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ON A SET OF GENERALIZED JANOWSKI-TYPE STARLIKE FUNCTIONS CONNECTED WITH MATHIEU-TYPE SERIES AND OPOOLA DIFFERENTIAL OPERATOR

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ABSTRACT. The study carried out in this paper is on a set of analytic-univalent functions whose definition is connected with the applications of the normalized Mathieu-type series, Opoola differential operator, Oyekan-Swamy-Opoola operator and Janowski-type starlike functions. More so, the techniques of convolution and subordination are embraced in the definition of our set of functions. Some of our investigations include the coefficient inequality, radii problems, growth, distortion, closure and inclusion properties.

1. INTRODUCTION AND PRELIMINARIES

This investigation is on a set of normalized analytic functions herein denoted by $\mathcal{A}$ and defined in the unit disk: $|z| < 1$. Further, let $\mathcal{S}$ be a subset of $\mathcal{A}$ which consists of analytic and univalent functions defined such that f is expressed in series form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (|z| < 1) \tag{1.1}$$

and normalized such that $f(0) = f'(0) - 1 = 0$. Recall that the Hadamard product of

$$f \text{ in (1.1) and } F(z) = z + \sum_{k=2}^{\infty} c_k z^k \tag{1.2}$$

is defined by $(f \star F)(z) = z + \sum_{k=2}^{\infty} (a_k \times c_k) z^k = (F \star f)(z) \quad (|z| < 1)$. Likewise, f is subordinate to F (symbolized as $f \prec F$) if $f = F \circ s := F(s(z))$ for the analytic function

$$s(z) = s_1 z + s_2 z^2 + s_3 z^3 + \cdots \quad (s(0) = 0 \text{ and } |s(z)| < 1). \tag{1.3}$$

If by peradventure F is univalent for $|z| < 1$, then

$$f \prec F \text{ if and only if } f(0) = F(0) \text{ and } f(|z| < 1) \subset F(|z| < 1).$$

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An astounding application of subordination in Geometric Function Theory (GFT) is the peculiar way it is used to define some subsets of $\mathcal{S}$. For instance, the set $\mathcal{S}^\star$ of starlike functions consist of functions that fulfill the condition

$$zf'/f \prec \wp_0(z) = \frac{1+z}{1-z}$$

where $\wp_0(z)$, known as the Möbius function, serves as the extremal function for all function-type

$$y(z) = 1 + y_1 z + y_2 z^2 + \cdots \in Y \quad (\Re y(z) > 0,\ |z| < 1).$$

In 1973, Janowski [6] generalized functions in Y where the author introduced the sets

$$Y(A,B) = \left\{ y \in Y : y(z) \prec \frac{1+Az}{1+Bz} \text{ and } -1 \leqq B < A \leqq 1,\ |z| < 1 \right\}$$

and

$$\mathcal{S}^\star(A,B) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \text{ and } -1 \leqq B < A \leqq 1,\ |z| < 1 \right\}.$$

1.1. **Certain Operators and Series.** Let

$$\mathcal{M}(r;z) = \frac{(1+r^2)^2}{2} \sum_{k=1}^{\infty} \frac{2k}{(k^2+r^2)^2} z^k = z + \sum_{k=2}^{\infty} \frac{k(1+r^2)^2}{(k^2+r^2)^2} z^k \quad (r > 0,\ |z| < 1) \tag{1.4}$$

be the normalized form of the Mathieu-type series introduced by Bansal and Sokól [3]. The Mathieu-type series $\mathcal{M}(r) = \sum\limits_{k=1}^{\infty} \frac{2k}{(k^2+r^2)^2}$ was introduced by Mathieu [13] and it was dedicated to the study of elasticity of solid bodies. In fact, the bounds for the series $\mathcal{M}(r)$ were applied in the solution of boundary value problems for the biharmonic equations in a 2D rectangular domain. Also, let

$$\mathcal{D}^{m,\mu}_{\tau,\sigma} f(z) = z + \sum_{k=2}^{\infty} [1 + (k+\mu-\sigma-1)\tau]^m a_k z^k \tag{1.5}$$

$$(\tau, \sigma \geqq 0;\ 0 \leqq \mu \leqq \sigma,\ m \in \mathbb{N} \cup \{0\} := \mathbb{N}_0, \text{ and } |z| < 1) \tag{1.6}$$

be the Opoola differential operator introduced in [14]. This differential operator is well-known to generalize the Sălăgean [28] and Al-Oboudi [1] differential operators, see [7,8,17–19,22,24,29,30] for more information. In 2021, Oyekan et al. [25] studied the operator

$$\mathcal{J}^{m,\alpha,\beta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\alpha + k\beta}{\alpha + \beta} \right)^m a_k z^k \quad (\alpha \in \mathbb{R},\ \beta \geqq 0,\ \alpha + \beta > 0). \tag{1.7}$$

