20], Liu, Laine and Yang presented developments and new results on complex delay-differential equations, an area with important and interesting applications, which also gathers increasing attention (see, [4,5,8,24]. In [8], Chen and Zheng considered the following homogeneous complex delay-differential equation

$$\sum_{i=0}^{n}\sum_{j=0}^{m}A_{ij}(z)f^{(j)}(z+c_i)=0, \tag{1.1}$$

where $A_{ij}(z)$ $(i=0,1,\ldots,n,j=0,1,\ldots,m,n,m\in\mathbb{N})$ are entire or meromorphic functions of finite order, $c_i(i=0,\ldots,n)$ are distinct non-zero complex constants, and they proved the following results.

Theorem 1.5 ([8]). *Let $A_{ij}(z)$ $(i = 0, 1, \ldots, n, j = 0, 1, \ldots, m)$ be entire functions, and $a, l \in \{0, 1, ..., n\}$, $b \in \{0, 1, ..., m\}$ such that $(a, b) \neq (l, 0)$. If the following three assumptions hold simultaneously:*

(1) $\max\{\mu(A_{ab}), \rho(A_{ij})\colon (i, j) \neq (a, b), (l, 0)\} \leq \mu(A_{l0}) < \infty, \mu(A_{l0}) > 0$;

(2) $\underline{\tau}_M(A_{l0}) > \underline{\tau}_M(A_{ab})$, *when* $\mu(A_{l0}) = \mu(A_{ab})$;

(3) $\underline{\tau}_M(A_{l0}) > \max\{\tau_M((A_{ij}) : \rho(A_{ij}) = \mu(A_{l0})\colon (i, j) \neq (a, b), (l, 0)\}$, *when* $\mu(A_{l0}) = \max\{\rho(A_{ij})\colon (i, j) \neq (a, b), (l, 0)\}$,

then any meromorphic solution $f(z)(\not\equiv 0)$ of (1.1) satisfies $\rho(f) \geq \mu(A_{l0}) + 1$.

Theorem 1.6 ([8]). *Let $A_{ij}(z)$ $(i = 0, 1, \ldots, n, j = 0, 1, \ldots, m)$ be meromorphic functions, and $a, l \in \{0, 1, ..., n\}$, $b \in \{0, 1, ..., m\}$ such that $(a, b) \neq (l, 0)$. If the following four assumptions hold simultaneously:*

(1) $\delta(\infty, A_{l0}) = \delta > 0$;

(2) $\max\{\mu(A_{ab}), \rho(A_{ij})\colon (i, j) \neq (a, b), (l, 0)\} \leq \mu(A_{l0}) < \infty, \mu(A_{l0}) > 0$;

(3) $\delta\underline{\tau}(A_{l0}) > \underline{\tau}(A_{ab})$, *when* $\mu(A_{l0}) = \mu(A_{ab})$;

(4) $\delta\underline{\tau}(A_{l0}) > \max\{\tau((A_{ij}) : \rho(A_{ij}) = \mu(A_{l0})\colon (i, j) \neq (a, b), (l, 0)\}$, *when* $\mu(A_{l0}) = \max\{\rho(A_{ij})\colon (i, j) \neq (a, b), (l, 0)\}$,

then any meromorphic solution $f(z)(\not\equiv 0)$ of (1.1) satisfies $\rho(f) \geq \mu(A_{l0}) + 1$.

Further, Bellaama and Belaïdi in [5] extended the previous results to the non homogeneous delay differential equation

$$\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z)f^{(j)}(z + c_i) = F(z), \tag{1.2}$$

where $A_{ij}(z)$ $(i = 0, 1, \ldots, n, j = 0, 1, \ldots, m, n, m \in \mathbb{N})$, and $F(z)$ are entire or meromorphic functions of finite order, $c_i (i = 0, \ldots, n)$ are distinct non-zero complex constants, and obtained the following theorems for the homogeneous and non-homogeneous cases.

Theorem 1.7 ([5]). *Consider the delay differential equation (1.2) with entire coefficients. Suppose that one of the coefficients, say A_{l0} with $\mu(A_{l0}) > 0$, is dominate in the sense that:*

(1) $\max\{\mu(A_{ab}), \rho(S)\} \leq \mu(A_{l0}) < \infty$;

(2) $\underline{\tau}_M(A_{l0}) > \underline{\tau}_M(A_{ab})$, *whenever* $\mu(A_{l0}) = \mu(A_{ab})$;

(3) $\underline{\tau}_M(A_{l0}) > \max\{\tau_M(g) : \rho(g) = \mu(A_{l0})\colon g \in S\}$, *whenever* $\mu(A_{l0}) = \rho(S)$, *where* $S := \{F, A_{ij}\colon (i, j) \neq (a, b), (l, 0)\}$ *and* $\rho(S) := \max\{\rho(g)\colon g \in S\}$.

Then any meromorphic solution f of (1.2) satisfies $\rho(f) \geq \mu(A_{l0})$ if $F(z)(\not\equiv 0)$. Further if $F(z)(\equiv 0)$, then any meromorphic solution $f(z)(\not\equiv 0)$ of (1.1) satisfies $\rho(f) \geq \mu(A_{l0}) + 1$.

Theorem 1.8 ([5]). *Consider the delay differential equation (1.2) with meromorphic coefficients. Suppose that one of the coefficients, say A_{l0} with $\mu(A_{l0}) > 0$, is dominate in the sense that:*

(1) $\max\{\mu(A_{ab}), \rho(S)\} \leq \mu(A_{l0}) < \infty$;

(2) $\underline{\tau}(A_{l0}) > \underline{\tau}(A_{ab})$, *whenever* $\mu(A_{l0}) = \mu(A_{ab})$;

(3) $\sum_{\rho(A_{ij})=\mu(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau(A_{ij}) + \tau(F) < \underline{\tau}(A_{l0}) < \infty$, *whenever* $\mu(A_{l0}) = \rho(S)$;

(4) $\sum_{\rho(A_{ij})=\mu(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau(A_{ij}) + \underline{\tau}(A_{ab}) < \underline{\tau}(A_{l0}) < \infty$, *whenever* $\mu(A_{l0}) = \mu(A_{ab}) = \rho(S)$;

(5) $\lambda\left(\frac{1}{A_{l0}}\right) < \mu(A_{l0}) < \infty$.

Then any meromorphic solution f of (1.2) satisfies $\rho(f) \geq \mu(A_{l0})$ if $F(z)(\not\equiv 0)$. Further if $F(z)(\equiv 0)$, then any meromorphic solution $f(z)(\not\equiv 0)$ of (1.1) satisfies $\rho(f) \geq \mu(A_{l0}) + 1$.

