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REVISIT OF AN IMPROVED WILKER TYPE INEQUALITY

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ABSTRACT. In this article, we revisit an improved Wilker type inequality established in 2020. We fill some gaps in the existing proof and propose an alternative proof using the same mathematical ingredients. All the details are given for checking purposes.

1. INTRODUCTION

Trigonometric type inequalities hold significant importance in mathematics and its applications. These inequalities establish bounds and relationships among trigonometric functions and play a fundamental role in solving problems involving angles and periodic functions. They provide valuable tools for proving other mathematical theorems and inequalities, aiding in the development of mathematical analysis and calculus. Moreover, trigonometric type inequalities find practical applications in physics, engineering, and various scientific disciplines, enabling accurate modeling and prediction of phenomena involving periodic behavior. Emphasizing their significance contribute to a deeper understanding and utilization of trigonometry in diverse fields. Former and recent results on this topic can be found in [1–9].

In particular, the authors in [2] demonstrated the following inequalities:

$$\frac{16}{\pi^4}x^3\tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 < \frac{8}{45}x^3\tan x, \quad x \in \left(0, \frac{\pi}{2}\right).$$

It is known as the Wilker inequalities. Elegant proofs are given in [10] and [11]. In 2020, Theorem 1.1 in [12] stated the following improvement:

Theorem 1.1. *[12, Theorem 1.1] The inequality,*

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 > \frac{8}{45}x^4\left(\frac{\tan x}{x}\right)^{6/7}, \quad x \in \left(0, \frac{\pi}{2}\right)$$

is valid.

The proof of this sharp inequality given on page no. 4879 is mainly based on differentiation techniques, several power series formulae and the deep use of Bernoulli's numbers B_n (including a result established in [9]). However, most of the details in the developments are omitted, certainly for the sake of place; the developments are very demanding in their algebraic manipulations of numerous terms. Unfortunately, it

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seems that a term has been omitted at one step, and in all the other steps, making the proof perfectible from a mathematical viewpoint.
The aim of this paper is to revisit this proof. In Section 2, we revise some key results of the proof of Theorem 1.1 in [12]. The detailed proofs are given in Section 3.

2. Alternative Proof of Theorem 1.1 in [12]

The proof of Theorem 1.1 in [12] is mainly based on three complementary results: one on the derivative of a thoroughly selected function, one on a series expansion of a key function and another on an inequality involving the coefficient of the previous series expansion.

2.1. **Main results.**

The next lemma presents the obtained differentiation of the main function used in the first step in the proof of Theorem 1.1 in [12].

Lemma 2.1. *Let us consider the following function:*

$$F(x) = 7\ln\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right] - \ln\left[\left(\frac{8}{45}\right)^7 x^{22}\tan^6 x\right], \quad x \in \left(0, \frac{\pi}{2}\right).$$

Then we have

$$F'(x) = \frac{\cos^2 x}{x^3(\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}.f(x),$$

where

$$f(x) = 12x^3\sec^2 x + x^2\sec^2 x\tan^3 x + 12x^3\tan^2 x - 6x\tan^2 x + 14x\tan^2 x - 29x\tan^2 x\sec^2 x + 45x^2\tan x\sec^2 x - 36\tan^3 x - 6x\tan^4 x + 12x^3\tan^4 x. \tag{2.1}$$

In the proof of Theorem 1.1 in [12], it is found that

$$F'(x) = \frac{\cos^2 x}{x^3(\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}.f^\dagger(x),$$

where

$$f^\dagger(x) = f(x) - 12x^3\tan^4 x. \tag{2.2}$$

In other words, the term $12x^3\tan^4 x$ is missing in the derivative of $F(x)$ in the proof of Theorem 1.1 in [12]. And this term is not considered in the rest of the proof; it is not just a local miss (see Subsection 2.2).
The rest of the proof of Theorem 1.1 in [12] consists in showing that $f^\dagger(x) > 0$ for all $x \in (0, \pi/2)$. This is however not possible; a counter example is given as follows:

$$f^\dagger\left(\frac{\pi}{4}\right) = -5.79104121132 < 0.$$

The term $12x^3\tan^4 x$ is thus crucial to obtain the valid proof.

The following lemma present a valid power series expansion for $f(x)$ as described in Equation (2.1), with the same notations as in the proof of Theorem 1.1 in [12], which allows us to conclude properly the proof of Theorem 1.1 in [12].

Lemma 2.2. *The following power series expansion holds:*

$$f(x) = \sum_{n=6}^{+\infty} l_n x^{2n-1},$$

where

$$l_n = \frac{2.(2n-3)2^{2n}(2^{2n-2}-1)}{(2n-2)!}|B_{2n-2}| + \frac{2^{2n}(2^{2n}-1)}{3(2n)!}\left(48n^3+122n^2-113n+128\right)|B_{2n}|$$

$$+\frac{2^{2n}(2^{2n+2}-1)(2n+1)}{(2n+2)!}\left(\frac{2n(2n-145)(n+1)}{3}\right)|B_{2n+2}|. \tag{2.3}$$

The difference obtained with the series expansion of $f^{\dagger}(x)$ as obtained in the proof of Theorem 1.1 in [12] is substantial (see Subsection 2.2). To end the proof of Theorem 1.1 in [12], an inequality involving l_n must be established, and it is in the next result.

Lemma 2.3. *For $n \geq 6$, the term l_n defined in Equation* (2.3) *satisfies*

$$\frac{3.l_n.(2n)!}{2^{2n}|B_{2n}|} > \frac{h(n)}{\pi^2(2^{2n+1}-1)},$$

where

$$h(n) = \left[u_1(n).2^{2n} - v_1(n)\right]2^{2n} + w(n)$$

with

$$u_1(n) = 12\pi^4(2n-3) + 2\pi^2\left(48n^3+122n^2-113n+128\right) + 4n.(2n+1)(2n-145)(n+1),$$

$$v_1(n) = 18\pi^4(2n-3) + 3\pi^2\left(48n^3+122n^2-113n+128\right) + 9n.2^{2n}(2n+1)(2n-145)(n+1)$$

and

$$w(n) = 6\pi^4(2n-3) + \pi^2\left(48n^3+122n^2-113n+128\right) + (2n+1)2n(2n-145)(n+1).$$

If we combine Lemmas 2.1, 2.2 and 2.3, as in the proof of Theorem 1.1 in [12], by noticing that

- $w(n) > 0$ for any $n \geq 6$; a numerical support of this claim is given in Table 1 for the integers 6 to 100, with results divided by 100000 and rounded to the third decimal for the sake of place (in fact, we have $w(6) = 271.169455316$, $w(7) = 1299.11425957$, $w(8) = 3764.36501028$, etc.)
- $2^{2n} > \frac{v_1(n)}{u_1(n)}$ for all $n \geq 6$ by using mathematical induction; a numerical support of this claim is given in Table 2, where the function $g(n)/1000000$ with

  $$g(n) = 2^{2n} - \frac{v_1(n)}{u_1(n)}$$

  is considered for $n = 6, 7, \ldots, 20$.