The operator in (1.7) generalized the well-known Ruscheweyh operator in [26]. Using the Hadamard product in connection with (1.4), (1.5) and (1.7), we therefore define a novel multiplier operator $\mathcal{X}^{m,\alpha,\beta}_{\sigma,\mu,t} : \mathcal{A} \longrightarrow \mathcal{A}$ as follows.

Definition 1.1. Let

$$\mathcal{X}^{m,\alpha,\beta,\mu}_{\tau,\sigma,r} f(z) = \mathcal{M}(r;z) \star \mathcal{D}^{m,\mu}_{\tau,\sigma} f(z) \star \mathcal{J}^{m,\alpha,\beta,\mu}_{\tau,\sigma} f(z) = z + \sum_{k=2}^{\infty} \frac{k(1+r^2)^2}{(k^2+r^2)^2} \left(\frac{\alpha + k\beta}{\alpha + \beta} \right)^m [1 + (k+\mu-\sigma-1)\tau]^m a_k z^k \tag{1.8}$$

or for instance we may write

$$\mathcal{X}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu} f(z) = z + \sum_{k=2}^{\infty} \varpi_r^m(k,\alpha,\beta,\sigma,\tau,\mu) a_k z^k \tag{1.9}$$

where

$$\varpi_r^m(k,\alpha,\beta,\sigma,\tau,\mu) = \frac{k(1+r^2)^2}{(k^2+r^2)^2}\left(\frac{\alpha+k\beta}{\alpha+\beta}\right)^m [1+(k+\mu-\sigma-1)\tau]^m.$$

2. Main Results

Definition 2.1. From now on, let $\tau, \sigma \geqq 0$, $0 \leqq \mu \leqq \sigma$, $m \in \mathbb{N}_0$, $r > 0$, and $-1 \leqq B < A \leqq 1$, then a function f of the form (1.1) is an element of the set $\mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$ if it fulfills the condition

$$\frac{z(\mathcal{X}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu} f(z))'}{\mathcal{X}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu} f(z)} \prec \frac{1+Az}{1+Bz} \quad (|z|<1) \tag{2.1}$$

for $\mathcal{X}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu} f(z)$ defined in (1.8).

Suppose $m = 0$, then set $\mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B) = \mathcal{S}_r(A,B)$, where $\mathcal{S}_r(A,B)$ is the set studied by Liu et al. [12].

Geometric function theorists have studied several geometric properties of many subsets of analytic functions defined by devise number of operators, for instance see [7–9]. In this investigation, many geometric properties of functions that fulfill condition (2.1) are presented. Some of the investigated properties are the coefficient inequality, radii problems and subordinating factor sequence. Others are distortion, growth, covering, closure, inclusion and some integral operators that are preserved in the new class. Some contextual work relevant to this properties are cited in [8, 10, 14–16, 19–21, 23].

Theorem 2.2. *A function $f \in \mathcal{A}$ belongs to the set $\mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$ if it fulfills the inequality*

$$\sum_{k=2}^{\infty} \mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)|a_k| \leqq A - B \tag{2.2}$$

where

$$\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu) = \frac{k(1+r^2)^2}{(k^2+r^2)^2}\left(\frac{\alpha+k\beta}{\alpha+\beta}\right)^m [1+(k+\mu-\sigma-1)\tau]^m [k-1+|Bk-A|].$$