Note that the case when the coefficients are of order zero is not included in the above results and because the logarithmic order is an effective technique to express the growth of solutions of the linear difference

equations and the linear differential equations even when the coefficients are zero order entire or meromorphic functions, in this article, our main aim is to investigate the logarithmic lower order of meromorphic solutions of equations (1.1) and (1.2) to extend and improve the above theorems. When the coefficients of (1.1) and (1.2) are meromorphic functions and there is one dominating coefficient by its logarithmic lower order or by its logarithmic lower type, we get the following two theorems.

Theorem 1.9. *Let $A_{ij}(z)$ $(i = 0,1,\ldots,n, j = 0,1,\ldots,m, n, m \in \mathbb{N})$ be meromorphic functions, and $a, l \in \{0,1,...,n\}$, $b \in \{0,1,...,m\}$ such that $(a,b) \neq (l,0)$. Suppose that one of the coefficients, say A_{l0} with $\lambda_{\log}\left(\frac{1}{A_{l0}}\right) + 1 < \mu_{\log}(A_{l0}) < \infty$, is dominate in the sense that:*

(i) $\max\{\mu_{\log}(A_{ab}), \rho_{\log}(S)\} \leq \mu_{\log}(A_{l0}) < \infty$;

(ii) $\underline{\tau}_{\log}(A_{l0}) > \underline{\tau}_{\log}(A_{ab})$, *whenever* $\mu_{\log}(A_{l0}) = \mu_{\log}(A_{ab})$;

(iii) $\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}) + \tau_{\log}(F) < \underline{\tau}_{\log}(A_{l0}) < \infty$, *whenever* $\mu_{\log}(A_{l0}) = \rho_{\log}(S)$;

(iv) $\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}) + \tau_{\log}(F) + \underline{\tau}_{\log}(A_{ab}) < \underline{\tau}_{\log}(A_{l0}) < \infty$, *whenever* $\mu_{\log}(A_{l0}) = \mu_{\log}(A_{ab}) = \rho_{\log}(S)$, *where* $S := \{F, A_{ij} : (i,j) \neq (a,b), (l,0)\}$ *and* $\rho_{\log}(S) := \max\{\rho_{\log}(g) : g \in S\}$.

Then any meromorphic solution f of (1.2) satisfies $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$ if $F(z)(\not\equiv 0)$. Further if $F(z)(\equiv 0)$, then any meromorphic solution $f(z)(\not\equiv 0)$ of (1.1) satisfies $\rho_{\log}(f) \geq \mu_{\log}(A_{l0}) + 1$.

Theorem 1.10. *Let $A_{ij}(z)$ $(i = 0,1,\ldots,n, j = 0,1,\ldots,m, n, m \in \mathbb{N})$ be meromorphic functions, and $a, l \in \{0,1,...,n\}$, $b \in \{0,1,...,m\}$ such that $(a,b) \neq (l,0)$. Suppose that one of the coefficients, say A_{l0} with $\mu(A_{l0}) > 0$ and $\delta(\infty, A_{l0}) > 0$, is dominate in the sens that:*

(i) $\max\{\mu_{\log}(A_{ab}), \rho_{\log}(S)\} \leq \mu_{\log}(A_{l0}) < \infty$;

(ii) $\delta\underline{\tau}_{\log}(A_{l0}) > \underline{\tau}_{\log}(A_{ab})$, *whenever* $\mu_{\log}(A_{l0}) = \mu_{\log}(A_{ab})$;

(iii) $\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}) + \tau_{\log}(F) < \delta\underline{\tau}_{\log}(A_{l0}) < \infty$, *whenever* $\mu_{\log}(A_{l0}) = \rho_{\log}(S)$;

(iv) $\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}) + \tau_{\log}(F) + \underline{\tau}_{\log}(A_{ab}) < \delta\underline{\tau}_{\log}(A_{l0}) < \infty$, *whenever* $\mu_{\log}(A_{l0}) = \mu_{\log}(A_{ab}) = \rho_{\log}(S)$, *where* $S := \{F, A_{ij} : (i,j) \neq (a,b), (l,0)\}$ *and* $\rho_{\log}(S) := \max\{\rho_{\log}(g) : g \in S\}$.

Then any meromorphic solution f of (1.2) satisfies $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$ if $F(z)(\not\equiv 0)$. Further if $F(z)(\equiv 0)$, then any meromorphic solution $f(z)(\not\equiv 0)$ of (1.1) satisfies $\rho_{\log}(f) \geq \mu_{\log}(A_{l0}) + 1$.

2. Some Lemmas

The following lemmas are important to our proofs.

Lemma 2.1 ([16]). *Let k and j be integers such that $k > j \geq 0$. Let f be a meromorphic function in the plane $\mathbb{C}$ such that $f^{(j)}$ does not vanish identically. Then, there exists an $r_0 > 1$ such that*

$$m(r, \frac{f^{(k)}}{f^{(j)}}) \leq (k-j)\log^+ \frac{\rho(T(\rho, f))}{r(\rho - r)} + \log\frac{k!}{j!} + 5.3078(k-j),$$

for all $r_0 < r < \rho < +\infty$. If f is of finite order s, then

$$\limsup_{r\to+\infty} \frac{m(r, \frac{f^{(k)}}{f^{(j)}})}{\log r} \leq \max\{0, (k-j)(s-1)\}.$$

Remark 2.1. It is shown in [13, p. 66], that for an arbitrary complex number $c \neq 0$, the following inequalities

$$(1+o(1))\,T(r-|c|, f(z)) \leq T(r, f(z+c)) \leq (1+o(1))\,T(r+|c|, f(z))$$

hold as $r \to +\infty$ for a general meromorphic function $f(z)$. Therefore, it is easy to obtain that

$$\rho_{\log}(f+c) = \rho_{\log}(f),\ \mu_{\log}(f+c) = \mu_{\log}(f).$$

Lemma 2.2 ([3]). *Let f be a meromorphic function with $1 \leq \mu_{\log}(f) < +\infty$. Then there exists a set $E_1 \subset (1, +\infty)$ with infinite logarithmic measure such that for any given $\varepsilon > 0$ and $r \in E_1 \subset (1, +\infty)$, we have*

$$T(r, f) < (\log r)^{\mu_{\log}(f)+\varepsilon}.$$

Lemma 2.3. *Let f be a meromorphic function with $1 \leq \mu_{\log}(f) < +\infty$. Then there exists a set $E_2 \subset (1, +\infty)$ with infinite logarithmic measure such that*

$$\underline{\tau}_{\log}(f) = \lim_{\substack{r\to+\infty \\ r\in E_2}} \frac{T(r,f)}{(\log r)^{\mu_{\log}(f)}}.$$