TABLE 1. Values of $w(n)/100000$ for $n = 6, 7, \dots, 100$

$n = 6, 7, \dots \to 100$	0.003	0.013	0.038	0.085	0.167	0.295	0.484
	0.750	1.111	1.589	2.205	2.983	3.949	5.131
	6.559	8.266	10.283	12.648	15.397	18.571	22.210
	26.358	31.060	36.364	42.317	48.972	56.381	64.599
	73.682	83.690	94.683	106.724	119.876	134.207	149.784
	166.678	184.961	204.706	225.991	248.892	273.489	299.864
	328.102	358.286	390.505	424.849	461.408	500.275	541.547
	585.319	631.691	680.764	732.641	787.425	845.225	906.149
	970.306	1037.810	1108.775	1183.317	1261.554	1343.607	1429.597
	1519.649	1613.888	1712.442	1815.440	1923.015	2035.300	2152.430
	2274.543	2401.778	2534.277	2672.182	2815.638	2964.793	3119.795
	3280.795	3447.946	3621.403	3801.322	3987.862	4181.183	4381.448
	4588.820	4803.466	5025.554	5255.254	5492.738	5738.180	5991.756
	6253.643	6524.021	6803.072	7090.978			

TABLE 2. Values of $g(n)/1000000$ for $n = 6, 7, \dots, 20$

$n = 6, 7, \dots \to 20$	4.939	6.261	12.435	30.255	82.652 243.211
	753.374	2422.447	8012.212	27089.038	93206.628
	325306.639	1148844.509	4097566.015	14737981.067	

2.2. **Complementary results.** The following lemma shows that the missing term $12x^3 \tan^4 x$ is well not considered in the rest of the proof of Theorem 1.1 in [12]; it is not just a local miss.

Lemma 2.4. *For any $x \in (0, \pi/2)$, we can decompose $f^\dagger(x)$ as defined in* (2.2) *as*

$$f^\dagger(x) = \sum_{n=6}^{\infty} l_n^\dagger x^{2n-1},$$

where

$$l_n^\dagger = \frac{6(2n-3)2^{2n}(2^{2n-2}-1)}{(2n-2)!}|B_{2n-2}| + \frac{2^{2n}(2^{2n}-1)}{3(2n)!}\left(266n^2 - 245n + 164\right)|B_{2n}|$$
$$+ \frac{2^{2n}(2^{2n+2}-1)(2n+1)}{(2n+2)!}\left(\frac{2n(2n-145)(n+1)}{3}\right)|B_{2n+2}|.$$

This result follows the one in the proof of Theorem 1.1 in [12] except the denominator of the first term of $l_n^\dagger$ (when we solve step by step, we get $(2n-2)!$ instead of $(2n)!$ as we actually see in the published proof) and the fact that the term $12x^3\tan^4 x$ is omitted, which is a crucial gap.

The conclusion that $F'(x)>0$ in the proof of Theorem 1.1 in [12] comes from the fact that the analogous term of $w(n)$ defined by

$$\begin{aligned} w^\dagger(n) &= 16n^6 - 1144n^5 + \left(532\pi^2 - 1164\right) n^4 - \left(756\pi^2 - 286\right) n^3 \\ &\quad + \left(573\pi^2 + 290\right) n^2 + \left(18\pi^4 - 164\pi^2\right) n - 27\pi^4 \end{aligned}$$

is claimed to be strictly positive when $n \geq 6$, whereas it is not, as proved by the following counter example:

$$w^\dagger(6) = -4190658.38942 < 0,$$

(or $w^\dagger(7) = -9704363.77226 < 0$, among others).

The rest of the article is devoted to the proof of all the presented lemmas.

3. Proofs

For the sake of correctness, check and reproducibility, the proof contains the maximum details.

3.1. **Proof of Lemma 2.1.** The proof is based on differentiation and technical arrangements. For the sake of correctness, all the details are provided below. We have

$$F(x) = 7\ln\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right] - \ln\left[\left(\frac{8}{45}\right)^7 x^{22}(\tan x)^6\right].$$

Therefore, we establish that

$$\begin{aligned} F'(x) &= \frac{7}{\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}\left[\frac{2\sin x}{x}\left(\frac{x\cos x - \sin x}{x^2}\right) + \frac{x\sec^2 x - \tan x}{x^2}\right] \\ &\quad - \frac{1}{x^{22}(\tan x)^6}\left[22x^{21}(\tan x)^6 + 6x^{22}(\tan x)^5\sec^2 x\right] \\ &= \frac{1}{\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}\left[\frac{14x\sin x\cos x - 14\sin^2 x}{x^3} + \frac{7x\sec^2 x - 7\tan x}{x^2}\right] \\ &\quad - \frac{x^2}{x^3(\tan x)}\left[22\tan x + 6x\sec^2 x\right] \\ &= \frac{1}{x^3(\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]} \\ &\quad \left(\left[x^3\tan x\left(\frac{14\sin x\cos x - 14\sin^2 x}{x^3} + \frac{7x\sec^2 x - 7\tan x}{x^2}\right)\right]\right. \\ &\qquad \left. - x^2\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right](22\tan x + 6x\sec^2 x)\right) \\ &= \frac{1}{x^3(\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}\left(14x\sin^2 x - 14\sin^2 x\tan x + 7x^2\sec^2 x\tan x - 7x\tan^2 x\right. \\ &\qquad \left. - \left[\frac{\sin^2 x}{x^2} + \frac{\tan x}{x} - 2\right](22x^2\tan x + 6x^3\sec^2 x)\right) \\ &= \frac{1}{x^3(\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}\left(14x\sin^2 x - 14\sin^2 x\tan x + 7x^2\sec^2 x\tan x - 7x\tan^2 x\right. \\ &\qquad \left. - 22\sin^2 x\tan x - 6x\sin^2 x\sec^2 x - 22x\tan^2 x - 6x^2\sec^2 x\tan x + 44x^2\tan x + 12x^3\sec^2 x\right) \end{aligned}$$