Proof. Assume condition (2.2) is fulfilled, then by subordination technique, (2.1) can be written as

$$\frac{z(\mathcal{X}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu} f(z))'}{\mathcal{X}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu} f(z)} = \frac{1+As(z)}{1+Bs(z)}$$

which by equivalence shows that

$$\left|\frac{z(\mathcal{X}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu} f(z))' - \mathcal{X}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu} f(z)}{A\mathcal{X}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu} f(z) - zB(\mathcal{X}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu} f(z))'}\right| = |s(z)| < 1$$

for $s(z)$ in (1.3). The application of (1.8) and further simplification shows that

$$\left|\frac{\sum\limits_{k=2}^{\infty} \frac{k(1+r^2)^2}{(k^2+r^2)^2}\left(\frac{\alpha+k\beta}{\alpha+\beta}\right)^m [1+(k+\mu-\sigma-1)\tau]^m (k-1) a_k z^{k-1}}{(A-B) - \sum\limits_{k=2}^{\infty} \frac{k(1+r^2)^2}{(k^2+r^2)^2}\left(\frac{\alpha+k\beta}{\alpha+\beta}\right)^m [1+(k+\mu-\sigma-1)\tau]^m (Bk-A) a_k z^{k-1}}\right|$$

$$= \frac{\sum\limits_{k=2}^{\infty} \frac{k(1+r^2)^2}{(k^2+r^2)^2}\left(\frac{\alpha+k\beta}{\alpha+\beta}\right)^m [1+(k+\mu-\sigma-1)\tau]^m (k-1)|a_k|}{(A-B) - \sum\limits_{k=2}^{\infty} \frac{k(1+r^2)^2}{(k^2+r^2)^2}\left(\frac{\alpha+k\beta}{\alpha+\beta}\right)^m [1+(k+\mu-\sigma-1)\tau]^m |Bk-A||a_k|} \leqq 1.$$

Clearly, the LHS is bounded above by 1 if

$$\sum_{k=2}^{\infty} \frac{k(1+r^2)^2}{(k^2+r^2)^2} \left(\frac{\alpha+k\beta}{\alpha+\beta}\right)^m [1+(k+\mu-\sigma-1)\tau]^m (k-1)|a_k|$$

$$\leqq (A-B) - \sum_{k=2}^{\infty} \frac{k(1+r^2)^2}{(k^2+r^2)^2} \left(\frac{\alpha+k\beta}{\alpha+\beta}\right)^m [1+(k+\mu-\sigma-1)\tau]^m |Bk-A||a_k|.$$

while some rearrangement and simplification show that (2.2). □

Corollary 2.3. *Let* $f \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$. *Then*

$$|a_k| \leqq \frac{A-B}{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)} \quad (k=2,3,4,\ldots)$$

and inequality (2.2) *is sharp for the function*

$$f_k(z) = z + \frac{A-B}{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)} z^k \quad (k=2,3,4,\ldots,\ |z|<1). \tag{2.3}$$

Remark 2.4. Setting $m=0$ shows that $f \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$ if

$$\sum_{k=2}^{\infty} \frac{k(1+r^2)^2}{(k^2+r^2)^2}[(k-1)+|Bk-A|]|a_k| \leqq A-B.$$

This is the result of Liu et al. [12].

2.1. **Growth Theorem.**

Theorem 2.5. *Let* $f \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$. *Then*

$$|z| - \frac{|z|^2\varpi_r^m(2,\alpha,\beta,\sigma,\tau,\mu)(A-B)}{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)} \leqq |\mathcal{X}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu} f(z)| \leqq |z| + \frac{|z|^2\varpi_r^m(2,\alpha,\beta,\sigma,\tau,\mu)(A-B)}{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}. \tag{2.4}$$

The result is sharp for the function

$$f_2(z) = z + \frac{A-B}{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)} z^2 \quad (|z|<1). \tag{2.5}$$

Proof. From (2.2) we get

$$\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)\sum_{k=2}^{\infty}|a_k| \leqq \sum_{k=2}^{\infty}\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)|a_k| \leqq A-B$$

so

$$\sum_{k=2}^{\infty}|a_k| \leqq \frac{A-B}{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}. \tag{2.6}$$

Clearly, $|z|^k < |z| < 1$ so that from (1.9) we get

$$|\mathcal{X}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu} f(z)| = \left|z + \sum_{k=2}^{\infty}\varpi_r^m(k,\alpha,\beta,\sigma,\tau,\mu)a_k z^k\right| \leqq |z| + |z|^2\varpi_r^m(2,\alpha,\beta,\sigma,\tau,\mu)\sum_{k=2}^{\infty}|a_k| \tag{2.7}$$

and putting (2.6) into (2.7) shows that

$$|\mathcal{X}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu} f(z)| \leqq |z| + \frac{|z|^2\varpi_r^m(2,\alpha,\beta,\sigma,\tau,\mu)(A-B)}{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}.$$