Consequently, for any given $\varepsilon > 0$ and all sufficiently large $r \in E_2$, we have

$$T(r, f) < \left(\underline{\tau}_{\log}(f) + \varepsilon\right)(\log r)^{\mu_{\log}(f)}.$$

Proof. By the definition of the logarithmic lower type, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $\left(1 + \frac{1}{n}\right) r_n < r_{n+1}$, and

$$\underline{\tau}_{\log}(f) = \lim_{r_n\to+\infty} \frac{T(r_n,f)}{(\log r_n)^{\mu_{\log}(f)}}.$$

Then for any given $\varepsilon > 0$, there exists an integer n_1 such that for $n \geq n_1$ and any $r \in \left[\frac{n}{n+1}r_n, r_n\right]$, we have

$$\frac{T(\frac{n}{n+1}r_n,f)}{(\log r_n)^{\mu_{\log}(f)}} \leq \frac{T(r,f)}{(\log r)^{\mu_{\log}(f)}} \leq \frac{T(r_n,f)}{(\log \frac{n}{n+1}r_n)^{\mu_{\log}(f)}}.$$

It follows that

$$\left(\frac{\log \frac{n}{n+1}r_n}{\log r_n}\right)^{\mu_{\log}(f)} \frac{T(\frac{n}{n+1}r_n,f)}{(\log \frac{n}{n+1}r_n)^{\mu_{\log}(f)}} \leq \frac{T(r,f)}{(\log r)^{\mu_{\log}(f)}}$$

$$\leq \frac{T(r_n,f)}{(\log r_n)^{\mu_{\log}(f)}} \left(\frac{\log r_n}{\log \frac{n}{n+1}r_n}\right)^{\mu_{\log}(f)}. \tag{2.1}$$

Set

$$E_2 = \bigcup_{n=n_1}^{+\infty} \left[\frac{n}{n+1}r_n, r_n\right].$$

Then from (2.1), we obtain

$$\lim_{\substack{r\to+\infty \\ r\in E_2}} \frac{T(r,f)}{(\log r)^{\mu_{\log}(f)}} = \lim_{r_n\to+\infty} \frac{T(r_n,f)}{(\log r_n)^{\mu_{\log}(f)}} = \underline{\tau}_{\log}(f),$$

so for any given $\varepsilon > 0$ and all sufficiently large $r \in E_2$, we get

$$T(r, f) < \left(\underline{\tau}_{\log}(f) + \varepsilon\right)(\log r)^{\mu_{\log}(f)},$$

where $lm(E_2) = \int_{E_2} \frac{dr}{r} = \sum_{n=n_1}^{+\infty} \int_{\frac{n}{n+1}r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_1}^{+\infty} \log\left(1 + \frac{1}{n}\right) = +\infty.$ □

Lemma 2.4 ([3]). *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let f be a finite logarithmic order meromorphic function. Let ρ be the logarithmic order of f. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O\left((\log r)^{\rho-1+\varepsilon}\right).$$

3. Proof of Theorem 1.9

Let f be a meromorphic solution of (1.2). If f has infinite logarithmic order, then the result holds. Now, we suppose that $\rho_{\log}(f) < \infty$. We divide (1.2) by $f(z+c_l)$ to get

$$\begin{aligned}-A_{l0}(z) = &\sum_{i=0,i\neq l,a}^{n}\sum_{j=0}^{m}A_{ij}\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\frac{f(z+c_i)}{f(z+c_l)}\\ &+\sum_{j=0,j\neq b}^{m}A_{aj}\frac{f^{(j)}(z+c_a)}{f(z+c_a)}\frac{f(z+c_a)}{f(z+c_l)}+\sum_{j=1}^{m}A_{lj}\frac{f^{(j)}(z+c_l)}{f(z+c_l)}\\ &+A_{ab}\frac{f^{(b)}(z+c_a)}{f(z+c_a)}\frac{f(z+c_a)}{f(z+c_l)}-\frac{F(z)}{f(z+c_l)}.\end{aligned}\tag{3.1}$$

By (3.1) and Remark 2.1, for sufficiently large r, we have

$$\begin{aligned}T(r,A_{l0}) = m(r,A_{l0})+N(r,A_{l0}) \leq &\sum_{i=0,i\neq l,a}^{n}\sum_{j=0}^{m}m(r,A_{ij})+m(r,A_{ab})\\ &+\sum_{j=1}^{m}m(r,A_{lj})+\sum_{j=0,j\neq b}^{m}m(r,A_{aj})+\sum_{i=0,i\neq l,a}^{n}\sum_{j=0}^{m}m\left(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right)\\ &+\sum_{i=0,i\neq l,a}^{n}m\left(r,\frac{f(z+c_i)}{f(z+c_l)}\right)+\sum_{j=1}^{m}m\left(r,\frac{f^{(j)}(z+c_a)}{f(z+c_a)}\right)+2m\left(r,\frac{f(z+c_a)}{f(z+c_l)}\right)\\ &+\sum_{j=1}^{m}m\left(r,\frac{f^{(j)}(z+c_l)}{f(z+c_l)}\right)+m\left(r,F\right)+m\left(r,\frac{1}{f(z+c_l)}\right)+N(r,A_{l0})+O(1)\\ \leq &\sum_{i=0,i\neq l,a}^{n}\sum_{j=0}^{m}T(r,A_{ij})+T(r,A_{ab})+\sum_{j=1}^{m}T(r,A_{lj})+\sum_{j=0,j\neq b}^{m}T(r,A_{aj})\\ &+\sum_{i=0,i\neq l,a}^{n}\sum_{j=1}^{m}m\left(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right)+\sum_{i=0,i\neq l,a}^{n}m\left(r,\frac{f(z+c_i)}{f(z+c_l)}\right)\\ &+\sum_{j=1}^{m}m\left(r,\frac{f^{(j)}(z+c_a)}{f(z+c_a)}\right)+2m\left(r,\frac{f(z+c_a)}{f(z+c_l)}\right)\\ &+\sum_{j=1}^{m}m\left(r,\frac{f^{(j)}(z+c_l)}{f(z+c_l)}\right)+T\left(r,F\right)+2T\left(2r,f\right)+N(r,A_{l0})+O(1).\end{aligned}\tag{3.2}$$

From Lemma 2.1, for sufficiently large r, we obtain

$$m\left(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right)\leq 2j\log^{+}T\left(2r,f\right),\ (i=0,1,...,n,j=1,...,m).\tag{3.3}$$

By Lemma 2.4, for any given $\varepsilon>0$ and all sufficiently large r, we have

$$m\left(r,\frac{f(z+c_i)}{f(z+c_l)}\right)=O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right),\ (i=0,1,...,n,i\neq l).\tag{3.4}$$