$$= \frac{\cos^2 x}{x^3(\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}\Big(14x\tan^2 x - 14\tan^3 x + 7x^2\sec^4 x\tan x - 7x\tan^2 x\sec^2 x$$
$$- 22\tan^3 x - 6x\tan^2 x\sec^2 x - 22x\tan^2 x\sec^2 x - 6x^2\sec^4 x\tan x + 44x^2\tan x\sec^2 x + 12x^3\sec^4 x\Big)$$
$$= \frac{\cos^2 x}{x^3(\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}\Big(14x\tan^2 x - 36\tan^3 x + x^2\sec^4 x\tan x$$
$$- 35x\tan^2 x\sec^2 x + 44x^2\tan x\sec^2 x + 12x^3\sec^4 x\Big).$$

We can develop this expression as follows:

$$F'(x) = \frac{\cos^2 x}{x^3(\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}\Big(14x\tan^2 x - 36\tan^3 x + x^2\sec^2 x\tan x(1+\tan^2 x)$$
$$- 35x\tan^2 x\sec^2 x + 44x^2\tan x\sec^2 x + 12x^3\sec^2 x(1+\tan^2 x)\Big)$$
$$= \frac{\cos^2 x}{x^3(\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}\Big(14x\tan^2 x - 36\tan^3 x + x^2\sec^2 x\tan x + x^2\sec^2 x\tan^3 x -$$
$$35x\tan^2 x\sec^2 x + 44x^2\tan x\sec^2 x + 12x^3\sec^2 x + 12x^3\sec^2 x\tan^2 x\Big)$$
$$= \frac{\cos^2 x}{x^3(\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}\Big(12x^3\sec^2 x + x^2\sec^2 x\tan^3 x + 14x\tan^2 x -$$
$$36\tan^3 x + 45x^2\tan x\sec^2 x - 35x\tan^2 x\sec^2 x + 12x^3\sec^2 x\tan^2 x\Big)$$
$$= \frac{\cos^2 x}{x^3(\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}\Big(12x^3\sec^2 x + x^2\sec^2 x\tan^3 x + 14x\tan^2 x - 36\tan^3 x +$$
$$45x^2\tan x\sec^2 x - 29x\tan^2 x\sec^2 x - 6x\tan^2 x(1+\tan^2 x) + 12x^3\tan^2 x(1+\tan^2 x)\Big)$$
$$= \frac{\cos^2 x}{x^3(\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}\Big(12x^3\sec^2 x + x^2\sec^2 x\tan^3 x + 14x\tan^2 x - 36\tan^3 x +$$
$$45x^2\tan x\sec^2 x - 29x\tan^2 x\sec^2 x - 6x\tan^4 x - 6x\tan^2 x + 12x^3\tan^4 x + 12x^3\tan^2 x\Big)$$
$$= \frac{\cos^2 x}{x^3(\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}\Big(12x^3\sec^2 x + x^2\sec^2 x\tan^3 x + 12x^3\tan^2 x - 6x\tan^2 x$$
$$+ 14x\tan^2 x - 29x\tan^2 x\sec^2 x + 45x^2\tan x\sec^2 x - 36\tan^3 x - 6x\tan^4 x + 12x^3\tan^4 x\Big).$$

Hence, we have

$$F'(x) = \frac{\cos^2 x}{x^3(\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]}.f(x),$$

where

$$f(x) = 12x^3\sec^2 x + x^2\sec^2 x\tan^3 x + 12x^3\tan^2 x - 6x\tan^2 x + 14x\tan^2 x - 29x\tan^2 x\sec^2 x$$
$$+ 45x^2\tan x\sec^2 x - 36\tan^3 x - 6x\tan^4 x + 12x^3\tan^4 x.$$

This ends the proof of Lemma 2.1. □

3.2. **Proof of Lemma 2.2.** The proof is based on the same series expansions used in the proof of Theorem 1.1 in [12]. We have

$$\begin{aligned}
f(x) = {} & 12x^3 \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n-2} \\
& + x^2 \Bigg(\sum_{n=2}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)(2n-4)|B_{2n}|}{24(2n)!} x^{2n-5} \\
& - \sum_{n=2}^{\infty} \frac{(2n-1)2^{2n+2}(2^{2n}-1)(2n-2)|B_{2n}|}{12(2n)!} x^{2n-3} \Bigg) \\
& + 12x^3 \sum_{n=2}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n-2} - 6x \sum_{n=2}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n-2} \\
& + 14x \sum_{n=2}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n-2} \\
& - 29x \Bigg(\sum_{n=2}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)|B_{2n}|}{6(2n)!} x^{2n-4} \\
& - \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{3(2n)!} x^{2n-2} \Bigg) \\
& + 45x^2 \sum_{n=2}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)|B_{2n}|}{2(2n)!} x^{2n-3} \\
& - 36 \Bigg(\sum_{n=3}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)|B_{2n}|}{2(2n)!} x^{2n-3} - \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n-1} \Bigg) \\
& - 6x \Bigg(\sum_{n=4}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)|B_{2n}|}{6(2n)!} x^{2n-4} \\
& - \sum_{n=3}^{\infty} \frac{(2n-1)2^{2n+2}(2^{2n}-1)|B_{2n}|}{3(2n)!} x^{2n-2} \Bigg) \\
& + 12x^3 \Bigg(\sum_{n=4}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)|B_{2n}|}{6(2n)!} x^{2n-4} \\
& - \sum_{n=3}^{\infty} \frac{(2n-1)2^{2n+2}(2^{2n}-1)|B_{2n}|}{3(2n)!} x^{2n-2} \Bigg).
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}
f(x) = {} & 12 \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n+1} \\
& + \sum_{n=2}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)(2n-4)|B_{2n}|}{24(2n)!} x^{2n-3} \\
& - \sum_{n=2}^{\infty} \frac{(2n-1)2^{2n+2}(2^{2n}-1)(2n-2)|B_{2n}|}{12(2n)!} x^{2n-1} \\
& + 12 \sum_{n=2}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n+1} - 6 \sum_{n=2}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n-1} \\
& + 14 \sum_{n=2}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n-1}
\end{aligned}$$