Equally,

$$|\mathcal{X}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu} f(z)| \geqq |z| - \frac{|z|^2\varpi_r^m(2,\alpha,\beta,\sigma,\tau,\mu)(A-B)}{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}$$

which completes the proof. □

2.2. **Distortion Theorem.**

Theorem 2.6. *Let $f \in \mathcal{S}^{m,\alpha,\beta,\mu}_{\tau,\sigma,r}(A,B)$. Then*

$$1 - \frac{2|z|\varpi^m_r(2,\alpha,\beta,\sigma,\tau,\mu)(A-B)}{\mho^m_r(2,\alpha,\beta,\sigma,\tau,\mu)} \leqq \left|(\mathcal{X}^{m,\alpha,\beta,\mu}_{\tau,\sigma,r} f(z))'\right| \leqq 1 + \frac{2|z|\varpi^m_r(2,\alpha,\beta,\sigma,\tau,\mu)(A-B)}{\mho^m_r(2,\alpha,\beta,\sigma,\tau,\mu)}.$$

The inequality is sharp for the extremal function in (2.5).

Proof. Clearly, $|z|^k < |z| < 1$ so that from (1.9) we have

$$\left|(\mathcal{X}^{m,\alpha,\beta,\mu}_{\tau,\sigma,r} f(z))'\right| = \left|1 + \sum_{k=2}^{\infty} \varpi^m_r(k,\alpha,\beta,\sigma,\tau,\mu) k a_k z^{k-1}\right| \leqq 1 + 2|z|\varpi^m_r(2,\alpha,\beta,\sigma,\tau,\mu)\sum_{k=2}^{\infty}|a_k| \tag{2.8}$$

and putting (2.6) into (2.8) gives

$$\left|(\mathcal{X}^{m,\alpha,\beta,\mu}_{\tau,\sigma,r} f(z))'\right| \leqq 1 + \frac{2|z|\varpi^m_r(2,\alpha,\beta,\sigma,\tau,\mu)(A-B)}{\mho^m_r(2,\alpha,\beta,\sigma,\tau,\mu)}.$$

Equally,

$$\left|(\mathcal{X}^{m,\alpha,\beta,\mu}_{\tau,\sigma,r} f(z))'\right| \geqq 1 - \frac{2|z|\varpi^m_r(2,\alpha,\beta,\sigma,\tau,\mu)(A-B)}{\mho^m_r(2,\alpha,\beta,\sigma,\tau,\mu)}$$

which completes the proof. □

2.3. **Covering Theorem.**

Theorem 2.7. *Let $f \in \mathcal{S}^{m,\alpha,\beta,\mu}_{\tau,\sigma,r}(A,B)$. Then $\mathcal{X}^{m,\alpha,\beta,\mu}_{\tau,\sigma,r} f(z)$ maps $|z| < 1$ onto a domain that contains the disk*

$$|\mathcal{X}^{m,\alpha,\beta,\mu}_{\tau,\sigma,r} f(z)| < 1 - \frac{\varpi^m_r(2,\alpha,\beta,\sigma,\tau,\mu)(A-B)}{\mho^m_r(2,\alpha,\beta,\sigma,\tau,\mu)}$$

The inequality is sharp for the extremal function in (2.5).

Proof. Letting $|z| \longrightarrow 1^-$ in (2.4) completes the proof. □

2.4. **Radii Problems.**

Theorem 2.8. *Let $f \in \mathcal{S}^{m,\alpha,\beta,\mu}_{\tau,\sigma,r}(A,B)$. Then f is close-to-convex of order λ $(0 \leqq \lambda < 1)$ in the disk*

$$|z| < \inf_{k \geqq 2} \left\{ \frac{\mho^m_r(k,\alpha,\beta,\sigma,\tau,\mu)(1-\lambda)}{k(A-B)} \right\}^{\frac{1}{k-1}}.$$

The inequality is sharp for the function in (2.3).