From the definition of $\lambda_{\log}\left(\frac{1}{A_{l0}}\right)$, for any given $\varepsilon>0$ with sufficiently large r, we have

$$N(r,A_{l0})\leq(\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}.\tag{3.5}$$

By using the assumptions (3.3)-(3.5), we may rewrite (3.2) as

$$T(r,A_{l0})\leq\sum_{i=0,i\neq l,a}^{n}\sum_{j=0}^{m}T(r,A_{ij})+T(r,A_{ab})$$

$$+\sum_{j=1}^{m} T(r, A_{lj}) + \sum_{j=0, j\neq b}^{m} T(r, A_{aj}) + O\left(\log^{+} T\left(2r, f\right)\right)$$

$$+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) + T\left(r, F\right) + 2T\left(2r, f\right) + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \tag{3.6}$$

This proof is also divided into four cases:

Case (i): If $\max\{\mu_{\log}(A_{ab}), \rho_{\log}(S)\} < \mu_{\log}(A_{l0})$, then by the definitions of $\mu_{\log}(A_{l0})$ and $\rho_{\log}(S)$ for any given $\varepsilon > 0$ and all sufficiently large r, we have

$$T(r, A_{l0}) \geq (\log r)^{\mu_{\log}(A_{l0})-\varepsilon}, \tag{3.7}$$

$$T(r, g) \leq (\log r)^{\rho_{\log}(S)+\varepsilon}, \quad g \in S. \tag{3.8}$$

By the definition of $\mu_{\log}(A_{ab})$ and Lemma 2.2, there exists a subset $E_1 \subset (1, +\infty)$ of infinite logarithmic measure such that for any given $\varepsilon > 0$ and for all sufficiently large $r \in E_1$, we have

$$T(r, A_{ab}) \leq (\log r)^{\mu_{\log}(A_{ab})+\varepsilon}. \tag{3.9}$$

We set $\rho = \max\{\mu_{\log}(A_{ab}), \rho_{\log}(S)\}$, then from (3.8) and (3.9), it follows

$$\max\left\{T(r, A_{ab}), T(r, g)\right\} \leq (\log r)^{\rho+\varepsilon}. \tag{3.10}$$

Also, from the definition of $\rho_{\log}(f)$ for any given $\varepsilon > 0$ and all sufficiently large r, we have

$$T(r, f) \leq (\log r)^{\rho_{\log}(f)+\varepsilon}. \tag{3.11}$$

By substituting (3.7), (3.10) and (3.11) into (3.6), for any given $\varepsilon > 0$ and all sufficiently large $r \in E_1$, we get

$$(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq O\left((\log r)^{\rho+\varepsilon}\right) + O\left(\log\left(\log r\right)\right) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)$$

$$+O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right) + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \tag{3.12}$$

Now, we choose sufficiently small ε satisfying

$$0 < 3\varepsilon < \min\left\{\mu_{\log}(A_{l0}) - \rho, \mu_{\log}(A_{l0}) - \lambda_{\log}\left(\frac{1}{A_{l0}}\right) - 1\right\},$$

for all sufficiently large $r \in E_1$, it follows from (3.12) that

$$(\log r)^{\mu_{\log}(A_{l0})-2\varepsilon} \leq (\log r)^{\rho_{\log}(f)+\varepsilon},$$

that means, $\mu_{\log}(A_{l0}) - 3\varepsilon \leq \rho_{\log}(f)$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$.
Similarly, for the homogeneous case, by (1.1) and (3.3)-(3.5), we obtain

$$T(r, A_{l0}) \leq \sum_{i=0, i\neq l, a}^{n} \sum_{j=0}^{m} T(r, A_{ij}) + T(r, A_{ab}) + \sum_{j=1}^{m} T(r, A_{lj}) + \sum_{j=0, j\neq b}^{m} T(r, A_{aj})$$

$$+O\left(\log\left(\log r\right)\right) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \tag{3.13}$$

Then, by substituting (3.7) and (3.10) into (3.13), for all sufficiently large $r \in E_1$, we have

$$(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq O\left((\log r)^{\rho+\varepsilon}\right) + O\left(\log\left(\log r\right)\right)$$

$$+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \tag{3.14}$$

For sufficiently small ε satisfying

$$0 < 3\varepsilon < \min\left\{\mu_{\log}(A_{l0}) - \rho, \mu_{\log}(A_{l0}) - \lambda_{\log}\left(\frac{1}{A_{l0}}\right) - 1\right\},$$

and all sufficiently large $r \in E_1$, we deduce from (3.14) that

$$(\log r)^{\mu_{\log}(A_{l0})-2\varepsilon} \leq \log r)^{\rho_{\log}(f)-1+\varepsilon},$$

that is, $\mu_{\log}(A_{l0}) - 3\varepsilon \leq \rho_{\log}(f) - 1$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0}) + 1$.
Case (ii): If $\beta = \rho_{\log}(S) < \mu_{\log}(A_{l0}) = \mu_{\log}(A_{ab})$ and $\underline{\tau}_{\log}(A_{l0}) > \underline{\tau}_{\log}(A_{ab})$, then by the definition of $\underline{\tau}_{\log}(A_{l0})$, for any given $\varepsilon > 0$ and all sufficiently large r, we have

$$T(r, A_{l0}) \geq (\underline{\tau}_{\log}(A_{l0}) - \varepsilon)(\log r)^{\mu_{\log}(A_{l0})}. \tag{3.15}$$

Also from the definition of $\underline{\tau}_{\log}(A_{ab})$ and Lemma 2.3 there exists a subset $E_2 \subset (1, +\infty)$ of infinite logarithmic measure such that for any given $\varepsilon > 0$ and for all sufficiently large $r \in E_2$, we obtain

$$T(r, A_{ab}) \leq (\underline{\tau}_{\log}(A_{ab}) + \varepsilon)(\log r)^{\mu_{\log}(A_{ab})} = (\underline{\tau}_{\log}(A_{ab}) + \varepsilon)(\log r)^{\mu_{\log}(A_{l0})}. \tag{3.16}$$

By substituting (3.8), (3.11), (3.15) and (3.16) into (3.6), for all sufficiently large $r \in E_2$, we get

$$(\underline{\tau}_{\log}(A_{l0}) - \varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \leq O\left((\log r)^{\beta+\varepsilon}\right)$$
$$+(\underline{\tau}_{\log}(A_{ab}) + \varepsilon)(\log r)^{\mu_{\log}(A_{l0})} + O\left(\log\left(\log r\right)\right) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)$$

$$+O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right) + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \tag{3.17}$$