$$-29\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)|B_{2n}|}{6(2n)!}x^{2n-3}$$
$$+29\sum_{n=1}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{3(2n)!}x^{2n-1}$$
$$+45\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)|B_{2n}|}{2(2n)!}x^{2n-1}$$
$$-36\sum_{n=3}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)|B_{2n}|}{2(2n)!}x^{2n-3}+36\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-1}$$
$$-6\sum_{n=4}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)|B_{2n}|}{6(2n)!}x^{2n-3}$$
$$+6\sum_{n=3}^{\infty}\frac{(2n-1)2^{2n+2}(2^{2n}-1)|B_{2n}|}{3(2n)!}x^{2n-1}$$
$$+12\sum_{n=4}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)|B_{2n}|}{6(2n)!}x^{2n-1}$$
$$-12\sum_{n=3}^{\infty}\frac{(2n-1)2^{2n+2}(2^{2n}-1)|B_{2n}|}{3(2n)!}x^{2n+1}.$$

After some arrangements, we get

$$f(x)=12\sum_{n=2}^{\infty}\frac{(2n-3)2^{2n-2}(2^{2n-2}-1)|B_{2n-2}|}{(2n-2)!}x^{2n-1}$$
$$+\sum_{n=1}^{\infty}\frac{(2n+1)2^{2n+2}(2^{2n+2}-1)(2n)(2n-1)(2n-2)|B_{2n+2}|}{24(2n+2)!}x^{2n-1}$$
$$-\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n+2}(2^{2n}-1)(2n-2)|B_{2n}|}{12(2n)!}x^{2n-1}$$
$$+12\sum_{n=3}^{\infty}\frac{(2n-3)2^{2n-2}(2^{2n-2}-1)|B_{2n-2}|}{(2n-2)!}x^{2n-1}$$
$$-6\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-1}+14\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-1}$$
$$-29\sum_{n=1}^{\infty}\frac{(2n+1)2^{2n+2}(2^{2n+2}-1)(2n)(2n-1)|B_{2n+2}|}{6(2n+2)!}x^{2n-1}$$
$$+29\sum_{n=1}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{3(2n)!}x^{2n-1}$$
$$+45\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)|B_{2n}|}{2(2n)!}x^{2n-1}$$

$$-36\sum_{n=2}^{\infty}\frac{(2n+1)2^{2n+2}(2^{2n+2}-1)(2n)|B_{2n+2}|}{2(2n+2)!}x^{2n-1}+36\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-1}$$
$$-6\sum_{n=3}^{\infty}\frac{(2n-1)2^{2n+2}(2^{2n+2}-1)(2n)(2n+1)|B_{2n+2}|}{6(2n+2)!}x^{2n-1}$$
$$+6\sum_{n=3}^{\infty}\frac{(2n-1)2^{2n+2}(2^{2n}-1)|B_{2n}|}{3(2n)!}x^{2n-1}$$
$$+12\sum_{n=4}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)|B_{2n}|}{6(2n)!}x^{2n-1}$$
$$-12\sum_{n=4}^{\infty}\frac{(2n-3)2^{2n}(2^{2n-2}-1)|B_{2n-2}|}{3(2n-2)!}x^{2n-1}.$$

We can write

$$f(x)=\sum_{n=6}^{\infty}\frac{24(2n-3)2^{2n-2}(2^{2n-2}-1)|B_{2n-2}|}{(2n-2)!}x^{2n-1}-12\sum_{n=6}^{\infty}\frac{(2n-3)2^{2n}(2^{2n-2}-1)|B_{2n-2}|}{3.(2n-2)!}x^{2n-1}$$
$$+\sum_{n=6}^{\infty}\Bigg[-\frac{2^{2n+2}(2^{2n}-1)(2n-1)(2n-2)}{12(2n)!}$$
$$-\frac{6.2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}+\frac{14.2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}+\frac{29.2^{2n}(2^{2n}-1)(2n-1)}{3.(2n)!}$$
$$+\frac{45.2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2.(2n)!}+\frac{36.2^{2n}(2^{2n}-1)}{(2n)!}+\frac{6.2^{2n+2}(2^{2n}-1)(2n-1)}{3.(2n)!}$$
$$+\frac{12(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)}{6(2n)!}\Bigg]|B_{2n}|x^{2n-1}$$
$$+\sum_{n=6}^{\infty}\Bigg[\frac{2^{2n+2}(2^{2n+2}-1)(2n+1)(2n)(2n-1)(2n-2)}{24.(2n+2)!}-\frac{29.2^{2n+2}(2^{2n+2}-1)(2n+1)(2n)(2n-1)}{6.(2n+2)!}$$
$$-\frac{36.2^{2n+2}(2^{2n+2}-1)(2n)(2n+1)}{2.(2n+2)!}-\frac{6.2^{2n+2}(2^{2n+2}-1)(2n+1)(2n)(2n-1)}{6.(2n+2)!}\Bigg]|B_{2n+2}|x^{2n-1}.$$

Therefore, we have

$$f(x)$$
$$=\sum_{n=6}^{\infty}\frac{24(2n-3)2^{2n}(2^{2n-2}-1)|B_{2n-2}|}{4.(2n-2)!}x^{2n-1}$$
$$-\sum_{n=6}^{\infty}\frac{12.(2n-3)2^{2n}(2^{2n-2}-1)|B_{2n-2}|}{3.(2n-2)!}x^{2n-1}$$
$$+\sum_{n=6}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}\Bigg[-\frac{4(2n-2)(2n-1)}{12}-6(2n-1)$$
$$+14(2n-1)+\frac{29}{3}(2n-1)+\frac{45(2n-2)(2n-1)}{2}+36+\frac{24(2n-1)}{3}+\frac{12(2n-1)(2n-2)(2n-3)}{6}\Bigg]|B_{2n}|x^{2n-1}$$
$$+\sum_{n=6}^{\infty}\frac{2^{2n+2}(2^{2n+2}-1)(2n+1)}{(2n+2)!}\Bigg[\frac{(2n)(2n-1)(2n-2)}{24}-29\frac{(2n)(2n-1)}{6}-36\frac{(2n)}{2}$$
$$-6\frac{(2n)(2n-1)}{6}\Bigg]|B_{2n+2}|x^{2n-1}.$$