Proof. It is sufficient to show that

$$|f' - 1| < 1 - \lambda.$$

Using (1.1) shows that

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\lambda}\right) |a_k||z|^{k-1} < 1. \tag{2.9}$$

Evidently, inequalities (2.2) and (2.9) is only valid if

$$\frac{k}{1-\lambda}|z|^{k-1} < \frac{\mho^m_r(k,\alpha,\beta,\sigma,\tau,\mu)}{A-B}$$

while isolating $|z|^{k-1}$ completes the proof. □

Theorem 2.9. *Let $f \in \mathcal{S}^{m,\alpha,\beta,\mu}_{\tau,\sigma,r}(A,B)$. Then f is starlike of order λ $(0 \leqq \lambda < 1)$ in the disk*

$$|z| < \inf_{k \geqq 2} \left\{ \frac{\mho^m_r(k,\alpha,\beta,\sigma,\tau,\mu)(1-\lambda)}{(k-\lambda)(A-B)} \right\}^{\frac{1}{k-1}}.$$

The inequality is sharp for the extremal function in (2.3).

Proof. It is sufficient to show that

$$\left| \frac{zf'(z) - f(z)}{zf'(z) + (1-2\lambda)f(z)} \right| < 1.$$

Using (1.1) shows that

$$\sum_{k=2}^{\infty} \left(\frac{k-\lambda}{1-\lambda} \right) |a_k||z|^{k-1} < 1. \tag{2.10}$$

Evidently, inequalities (2.2) and (2.10) is valid if

$$\frac{k-\lambda}{1-\lambda}|z|^{k-1} < \frac{\mho^m_r(k,\alpha,\beta,\sigma,\tau,\mu)}{A-B}$$

where some rearrangements complete the proof. □

Theorem 2.10. *Let $f \in \mathcal{S}^{m,\alpha,\beta,\mu}_{\tau,\sigma,r}(A,B)$. Then f is convex of order λ $(0 \leqq \lambda < 1)$ in the disk*

$$|z| < \inf_{k \geqq 2} \left\{ \frac{\mho^m_r(k,\alpha,\beta,\sigma,\tau,\mu)(1-\lambda)}{k(k-\lambda)(A-B)} \right\}^{\frac{1}{k-1}}.$$

The inequality is sharp for the extremal function in (2.3).

Proof. It is sufficient to show that

$$\left| \frac{zf''(z)}{f'(z)} \right| < 1-\lambda.$$

Using (1.1) gives

$$\sum_{k=2}^{\infty} \left(\frac{k(k-\lambda)}{1-\lambda} \right) |a_k||z|^{k-1} < 1. \tag{2.11}$$

Evidently, inequalities (2.2) and (2.11) is only valid if

$$\frac{k(k-\lambda)}{1-\lambda}|z|^{k-1} < \frac{\mho^m_r(k,\alpha,\beta,\sigma,\tau,\mu)}{(A-B)}$$

where some rearrangements complete the proof. □

2.5. **Integral Preserving Theorem.**

Definition 2.11 ([4]). Let $f \in \mathcal{A}$. Then $\mathcal{I}_\varkappa : \mathcal{A} \longrightarrow \mathcal{A}$ $\quad (\varkappa > -1)$ defined by

$$\mathcal{I}_\varkappa f(z) = \frac{1+\varkappa}{z^\varkappa} \int_0^z \zeta^{\varkappa-1} f(\zeta) d\zeta = z + \sum_{k=2}^{\infty} \frac{1+\varkappa}{k+\varkappa} a_k z^k \quad (|z| < 1). \tag{2.12}$$

is the well-known Bernardi integral operator.

Definition 2.12 ([2,9]). Let $f \in \mathcal{A}$. Then $\mathcal{I}^n_q : \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$\mathcal{I}^0_q f(z) = f(z)$$
$$\mathcal{I}^1_q f(z) = \frac{1}{qz^{(q^{-1}-1)}} \int_0^z \zeta^{(q^{-1}-2)} f(\zeta) d\zeta = I_q f(z) \quad (q > 0)$$
$$\mathcal{I}^2_q f(z) = I_q(\mathcal{I}^1_q f(z))$$

which in general shows that

$$\mathcal{I}_q^n f(z) = I_q(\mathcal{I}_q^{n-1} f(z))$$

and in particular

$$\mathcal{I}_q^n f(z) = z + \sum_{k=2}^{\infty} \frac{1}{(1+(k-1)q)^n} a_k z^k \quad (q \geqq 0,\ n = 0, 1, 2, \ldots,\ |z| < 1) \tag{2.13}$$

is the Al-Oboudi-Al-Qahtani integral operator.