Now, we choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min\left\{\mu_{\log}(A_{l0}) - \beta, \mu_{\log}(A_{l0}) - \lambda_{\log}\left(\frac{1}{A_{l0}}\right) - 1, \underline{\tau}_{\log}(A_{l0}) - \underline{\tau}_{\log}(A_{ab})\right\},$$

for all sufficiently large $r \in E_2$, it follows from (3.17) that

$$(\underline{\tau}_{\log}(A_{l0}) - \underline{\tau}_{\log}(A_{ab}) - 2\varepsilon)(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq (\log r)^{\rho_{\log}(f)+\varepsilon},$$

this means, $\mu_{\log}(A_{l0}) - 2\varepsilon \leq \rho_{\log}(f)$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$.
Next, for the homogeneous case, by substituting (3.8), (3.15) and (3.16) into (3.13), for all sufficiently large $r \in E_2$, we have

$$(\underline{\tau}_{\log}(A_{l0}) - \varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \leq O\left((\log r)^{\beta+\varepsilon}\right) + (\underline{\tau}_{\log}(A_{ab}) + \varepsilon)(\log r)^{\mu_{\log}(A_{l0})}$$

$$+O\left(\log\left(\log r\right)\right) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \tag{3.18}$$

Now, we choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min\left\{\mu_{\log}(A_{l0}) - \beta, \mu_{\log}(A_{l0}) - \lambda_{\log}\left(\frac{1}{A_{l0}}\right) - 1, \underline{\tau}_{\log}(A_{l0}) - \underline{\tau}_{\log}(A_{ab})\right\},$$

for all sufficiently large $r \in E_2$, we deduce from (3.18) that

$$(\underline{\tau}_{\log}(A_{l0}) - \underline{\tau}_{\log}(A_{ab}) - 2\varepsilon)(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq (\log r)^{\rho_{\log}(f)-1+\varepsilon},$$

that is, $\mu_{\log}(A_{l0}) - 2\varepsilon \leq \rho_{\log}(f) - 1$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0}) + 1$.
Case (iii): When $\mu_{\log}(A_{ab}) < \mu_{\log}(A_{l0}) = \rho_{\log}(S)$ and

$$\tau_1 = \sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}) + \tau_{\log}(F)$$

$$= \tau + \tau_{\log}(F) < \underline{\tau}_{\log}(A_{l0}),\ \tau = \sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}).$$

Then, there exists a subset $J \subseteq \{0,1,\ldots,n\} \times \{0,1,\ldots,m\} \setminus \{(l,0),(a,b)\}$ such that for all $(i,j) \in J$, when $\rho_{\log}(A_{ij}) = \mu_{\log}(A_{l0})$, we have $\sum_{(i,j)\in J} \tau_{\log}(A_{ij}) < \underline{\tau}_{\log}(A_{l0}) - \tau_{\log}(F)$, and for $(i,j) \in \Pi = \{0,1,\ldots,n\} \times \{0,1,\ldots,m\} \setminus (J \cup \{(l,0),(a,b)\})$ we have $\rho_{\log}(A_{ij}) < \mu_{\log}(A_{l0})$. Hence, for any given $\varepsilon > 0$ and all sufficiently large r, we get

$$T(r,A_{ij}) \leq \begin{cases} (\tau_{\log}(A_{ij})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}, \text{ if } (i,j)\in J, \\ (\log r)^{\rho_{\log}(A_{ij})+\varepsilon} \leq (\log r)^{\mu_{\log}(A_{l0})-\varepsilon}, \text{ if } (i,j)\in \Pi \end{cases} \tag{3.19}$$

and

$$T(r,F) \leq \begin{cases} (\tau_{\log}(F)+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}, \text{ if } \rho_{\log}(F)=\mu_{\log}(A_{l0}), \\ (\log r)^{\rho_{\log}(F)+\varepsilon} \leq (\log r)^{\mu_{\log}(A_{l0})-\varepsilon}, \text{ if } \rho_{\log}(F)<\mu_{\log}(A_{l0}). \end{cases} \tag{3.20}$$

By substituting (3.9), (3.11), (3.15), (3.19) and (3.20) into (3.6), for all sufficiently large $r \in E_1$, we get

$$\begin{aligned} \left(\underline{\tau}_{\log}(A_{l0})-\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})} &\leq \sum_{(i,j)\in J} (\tau_{\log}(A_{ij})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \\ &+ \sum_{(i,j)\in\Pi} (\log r)^{\mu_{\log}(A_{l0})-\varepsilon} + (\log r)^{\mu_{\log}(A_{ab})+\varepsilon} + O(\log(\log r)) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) \\ &+ (\tau_{\log}(F)+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} + O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right) + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon} \\ &\leq (\tau_1 + (mn+m+n)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} + O(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \\ &+ (\log r)^{\mu_{\log}(A_{ab})+\varepsilon} + O(\log(\log r)) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) \\ &+ O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right) + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \end{aligned} \tag{3.21}$$

We may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min\left\{\mu_{\log}(A_{l0}) - \mu_{\log}(A_{ab}), \mu_{\log}(A_{l0}) - \lambda_{\log}\left(\frac{1}{A_{l0}}\right) - 1, \frac{\underline{\tau}_{\log}(A_{l0}) - \tau_1}{mn+m+n+1}\right\},$$

for all sufficiently large $r \in E_1$, by (3.21) we have

$$(\underline{\tau}_{\log}(A_{l0}) - \tau_1 - (mn+m+n+1)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq (\log r)^{\rho_{\log}(f)+\varepsilon},$$

this means, $\mu_{\log}(A_{l0}) - 2\varepsilon \leq \rho_{\log}(f)$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$.
Further, for the homogeneous case, by substituting (3.9), (3.15), (3.19) and (3.20) into (3.13), for all sufficiently large $r \in E_1$, we get

$$\begin{aligned} (\underline{\tau}_{\log}(A_{l0}) - \tau - (nm+m+n)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} &\leq O\left((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\right) \\ &+ (\log r)^{\mu_{\log}(A_{ab})+\varepsilon} + O(\log(\log r)) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \end{aligned} \tag{3.22}$$

We may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min\left\{\mu_{\log}(A_{l0}) - \mu_{\log}(A_{ab}), \mu_{\log}(A_{l0}) - \lambda_{\log}\left(\frac{1}{A_{l0}}\right) - 1, \frac{\underline{\tau}_{\log}(A_{l0}) - \tau}{nm+m+n}\right\},$$

for all sufficiently large $r \in E_1$, by (3.22) we have

$$(\underline{\tau}_{\log}(A_{l0}) - \tau - (nm+m+n)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq (\log r)^{\rho_{\log}(f)-1+\varepsilon},$$

that is, $\mu_{\log}(A_{l0}) - 2\varepsilon \leq \rho_{\log}(f) - 1$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0}) + 1$.