So, we can write

$$f(x) = \sum_{n=6}^{\infty} \frac{2(2n-3)2^{2n}(2^{2n-2}-1)}{(2n-2)!}|B_{2n-2}|x^{2n-1} + \sum_{n=6}^{\infty} \frac{2^{2n}(2^{2n}-1)}{3(2n)!}\left(48n^3+122n^2-113n+128\right)|B_{2n}|x^{2n-1} + \sum_{n=6}^{\infty} \frac{2^{2n}(2^{2n+2}-1)(2n+1)}{(2n+2)!}\left(\frac{2n(2n-145)(n+1)}{3}\right)|B_{2n+2}|x^{2n-1}.$$

Hence, we find that

$$f(x) = \sum_{n=6}^{\infty} l_n x^{2n-1},$$

where

$$l_n = \frac{2.(2n-3)2^{2n}(2^{2n-2}-1)}{(2n-2)!}|B_{2n-2}| + \frac{2^{2n}(2^{2n}-1)}{3(2n)!}\left(48n^3+122n^2-113n+128\right)|B_{2n}| + \frac{2^{2n}(2^{2n+2}-1)(2n+1)}{(2n+2)!}\left(\frac{2n(2n-145)(n+1)}{3}\right)|B_{2n+2}|.$$

This ends the proof of Lemma 2.2. □

3.3. **Proof of Lemma 2.3.** The proof is based on the same inequalities and manipulations of Bernoulli's numbers used in the proof of Theorem 1.1 in [12].
We have

$$\frac{3.l_n}{2^{2n}|B_{2n}|} = \frac{6.(2n-3)(2^{2n-2}-1)}{(2n-2)!}\frac{|B_{2n-2}|}{|B_{2n}|} + \frac{(2^{2n}-1)}{(2n)!}\left(48n^3+122n^2-113n+128\right) + \frac{3.(2^{2n+2}-1)(2n+1)}{(2n+2)!}\left(\frac{2n(2n-145)(n+1)}{3}\right)\frac{|B_{2n+2}|}{|B_{2n}|}.$$

We know that

$$\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{(2n+2)(2n+1)}{\pi^2} < \frac{|B_{2n+2}|}{|B_{2n}|} < \frac{2^{2n}-1}{2^{2n+2}-1}\frac{(2n+2)(2n+1)}{\pi^2}.$$

Hence, we have

$$\frac{2^{2n-3}-1}{2^{2n-1}-1}\frac{(2n)(2n-1)}{\pi^2} < \frac{|B_{2n}|}{|B_{2n-2}|} < \frac{2^{2n-2}-1}{2^{2n}-1}\frac{(2n)(2n-1)}{\pi^2}.$$

Therefore, we establish that

$$\frac{3.l_n}{2^{2n}|B_{2n}|} > \frac{6.(2n-3)(2^{2n-2}-1)}{(2n-2)!}\frac{(2^{2n}-1)}{(2^{2n-2}-1)}\frac{\pi^2}{(2n)(2n-1)} + \frac{(2^{2n}-1)}{(2n)!}\left(48n^3+122n^2-113n+128\right) + \frac{(2^{2n+2}-1)(2n+1)2n(2n-145)(n+1)}{(2n+2)!}\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{(2n+2)(2n+1)}{\pi^2}.$$

This implies that

$$\frac{3.l_n}{2^{2n}|B_{2n}|} > \frac{6.(2^{2n}-1)(2n-3)}{(2n)!}\pi^2 + \frac{(2^{2n}-1)}{(2n)!}\left(48n^3+122n^2-113n+128\right) + \frac{(2^{2n+2}-1)(2n+1)2n(2n-145)(n+1)}{\pi^2.(2n)!}\frac{(2^{2n-1}-1)}{(2^{2n+1}-1)}$$

and

$$\frac{3.l_n.(2n)!}{2^{2n}|B_{2n}|} > 6.(2^{2n}-1)(2n-3)\pi^2 + (2^{2n}-1)\left(48n^3+122n^2-113n+128\right) + \frac{(2^{2n+2}-1)(2n+1)2n(2n-145)(n+1)}{\pi^2}\frac{(2^{2n-1}-1)}{(2^{2n+1}-1)}.$$

As a result, we get

$$\frac{3.l_n.(2n)!}{2^{2n}|B_{2n}|} > \frac{1}{\pi^2(2^{2n+1}-1)}\Bigg(6\pi^4(2^{2n+1}-1)(2^{2n}-1)(2n-3) + \pi^2(2^{2n+1}-1)(2^{2n}-1)\left(48n^3+122n^2-113n+128\right) + (2^{2n+2}-1)(2n+1)2n(2n-145)(n+1)(2^{2n-1}-1)\Bigg).$$

Hence, we have

$$\frac{3.l_n.(2n)!}{2^{2n}|B_{2n}|} > \frac{h(n)}{\pi^2(2^{2n+1}-1)},$$

where

$$\begin{aligned}
h(n) &= 6\pi^4(2^{2n+1}-1)(2^{2n}-1)(2n-3) \\
&\quad + \pi^2(2^{2n+1}-1)(2^{2n}-1)\left(48n^3+122n^2-113n+128\right) \\
&\quad + (2^{2n+2}-1)(2n+1)2n(2n-145)(n+1)(2^{2n-1}-1) \\
&= 2^{2n}6\pi^4(2^{2n+1}-1)(2n-3) - 6\pi^4(2^{2n+1}-1)(2n-3) \\
&\quad + 2^{2n}\pi^2(2^{2n+1}-1)\left(48n^3+122n^2-113n+128\right) - \pi^2(2^{2n+1}-1)\left(48n^3+122n^2-113n+128\right) \\
&\quad + 8n.2^{2n}(2n+1)(2n-145)(n+1)(2^{2n-1}-1) - (2n+1)2n(2n-145)(n+1)(2^{2n-1}-1) \\
&= 2^{2n}12\pi^4.2^{2n}(2n-3) - 2^{2n}6\pi^4(2n-3) - 12\pi^4 2^{2n}(2n-3) + 6\pi^4(2n-3) \\
&\quad + 2^{2n}2\pi^2 2^{2n}\left(48n^3+122n^2-113n+128\right) - 3\pi^2 2^{2n}\left(48n^3+122n^2-113n+128\right) \\
&\quad + \pi^2\left(48n^3+122n^2-113n+128\right) + 2^{2n}4n.2^{2n}(2n+1)(2n-145)(n+1) \\
&\quad - 8n.2^{2n}(2n+1)(2n-145)(n+1) - (2n+1)n(2n-145)(n+1)2^{2n} + (2n+1)2n(2n-145)(n+1) \\
&= \Bigg(\left[12\pi^4(2n-3) + 2\pi^2\left(48n^3+122n^2-113n+128\right) + 4n.(2n+1)(2n-145)(n+1)\right]2^{2n} \\
&\qquad - \left[18\pi^4(2n-3) + 3\pi^2\left(48n^3+122n^2-113n+128\right) + 9n.2^{2n}(2n+1)(2n-145)(n+1)\right]\Bigg)2^{2n} \\
&\quad + \left(6\pi^4(2n-3) + \pi^2\left(48n^3+122n^2-113n+128\right) + (2n+1)2n(2n-145)(n+1)\right).
\end{aligned}$$