Theorem 2.13. *Let $f \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A, B)$. Then $\mathcal{I}_{\varkappa} f(z) \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A, B)$.*

Proof. Let $f \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A, B)$ and from (2.12), it is clear that

$$\frac{1+\varkappa}{k+\varkappa}|a_k| < |a_k| \quad (\forall\ k = \{2, 3, 4, \ldots\})$$

so that from (2.2) we get

$$\sum_{k=2}^{\infty} \frac{\mho_r^m(k, \alpha, \beta, \sigma, \tau, \mu)}{A - B} \frac{1+\varkappa}{k+\varkappa}|a_k| < \sum_{k=2}^{\infty} \frac{\mho_r^m(k, \alpha, \beta, \sigma, \tau, \mu)}{A - B}|a_k| \leqq 1$$

which completes the proof. □

Theorem 2.14. *Let $f \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A, B)$. Then $\mathcal{I}_q^n f(z) \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A, B)$.*

Proof. Let $f \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A, B)$, then the application of Theorem 2.13 completes the proof. □

Remark 2.15. Since the Al-Oboudi-Al-Qahtani integral operator $\mathcal{I}_q^n$ (see [2, 9]) generalized the well-known Sălăgean integral operator $\mathcal{I}^n$ (see [28]), then the following Corollary holds.

Corollary 2.16. *Let $f \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A, B)$. Then $\mathcal{I}^n f(z) \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A, B)$.*

Lemma 2.17 ([11]). *Let $f, F \in \mathcal{A}$ where $f \prec F$. Then*

$$\int_0^{2\pi} |f(z)|^y d\vartheta \leqslant \int_0^{2\pi} |F(z)|^y d\vartheta$$

for $z = re^{i\vartheta}$, $\vartheta > 0$ and $0 < r < 1$.

Theorem 2.18. *Let $f \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A, B)$ and*

$$F(z) = z + \frac{(A - B)}{\mho_r^m(2, \alpha, \beta, \sigma, \tau, \mu)} z^k \quad (k = 2, 3, \ldots)$$

from (2.3). *Then for $\vartheta > 0$ and $z = re^{i\vartheta}$, $0 < r < 1$,*

$$\int_0^{2\pi} |f(z)|^y d\vartheta \leqq \int_0^{2\pi} |F(z)|^y d\vartheta. \tag{2.14}$$

Proof. Let $f \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A, B)$. Then by Lemma 2.17, we get from (2.14) that

$$\int_0^{2\pi} \left|1 + \sum_{k=2}^{\infty} a_k z^{k-1}\right|^y d\vartheta \leqq \int_0^{2\pi} \left|1 + \frac{A - B}{\mho_r^m(k, \alpha, \beta, \sigma, \tau, \mu)} z^{k-1}\right|^y d\vartheta.$$

so it suffices by Lemma 2.17 to proof that

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 + \frac{A - B}{\mho_r^m(k, \alpha, \beta, \sigma, \tau, \mu)} z^{k-1}$$

which by implication means that

$$1+\sum_{k=2}^{\infty} a_k z^{k-1} = 1+\frac{A-B}{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)}[s(z)]^{k-1}$$

for $|s(z)|$ in (1.3). Simple simplification shows that

$$[s(z)]^{k-1} = \sum_{k=2}^{\infty} \frac{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)}{(1-\beta)} a_k z^{k-1}$$

and

$$|s(z)|^{k-1} = \left|\sum_{k=2}^{\infty} \frac{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)}{A-B} a_k z^{k-1}\right| \leqq \sum_{k=2}^{\infty} \frac{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)}{A-B}|a_k| < 1$$

which completes the proof. □

2.6. **Closure Properties.**

Theorem 2.19. *From* (1.2)*, let* $f, F \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$*. Then*

$$G_n(z) = (1-n)f(z) + nF(z) = z + \sum_{k=2}^{\infty}\{(1-n)a_k + nc_k\}z^k \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B) \tag{2.15}$$

for $n \in [0,1]$.

Proof. Using (1.2) in (2.15) and (2.2) shows that

$$\begin{aligned}
&\sum_{k=2}^{\infty}\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)\big|\{(1-n)a_k + nc_k\}\big| \\
&\leqq \sum_{k=2}^{\infty}\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)\{(1-n)|a_k| + n|c_k|\} \\
&= (1-n)\sum_{k=2}^{\infty}\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)|a_k| + n\sum_{k=2}^{\infty}\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)|c_k| \\
&\leqq (1-n)\{(A-B)\} + n\{(A-B)\} = (A-B).
\end{aligned}$$