Case (iv): When $\mu_{\log}(A_{l0}) = \mu_{\log}(A_{ab}) = \rho_{\log}(S)$ and

$$\tau_3 = \sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}) + \tau_{\log}(F) + \underline{\tau}_{\log}(A_{ab})$$
$$= \tau_2 + \tau_{\log}(F) < \underline{\tau}_{\log}(A_{l0}),$$
$$\tau_2 = \sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}) + \underline{\tau}_{\log}(A_{ab}).$$

Then, by substituting (3.11), (3.15), (3.16), (3.19) and (3.20) into (3.6), for all sufficiently large $r \in E_1$, we have

$$(\underline{\tau}_{\log}(A_{l0}) - \tau_3 - (mn+m+n+2)\,\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \leq O\left((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\right)$$
$$+O\left(\log\left(\log r\right)\right) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)$$

$$+O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right) + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \tag{3.23}$$

Now, we may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min\left\{\mu_{\log}(A_{l0}) - \lambda_{\log}\left(\frac{1}{A_{l0}}\right) - 1, \frac{\underline{\tau}_{\log}(A_{l0}) - \tau_3}{mn+m+n+2}\right\},$$

for all sufficiently large $r \in E_1$, we deduce from (3.23) that

$$(\underline{\tau}_{\log}(A_{l0}) - \tau_3 - (mn+m+n+2)\,\varepsilon)(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq (\log r)^{\rho_{\log}(f)+\varepsilon},$$

this means, $\mu_{\log}(A_{l0}) - 2\varepsilon \leq \rho_{\log}(f)$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$.
Further, for the homogeneous case, by substituting (3.15), (3.16), (3.19) and (3.20) into (3.13), for all sufficiently large $r \in E_1$, we get

$$(\underline{\tau}_{\log}(A_{l0}) - \tau_2 - (mn+m+n+1)\,\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \leq O\left((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\right)$$

$$+O\left(\log\left(\log r\right)\right) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \tag{3.24}$$

Therefore, for ε satisfying

$$0 < 2\varepsilon < \min\left\{\mu_{\log}(A_{l0}) - \lambda_{\log}\left(\frac{1}{A_{l0}}\right) - 1, \frac{\underline{\tau}_{\log}(A_{l0}) - \tau_2}{mn+m+n+1}\right\}$$

and for all sufficiently large $r \in E_1$, by (3.24) we have

$$(\underline{\tau}_{\log}(A_{l0}) - \tau_2 - (mn+m+n+1)\,\varepsilon)(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq (\log r)^{\rho_{\log}(f)-1+\varepsilon},$$

that is, $\mu_{\log}(A_{l0}) - 2\varepsilon \leq \rho_{\log}(f) - 1$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0}) + 1$. The proof of Theorem 1.9 is complete.

4. Proof of Theorem 1.10

Let f be a meromorphic solution of (1.2). If f has infinite logarithmic order, then the result holds. Now, we suppose that $\rho_{\log}(f) < \infty$. By (3.1) and Remark 2.1, for sufficiently large r, we have

$$m(r, A_{l0}) \leq \sum_{i=0, i\neq l, a}^{n} \sum_{j=0}^{m} m(r, A_{ij}) + m(r, A_{ab})$$
$$+\sum_{j=1}^{m} m(r, A_{lj}) + \sum_{j=0, j\neq b}^{m} m(r, A_{aj}) + \sum_{i=0, i\neq l, a}^{n} \sum_{j=0}^{m} m\left(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right)$$

$$+\sum_{i=0,i\neq l,a}^{n} m\left(r,\frac{f(z+c_i)}{f(z+c_l)}\right)+\sum_{j=1}^{m} m\left(r,\frac{f^{(j)}(z+c_a)}{f(z+c_a)}\right)+2m\left(r,\frac{f(z+c_a)}{f(z+c_l)}\right)$$
$$+\sum_{j=1}^{m} m\left(r,\frac{f^{(j)}(z+c_a)}{f(z+c_a)}\right)+m\left(r,\frac{F(z)}{f(z+c_l)}\right)+O(1)$$
$$\leq \sum_{i=0,i\neq l,a}^{n}\sum_{j=0}^{m} T(r,A_{ij})+T(r,A_{ab})+\sum_{j=1}^{m} T(r,A_{lj})$$
$$+\sum_{j=0,j\neq b}^{m} T(r,A_{aj})+\sum_{i=0,i\neq l,a}^{n}\sum_{j=0}^{m} m\left(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right)$$
$$+\sum_{i=0,i\neq l,a}^{n} m\left(r,\frac{f(z+c_i)}{f(z+c_l)}\right)+\sum_{j=1}^{m} m\left(r,\frac{f^{(j)}(z+c_a)}{f(z+c_a)}\right)+2m\left(r,\frac{f(z+c_a)}{f(z+c_l)}\right)$$

$$+\sum_{j=1}^{m} m\left(r,\frac{f^{(j)}(z+c_a)}{f(z+c_a)}\right)+T(r,F)+2T(2r,f)+O(1). \tag{4.1}$$

By substituting (3.3) and (3.4) into (4.1), for any given $\varepsilon>0$ and all sufficiently large r, we obtain

$$m(r,A_{l0})\leq \sum_{i=0,i\neq l,a}^{n}\sum_{j=0}^{m} T(r,A_{ij})+T(r,A_{ab})+\sum_{j=1}^{m} T(r,A_{lj})+\sum_{j=0,j\neq b}^{m} T(r,A_{aj})$$

$$+O\left(\log^{+}T\left(2r,f\right)\right)+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)+T(r,F)+2T(2r,f). \tag{4.2}$$

Let us set

$$\delta=\delta(\infty,A_{l0})>0. \tag{4.3}$$

Now, we divide this proof into four cases:

Case (i): If $\max\{\mu_{\log}(A_{ab}),\rho_{\log}(S)\}<\mu_{\log}(A_{l0})$, then by the definition of $\mu_{\log}(A_{l0})$ and (4.3), for any given $\varepsilon>0$ and all sufficiently large r, we have

$$m(r,A_{l0})\geq\frac{\delta}{2}T(r,A_{l0})\geq\frac{\delta}{2}(\log r)^{\mu_{\log}(A_{l0})-\frac{\varepsilon}{2}}\geq(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}. \tag{4.4}$$

By substituting (3.10), (3.11) and (4.4) into (4.2), for all sufficiently large r, we get

$$(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\leq O\left((\log r)^{\rho+\varepsilon}\right)+O(\log(\log r))$$

$$+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)+O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right). \tag{4.5}$$