Thus, we can write

$$h(n) = \Big[u_1(n).2^{2n} - v_1(n)\Big]2^{2n} + w(n),$$

where

$$\begin{aligned}u_1(n) &= 12\pi^4(2n-3) + 2\pi^2\Big(48n^3 + 122n^2 - 113n + 128\Big) + 4n.(2n+1)(2n-145)(n+1),\\ &v_1(n)\\ &= 18\pi^4(2n-3) + 3\pi^2\Big(48n^3 + 122n^2 - 113n + 128\Big) + 9n.2^{2n}(2n+1)(2n-145)(n+1),\\ &w(n)\\ &= 6\pi^4(2n-3) + \pi^2\Big(48n^3 + 122n^2 - 113n + 128\Big) + (2n+1)2n(2n-145)(n+1).\end{aligned}$$

This ends the proof of Lemma 2.3. □

3.4. **Proof of Lemma 2.4.** The proof used the same techniques as in the one of Lemma 2.2. We have

$$\begin{aligned}f^{\dagger}(x) &= 12x^3\sec^2 x + x^2\sec^2 x\tan^3 x + 12x^3\tan^2 x - 6x\tan^2 x + 14x\tan^2 x - 29x\tan^2 x\sec^2 x\\ &\quad + 45x^2\tan x\sec^2 x - 36\tan^3 x - 6x\tan^4 x.\end{aligned}$$

As a result, we have

$$\begin{aligned}f^{\dagger}(x) &= 12x^3\sum_{n=1}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-2}\\ &\quad + x^2\Bigg(\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)(2n-4)|B_{2n}|}{24(2n)!}x^{2n-5}\\ &\quad - \sum_{n=2}^{\infty}\frac{(2n-1)2^{2n+2}(2^{2n}-1)(2n-2)|B_{2n}|}{12(2n)!}x^{2n-3}\Bigg)\\ &\quad + 12x^3\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-2} - 6x\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-2}\\ &\quad + 14x\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-2}\\ &\quad - 29x\Bigg(\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)|B_{2n}|}{6(2n)!}x^{2n-4}\\ &\quad - \sum_{n=1}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{3(2n)!}x^{2n-2}\Bigg)\\ &\quad + 45x^2\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)|B_{2n}|}{2(2n)!}x^{2n-3}\\ &\quad - 36\Bigg(\sum_{n=3}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)|B_{2n}|}{2(2n)!}x^{2n-3}\\ &\quad - \sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-1}\Bigg)\\ &\quad - 6x\Bigg(\sum_{n=4}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)|B_{2n}|}{6(2n)!}x^{2n-4}\\ &\quad - \sum_{n=3}^{\infty}\frac{(2n-1)2^{2n+2}(2^{2n}-1)|B_{2n}|}{3(2n)!}x^{2n-2}\Bigg).\end{aligned}$$

Therefore, we establish that

$$
\begin{aligned}
f^{\dagger}(x) = {} & 12\sum_{n=1}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n+1} \\
& + \sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)(2n-4)|B_{2n}|}{24(2n)!}x^{2n-3} \\
& - \sum_{n=2}^{\infty}\frac{(2n-1)2^{2n+2}(2^{2n}-1)(2n-2)|B_{2n}|}{12(2n)!}x^{2n-1} \\
& + 12\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n+1} - 6\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-1} \\
& + 14\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-1} \\
& - 29\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)|B_{2n}|}{6(2n)!}x^{2n-3} \\
& + 29\sum_{n=1}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{3(2n)!}x^{2n-1} \\
& + 45\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)|B_{2n}|}{2(2n)!}x^{2n-1} \\
& - 36\sum_{n=3}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)|B_{2n}|}{2(2n)!}x^{2n-3} \\
& + 36\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-1} \\
& - 6\sum_{n=4}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)(2n-3)|B_{2n}|}{6(2n)!}x^{2n-3} \\
& + 6\sum_{n=3}^{\infty}\frac{(2n-1)2^{2n+2}(2^{2n}-1)|B_{2n}|}{3(2n)!}x^{2n-1}.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
f^{\dagger}(x) = {} & 12\sum_{n=2}^{\infty}\frac{(2n-3)2^{2n-2}(2^{2n-2}-1)|B_{2n-2}|}{(2n-2)!}x^{2n-1} \\
& + \sum_{n=1}^{\infty}\frac{(2n+1)2^{2n+2}(2^{2n+2}-1)(2n)(2n-1)(2n-2)|B_{2n+2}|}{24(2n+2)!}x^{2n-1} \\
& - \sum_{n=2}^{\infty}\frac{(2n-1)2^{2n+2}(2^{2n}-1)(2n-2)|B_{2n}|}{12(2n)!}x^{2n-1} \\
& + 12\sum_{n=3}^{\infty}\frac{(2n-3)2^{2n-2}(2^{2n-2}-1)|B_{2n-2}|}{(2n-2)!}x^{2n-1} \\
& - 6\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-1} + 14\sum_{n=2}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-1}
\end{aligned}
$$

$$
\begin{aligned}
&-29 \sum_{n=1}^{\infty} \frac{(2n+1)2^{2n+2}(2^{2n+2}-1)(2n)(2n-1)|B_{2n+2}|}{6(2n+2)!} x^{2n-1} \\
&+29 \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{3(2n)!} x^{2n-1} \\
&+45 \sum_{n=2}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)(2n-2)|B_{2n}|}{2(2n)!} x^{2n-1} \\
&-36 \sum_{n=2}^{\infty} \frac{(2n+1)2^{2n+2}(2^{2n+2}-1)(2n)|B_{2n+2}|}{2(2n+2)!} x^{2n-1} + 36 \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n-1} \\
&-6 \sum_{n=3}^{\infty} \frac{(2n-1)2^{2n+2}(2^{2n+2}-1)(2n)(2n+1)|B_{2n+2}|}{6(2n+2)!} x^{2n-1} \\
&+6 \sum_{n=3}^{\infty} \frac{(2n-1)2^{2n+2}(2^{2n}-1)|B_{2n}|}{3(2n)!} x^{2n-1}.
\end{aligned}
$$

We can arrange this function as follows:

$$
\begin{aligned}
f^{\dagger}(x) = 12\Bigg(&\frac{2^2(2^2-1)|B_2|}{2!} x^3 + \frac{3.2^4(2^4-1)|B_4|}{4!} x^5 + \frac{5.2^6(2^6-1)|B_6|}{6!} x^7 + \frac{7.2^8(2^8-1)|B_8|}{8!} x^9 \\
&+ \sum_{n=6}^{\infty} \frac{(2n-3)2^{2n-2}(2^{2n-2}-1)|B_{2n-2}|}{(2n-2)!} x^{2n-1} \Bigg) \\
&+ \Bigg(\frac{2^6(2^6-1).5.4.3.2|B_6|}{24.6!} x^3 + \frac{2^8(2^8-1).7.6.5.4|B_8|}{24.8!} x^5 + \frac{2^{10}(2^{10}-1).9.8.7.6|B_{10}|}{24.10!} x^7 \\
&+ \frac{2^{12}(2^{12}-1).11.10.9.8|B_{12}|}{24.12!} x^9 + \sum_{n=6}^{\infty} \frac{(2n-2)(2n)(2n-1)(2n+1)2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{24.(2n+2)!} x^{2n-1} \Bigg) \\
&- \Bigg(\frac{2^6(2^4-1).3.2|B_4|}{12.4!} x^3 + \frac{2^8(2^6-1)5.4|B_6|}{12.6!} x^5 + \frac{2^{10}(2^8-1).7.6|B_8|}{12.8!} x^7 \\
&+ \frac{2^{12}(2^{10}-1).9.8|B_{10}|}{12.10!} x^9 + \sum_{n=6}^{\infty} \frac{(2n-2)(2n-1)2^{2n+2}(2^{2n}-1)|B_{2n}|}{12.(2n)!} x^{2n-1} \Bigg) \\
&+ 12\Bigg(\frac{2^4(2^4-1).3.|B_4|}{4!} x^5 + \frac{5.2^6(2^6-1)|B_6|}{6!} x^7 + \frac{7.2^8(2^8-1)|B_8|}{8!} x^9 \\
&+ \sum_{n=6}^{\infty} \frac{(2n-3)2^{2n-2}(2^{2n-2}-1)|B_{2n-2}|}{(2n-2)!} x^{2n-1} \Bigg) \\
&- 6\Bigg(\frac{3.2^4(2^4-1)|B_4|}{4!} x^3 + \frac{5.2^6(2^6-1)|B_6|}{6!} x^5 + \frac{7.2^8(2^8-1)|B_8|}{8!} x^7 + \frac{9.2^{10}(2^{10}-1)|B_{10}|}{10!} x^9 \\
&+ \sum_{n=6}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!} x^{2n-1} \Bigg) \\
&+ 14\Bigg(\frac{2^4.(2^4-1).3.|B_4|x^3}{4!} + \frac{2^6.(2^6-1).5.|B_6|x^5}{6!} + \frac{2^8.(2^8-1).7.|B_8|x^7}{8!} \\
&+ \frac{2^{10}.(2^{10}-1).9.|B_{10}|x^9}{10!} + \sum_{n=6}^{\infty} \frac{2^{2n}.(2^{2n}-1).(2n-1).|B_{2n}|x^{2n-1}}{(2n)!} \Bigg)
\end{aligned}
$$

$$-29\Bigg(\frac{2^4.(2^4-1).3.2.1|B_4|x}{6.4!}+\frac{2^6.(2^6-1).5.4.3|B_6|x^3}{6.6!}+$$

$$\frac{2^8.(2^8-1).7.6.5.|B_8|x^5}{6.8!}+\frac{2^{10}.(2^{10}-1).9.8.7.|B_{10}|x^7}{6.10!}+\frac{2^{12}.(2^{12}-1).11.10.9.|B_{12}|x^9}{6.12!}$$

$$+\sum_{n=6}^{\infty}\frac{2^{2n+2}.(2^{2n+2}-1).(2n+1)(2n)(2n-1).|B_{2n+2}|x^{2n-1}}{6.(2n+2)!}\Bigg)$$

$$+29\Bigg(\frac{2^2.(2^2-1).1|B_2|x}{3.2!}+\frac{2^4.(2^4-1).3|B_4|x^3}{3.4!}+\frac{2^6.(2^6-1).5|B_6|x^5}{3.6!}$$

$$+\frac{2^8.(2^8-1).7|B_8|x^7}{3.8!}+\frac{2^{10}.(2^{10}-1).9|B_{10}|x^9}{3.10!}+\sum_{n=6}^{\infty}\frac{2^{2n}.(2^{2n}-1).(2n-1).|B_{2n}|x^{2n-1}}{3.(2n)!}\Bigg)$$

$$+45\Bigg(\frac{2^4.(2^4-1).3..2|B_4|x^3}{2.4!}+\frac{2^6.(2^6-1).5.4.|B_6|x^5}{2.6!}+\frac{2^8.(2^8-1).7.6.|B_8|x^7}{2.8!}+\frac{2^{10}.(2^{10}-1).9.8.|B_{10}|x^9}{2.10!}$$

$$+\sum_{n=6}^{\infty}\frac{2^{2n}.(2^{2n}-1).(2n-1).(2n-2).|B_{2n}|x^{2n-1}}{2.(2n)!}\Bigg)$$