Now, we choose sufficiently small ε satisfying $0<3\varepsilon<\mu_{\log}(A_{l0})-\rho$, for all sufficiently large r, it follows from (3.10) that

$$(\log r)^{\mu_{\log}(A_{l0})-2\varepsilon}\leq(\log r)^{\rho_{\log}(f)+\varepsilon},$$

this means, $\mu_{\log}(A_{l0})-3\varepsilon\leq\rho_{\log}(f)$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})$.
Similarly, for the homogeneous case, by (1.1) and (3.3) and (3.4), we obtain

$$m(r,A_{l0})\leq \sum_{i=0,i\neq l,a}^{n}\sum_{j=0}^{m} T(r,A_{ij})+T(r,A_{ab})+\sum_{j=1}^{m} T(r,A_{lj})+\sum_{j=0,j\neq b}^{m} T(r,A_{aj})$$

$$+O(\log(\log r))+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{4.6}$$

Then, by substituting (3.10) and (4.4) into (4.6), for all sufficiently large r, we have

$$(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq O\left((\log r)^{\rho+\varepsilon}\right) + O(\log(\log r)) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{4.7}$$

For the above ε and all sufficiently large r, we deduce from (4.7) that

$$(\log r)^{\mu_{\log}(A_{l0})-2\varepsilon} \leq (\log r)^{\rho_{\log}(f)-1+\varepsilon},$$

that is, $\mu_{\log}(A_{l0}) - 3\varepsilon \leq \rho_{\log}(f) - 1$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0}) + 1$.

Case (ii): If $\beta = \rho_{\log}(S) < \mu_{\log}(A_{l0}) = \mu_{\log}(A_{ab})$ and $\delta\underline{\tau}_{\log}(A_{l0}) > \underline{\tau}_{\log}(A_{ab})$, then by the definition of $\underline{\tau}_{\log}(A_{l0})$ and (4.3), for any given $\varepsilon > 0$ and all sufficiently large r, we have

$$\begin{aligned} m(r, A_{l0}) &\geq (\delta-\varepsilon)T(r, A_{l0}) \geq (\delta-\varepsilon)(\underline{\tau}_{\log}(A_{l0})-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \\ &\geq \left(\delta\underline{\tau}_{\log}(A_{l0}) - (\underline{\tau}_{\log}(A_{l0})+\delta)\varepsilon + \varepsilon^2\right)(\log r)^{\mu_{\log}(A_{l0})} \end{aligned}$$

$$\geq \left(\delta\underline{\tau}_{\log}(A_{l0}) - (\underline{\tau}_{\log}(A_{l0})+\delta)\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})}. \tag{4.8}$$

By substituting (3.8), (3.11), (3.16) and (4.8) into (4.2), for all sufficiently large $r \in E_2$, we get

$$\left(\delta\underline{\tau}_{\log}(A_{l0}) - \underline{\tau}_{\log}(A_{ab}) - (\underline{\tau}_{\log}(A_{l0})+\delta+1)\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})} \leq O\left((\log r)^{\beta+\varepsilon}\right)$$

$$+O(\log(\log r)) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) + O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right). \tag{4.9}$$

Now, we choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min\left\{\mu_{\log}(A_{l0}) - \beta, \frac{\delta\underline{\tau}_{\log}(A_{l0}) - \underline{\tau}_{\log}(A_{ab})}{\underline{\tau}_{\log}(A_{l0})+\delta+1}\right\},$$

for all sufficiently large $r \in E_2$, by (4.9), we obtain

$$\begin{aligned} &\left(\delta\underline{\tau}_{\log}(A_{l0}) - \underline{\tau}_{\log}(A_{ab}) - (\underline{\tau}_{\log}(A_{l0})+\delta+1)\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \\ &\leq (\log r)^{\rho_{\log}(f)+\varepsilon}, \end{aligned}$$

this means, $\mu_{\log}(A_{l0}) - 2\varepsilon \leq \rho_{\log}(f)$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$.

Next, for the homogeneous case, by substituting (3.8), (3.16) and (4.8) into (4.6), for all sufficiently large $r \in E_2$, we have

$$\left(\delta\underline{\tau}_{\log}(A_{l0}) - \underline{\tau}_{\log}(A_{ab}) - (\underline{\tau}_{\log}(A_{l0})+\delta+1)\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})} \leq O\left((\log r)^{\beta+\varepsilon}\right)$$

$$+O(\log(\log r)) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{4.10}$$

For the above ε and all sufficiently large $r \in E_2$, from (4.10), we obtain

$$\begin{aligned} &\left(\delta\underline{\tau}_{\log}(A_{l0}) - \underline{\tau}_{\log}(A_{ab}) - (\underline{\tau}_{\log}(A_{l0})+\delta+1)\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \\ &\leq (\log r)^{\rho_{\log}(f)-1+\varepsilon}, \end{aligned}$$

that is, $\mu_{\log}(A_{l0}) - 2\varepsilon \leq \rho_{\log}(f) - 1$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0}) + 1$.

Case (iii): When $\mu_{\log}(A_{ab}) < \mu_{\log}(A_{l0}) = \rho_{\log}(S)$ and

$$\tau_1 = \sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}) + \tau_{\log}(F) < \delta\underline{\tau}_{\log}(A_{l0}).$$

Then, by substituting (3.9), (3.11), (3.19), (3.20) and (4.8) into (4.2), for all sufficiently large $r \in E_1$, we get

$$(\delta\underline{\tau}_{\log}(A_{l0}) - \tau_1 - \left(\underline{\tau}_{\log}(A_{l0}) + \delta + mn + m + n + 1\right)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}$$

$$\leq O\left((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\right)+(\log r)^{\mu_{\log}(A_{ab})+\varepsilon}+O(\log(\log r))$$

$$+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)+O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right). \tag{4.11}$$

We may choose sufficiently small ε satisfying

$$0<2\varepsilon<\min\left\{\mu_{\log}(A_{l0})-\mu_{\log}(A_{ab}),\frac{\underline{\tau}_{\log}(A_{l0})-\tau_1}{\underline{\tau}_{\log}(A_{l0})+\delta+mn+m+n+1}\right\},$$

for all sufficiently large $r\in E_1$, by (4.11), we obtain

$$(\delta\underline{\tau}_{\log}(A_{l0})-\tau_1-\left(\underline{\tau}_{\log}(A_{l0})+\delta+mn+m+n+1\right)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}$$
$$\leq(\log r)^{\rho_{\log}(f)+\varepsilon},$$

this means, $\mu_{\log}(A_{l0})-2\varepsilon\leq\rho_{\log}(f)$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})$.
Further, for the homogeneous case, by substituting (3.9), (3.19), (3.20) and (4.8) into (4.6), for all sufficiently large $r\in E_1$, we get

$$(\delta\underline{\tau}_{\log}(A_{l0})-\tau-\left(\underline{\tau}_{\log}(A_{l0})+\delta+nm+m+n\right)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}$$

$$\leq O\left((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\right)+(\log r)^{\mu_{\log}(A_{ab})+\varepsilon}+O(\log(\log r))+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{4.12}$$

For ε sufficiently small satisfying

$$0<2\varepsilon<\min\left\{\mu_{\log}(A_{l0})-\mu_{\log}(A_{ab}),\frac{\underline{\tau}_{\log}(A_{l0})-\tau}{\underline{\tau}_{\log}(A_{l0})+\delta+nm+m+n}\right\},$$

and for all sufficiently large $r\in E_1$, from (4.12) we conclude

$$(\delta\underline{\tau}_{\log}(A_{l0})-\tau-\left(\underline{\tau}_{\log}(A_{l0})+\delta+nm+m+n\right)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\leq(\log r)^{\rho_{\log}(f)-1+\varepsilon},$$

that is, $\mu_{\log}(A_{l0})-2\varepsilon\leq\rho_{\log}(f)-1$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})+1$.
Case (iv): When $\mu_{\log}(A_{l0})=\mu_{\log}(A_{ab})=\rho_{\log}(S)$ and

$$\tau_3=\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)}\tau_{\log}(A_{ij})+\tau_{\log}(F)+\underline{\tau}_{\log}(A_{ab})<\underline{\tau}_{\log}(A_{l0}).$$

Then, by substituting (3.9), (3.11), (3.19), (3.20) and (4.8) into (4.2), for all sufficiently large $r\in E_1$, we get

$$(\delta\underline{\tau}_{\log}(A_{l0})-\tau_3-\left(\underline{\tau}_{\log}(A_{l0})+\delta+mn+m+n+2\right)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}$$

$$\leq O\left((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\right)+O(\log(\log r))+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)+O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right). \tag{4.13}$$

Now, we may choose sufficiently small ε satisfying $0<2\varepsilon<\frac{\delta\underline{\tau}_{\log}(A_{l0})-\tau_3}{\underline{\tau}_{\log}(A_{l0})+\delta+mn+m+n+2}$, for all sufficiently large $r\in E_1$, we deduce from (4.13) that

$$(\delta\underline{\tau}_{\log}(A_{l0})-\tau_3-\left(\underline{\tau}_{\log}(A_{l0})+\delta+mn+m+n+2\right)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}$$
$$\leq(\log r)^{\rho_{\log}(f)+\varepsilon},$$

this means, $\mu_{\log}(A_{l0})-2\varepsilon\leq\rho_{\log}(f)$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})$.
Also for the homogeneous case, by substituting (3.9), (3.19), (3.20) and (4.8) into (4.6), for all sufficiently large $r\in E_1$, we have

$$(\delta\underline{\tau}_{\log}(A_{l0})-\tau_2-\left(\underline{\tau}_{\log}(A_{l0})+\delta+mn+m+n+1\right))\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}$$

$$\leq O\left((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\right)+O(\log(\log r))+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{4.14}$$

Thus, for sufficiently small ε satisfying $0<2\varepsilon<\frac{\delta_{\underline{\tau}_{\log}}(A_{l0})-\tau_2}{\underline{\tau}_{\log}(A_{l0})+\delta+mn+m+n+1}$, for all sufficiently large $r\in E_1$, from (4.14) we obtain

$$\begin{aligned}&(\delta_{\underline{\tau}_{\log}}(A_{l0})-\tau_2-\left(\underline{\tau}_{\log}(A_{l0})+\delta+mn+m+n+1\right)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\\&\leq(\log r)^{\rho_{\log}(f)-1+\varepsilon},\end{aligned}$$

that is, $\mu_{\log}(A_{l0})-2\varepsilon\leq\rho_{\log}(f)-1$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})+1$ which completes the proof of Theorem 1.10.

5. Example

The following example is for illustrating the sharpness of some assertions in Theorem 1.10.

Example 5.1. For Theorem 1.10, we consider the meromorphic function

$$f(z)=\frac{1}{z^5} \tag{5.1}$$

which is a solution to the delay-differential equation

$$\begin{aligned}&A_{20}(z)f(z-2i)+A_{11}(z)f'(z+i)+A_{10}(z)f(z+i)\\&+A_{01}(z)f'(z)+A_{00}(z)f(z)=F(z),\end{aligned} \tag{5.2}$$

where $A_{20}(z)=\frac{2}{3}(z-2i)^4$, $A_{11}(z)=2e$, $A_{10}(z)=\frac{10e}{z+i}$, $A_{01}(z)=\frac{i}{2}$, $A_{00}(z)=\frac{5i}{2z}$ and $F(z)=\frac{2}{3(z-2i)}$. Obviously, $A_{ij}(z)$ $(i=0,1,2,j=0,1)$ and $F(z)$ satisfy the conditions in Case (iii) of Theorem 1.10 such that

$$\delta(\infty,A_{20})=1>0,$$

$$\mu_{\log}(A_{11})=0<\max\{\rho_{\log}(F),\rho_{\log}(A_{ij}),(i,j)\neq(1,1),(2,0)\}=\mu_{\log}(A_{20})=1$$

and

$$\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{20}),(i,j)\neq(1,1),(2,0)}\tau_{\log}(A_{ij})+\tau_{\log}(F)=3<\delta_{\underline{\tau}_{\log}}(A_{20})=4.$$

We see that f satisfies

$$\mu_{\log}(f)=1=\rho_{\log}(A_{20}).$$

The meromorphic function $f(z)=\frac{1}{z^5}$ is a solution of equation (5.2) for the coefficients $A_{20}(z)=3(z-2i)^7$, $A_{11}(z)=\frac{1}{z-i}$, $A_{10}(z)=\frac{5}{z^2+1}$, $A_{01}(z)=\frac{i}{2}$, $A_{00}(z)=\frac{5i}{2z}$ and $F(z)=3(z-2i)^2$. Clearly, $A_{ij}(z)$ $(i=0,1,2,j=0,1)$ and $F(z)$ satisfy the conditions in Case (iv) of Theorem 1.10 such that

$$\delta(\infty,A_{20})=1>0,$$

$$\mu_{\log}(A_{11})=\max\{\rho_{\log}(F),\rho_{\log}(A_{ij}),(i,j)\neq(1,1),(2,0)\}=\mu_{\log}(A_{20})=1$$

and

$$\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{20}),(i,j)\neq(1,1),(2,0)}\tau_{\log}(A_{ij})+\tau_{\log}(F)+\underline{\tau}_{\log}(A_{11})=6<\delta_{\underline{\tau}_{\log}}(A_{20})=7.$$

We see that f satisfies $\rho_{\log}(f)=1=\mu_{\log}(A_{20})$.

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