$$-36\Bigg(\frac{2^6.(2^6-1).5.4.|B_6|x^3}{2.6!}+\frac{2^8.(2^8-1).7.6.|B_8|x^5}{2.8!}+\frac{2^{10}.(2^{10}-1).9.8.|B_{10}|x^7}{2.10!}+\frac{2^{12}.(2^{12}-1).11.10.|B_{12}|x^9}{2.12!}$$

$$+\sum_{n=6}^{\infty}\frac{2^{2n+2}.(2^{2n+2}-1).(2n-1)(2n-2).|B_{2n+2}|x^{2n-1}}{2.(2n+2)!}\Bigg)$$

$$+36\Bigg(\frac{2^4.(2^4-1).|B_4|x^3}{4!}+\frac{2^6.(2^6-1).|B_6|x^5}{6!}+\frac{2^8.(2^8-1).|B_8|x^7}{8!}+\frac{2^{10}.(2^{10}-1).|B_{10}|x^9}{10!}$$

$$+\sum_{n=6}^{\infty}\frac{2^{2n}.(2^{2n}-1).|B_{2n}|x^{2n-1}}{(2n)!}\Bigg)$$

$$-6\Bigg(\frac{2^8.(2^8-1).7.6.5.|B_8|x^5}{6.8!}+\frac{2^{10}.(2^{10}-1).9.8.7.|B_{10}|x^7}{6.10!}+\frac{2^{12}.(2^{12}-1).11.10.8.|B_{12}|x^9}{6.12!}$$

$$+\sum_{n=6}^{\infty}\frac{2^{2n+2}.(2^{2n+2}-1).(2n+1)(2n)(2n-1).|B_{2n+2}|x^{2n-1}}{6.(2n+2)!}\Bigg)$$

$$+6\Bigg(\frac{2^8.(2^6-1).5.|B_6|x^5}{3.6!}+\frac{2^{10}.(2^8-1).7.|B_8|x^7}{3.8!}+\frac{2^{12}.(2^{10}-1).9.|B_{10}|x^9}{3.10!}$$

$$+\sum_{n=6}^{\infty}\frac{2^{2n+2}.(2^{2n}-1).(2n-1).|B_{2n}|x^{2n-1}}{3.(2n)!}\Bigg).$$

Therefore, we have

$$f^{\dagger}(x)=\sum_{n=6}^{\infty}\frac{24(2n-3)2^{2n-2}(2^{2n-2}-1)|B_{2n-2}|}{(2n-2)!}x^{2n-1}+\sum_{n=6}^{\infty}\Bigg[-\frac{2^{2n+2}(2^{2n}-1)(2n-1)(2n-2)}{12(2n)!}$$
$$-\frac{6.2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}+\frac{14.2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}+\frac{29.2^{2n}(2^{2n}-1)(2n-1)}{3.(2n)!}$$
$$+\frac{45.2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2.(2n)!}+\frac{36.2^{2n}(2^{2n}-1)}{(2n)!}+\frac{6.2^{2n+2}(2^{2n}-1)(2n-1)}{3.(2n)!}\Bigg]|B_{2n}|x^{2n-1}$$

$$+\sum_{n=6}^{\infty}\Big[\frac{2^{2n+2}(2^{2n+2}-1)(2n+1)(2n)(2n-1)(2n-2)}{24.(2n+2)!}-\frac{29.2^{2n+2}(2^{2n+2}-1)(2n+1)(2n)(2n-1)}{6.(2n+2)!}$$
$$-\frac{36.2^{2n+2}(2^{2n+2}-1)(2n)(2n+1)}{2.(2n+2)!}-\frac{6.2^{2n+2}(2^{2n+2}-1)(2n+1)(2n)(2n-1)}{6.(2n+2)!}\Big]|B_{2n+2}|x^{2n-1}.$$

We can write

$$f^{\dagger}(x)=\sum_{n=6}^{\infty}\frac{24(2n-3)2^{2n-2}(2^{2n-2}-1)|B_{2n-2}|}{(2n-2)!}x^{2n-1}+\sum_{n=6}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}\Big[-\frac{4(2n-2)(2n-1)}{12}-6(2n-1)$$
$$+14(2n-1)+\frac{29}{3}(2n-1)+\frac{45(2n-2)(2n-1)}{2}+36+\frac{24(2n-1)}{3}\Big]|B_{2n}|x^{2n-1}$$
$$+\sum_{n=6}^{\infty}\frac{2^{2n+2}(2^{2n+2}-1)(2n+1)}{(2n+2)!}\Big[\frac{(2n)(2n-1)(2n-2)}{24}-29\frac{(2n)(2n-1)}{6}-36\frac{(2n)}{2}$$
$$-6\frac{(2n)(2n-1)}{6}\Big]|B_{2n+2}|x^{2n-1}.$$

As a result, we obtain

$$f^{\dagger}(x)=\sum_{n=6}^{\infty}\frac{24(2n-3)2^{2n}(2^{2n-2}-1)}{4.(2n-2)!}|B_{2n-2}|x^{2n-1}$$
$$+\sum_{n=6}^{\infty}\frac{2^{2n}(2^{2n}-1)}{3(2n)!}\Big(266n^2-245n+164\Big)|B_{2n}|x^{2n-1}$$
$$+\sum_{n=6}^{\infty}\frac{2^{2n}(2^{2n+2}-1)(2n+1)}{(2n+2)!}\Big(\frac{2n(2n-145)(n+1)}{3}\Big)|B_{2n+2}|x^{2n-1}.$$

Hence, we have

$$f^{\dagger}(x)=\sum_{n=6}^{\infty}l_n^{\dagger}x^{2n-1},$$

where

$$l_n^{\dagger}=\frac{6(2n-3)2^{2n}(2^{2n-2}-1)}{(2n-2)!}|B_{2n-2}|+\frac{2^{2n}(2^{2n}-1)}{3(2n)!}\Big(266n^2-245n+164\Big)|B_{2n}|$$
$$+\frac{2^{2n}(2^{2n+2}-1)(2n+1)}{(2n+2)!}\Big(\frac{2n(2n-145)(n+1)}{3}\Big)|B_{2n+2}|.$$

This ends the proof of Lemma 2.4. □

Conflict of interest statement. All the authors declare that there are no conflict of interests.

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