Hence $G_n \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$. □

Theorem 2.20. *Let* $n = 1, 2, 3, \ldots, x$ *and*

$$f_n(z) = z + \sum_{k=2}^{\infty} a_{k,n} z^k \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B). \tag{2.16}$$

Then for $\sum_{n=1}^{x} \eta_n = 1$*, the function*

$$g(z) = \sum_{n=1}^{x} \eta_n f_n(z) \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B) \quad (|z|<1). \tag{2.17}$$

Proof. Note that for function $f_n(z)$ in (2.16)

$$\sum_{k=2}^{\infty} \frac{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)}{(A-B)}|a_{k,n}| \leqq 1 \tag{2.18}$$

and (2.17) can be written as

$$g(z) = \sum_{n=1}^{x}\eta_n\left(z + \sum_{k=2}^{\infty} a_{k,n}z^k\right) = z + \sum_{n=1}^{x}\sum_{k=2}^{\infty}\eta_n a_{k,n} z^k = z + \sum_{k=2}^{\infty}\left(\sum_{n=1}^{x}\eta_n a_{k,n}\right) z^k. \tag{2.19}$$

Putting (2.19) into (2.18) shows that

$$\sum_{k=2}^{\infty}\frac{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)}{A-B}\left|\sum_{n=1}^{x}\eta_n a_{k,n}\right| = \sum_{k=2}^{\infty}\frac{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)}{A-B}|a_{k,n}| \leqq 1.$$

Hence, $g \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$. □

Theorem 2.21. *Let $n = 1, 2, 3, \ldots, x$ for the functions*

$$f_n(z) = z + \sum_{k=2}^{\infty} a_{k,n} z^k \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B).$$

Then the arithmetic mean $m(z)$ of functions $f_n(z)$ defined by

$$m(z) = \frac{1}{x}\sum_{n=1}^{x} f_n(z) \quad (|z| < 1) \tag{2.20}$$

is in $\mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$.

Proof. From (2.20) we get

$$m(z) = \frac{1}{x}\sum_{n=1}^{x}\left(z + \sum_{k=2}^{\infty} a_{k,n} z^k\right) = z + \sum_{k=2}^{\infty}\left(\frac{1}{x}\sum_{n=1}^{x} a_{k,n}\right) z^k. \tag{2.21}$$

Since $f_n \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$ for all $n = 1, 2, 3, \ldots, x$, then putting (2.21) into (2.2) shows that

$$\sum_{k=2}^{\infty}\frac{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)}{(A-B)}\left|\frac{1}{x}\sum_{n=1}^{x} a_{k,n}\right| = \frac{1}{x}\sum_{n=1}^{x}\left\{\sum_{k=2}^{\infty}\frac{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)}{(A-B)}|a_{k,n}|\right\} \leqq \frac{1}{x}\sum_{n=1}^{x}(1) = 1$$

which implies that $m \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$. □

Theorem 2.22. *From (1.2), let $f, F \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$. Then the weighted mean w_n of functions f and F defined by*

$$w_n(z) = \frac{(1-n)f(z) + (1+n)F(z)}{2} \quad (n = 1, 2, \ldots,\ |z| < 1) \tag{2.22}$$

is also in $\mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$.

Proof. Using (1.2) in (2.22) shows that

$$w_n(z) = z + \sum_{k=2}^{\infty}\frac{(1-n)a_k + (1+n)c_k}{2} z^k. \tag{2.23}$$

To show that w_n is in $\mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$ is to show that

$$\sum_{k=2}^{\infty}\frac{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)}{A-B}\left|\frac{(1-n)a_k + (1+n)c_k}{2}\right| \leqq 1.$$

This follows by using (2.2) to give

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)}{A-B}\left\{\frac{(1-n)|a_k| + (1+n)|c_k|}{2}\right\} \\
&= \frac{(1-n)}{2}\sum_{k=2}^{\infty}\frac{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)}{A-B}|a_k| + \frac{(1+n)}{2}\sum_{k=2}^{\infty}\frac{\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)}{A-B}|c_k| \\
&\leqq \frac{(1-n)}{2} + \frac{(1+n)}{2} = 1
\end{aligned}$$

so $w_n \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$. □

2.7. **Inclusion Property.**

Definition 2.23. The δ-neighbourhood of $f \in \mathcal{A}$ is defined by the set

$$\mathcal{N}_\delta(f) = \left\{ F : F(z) = z + \sum_{k=2}^{\infty} c_k z^k \in \mathcal{A} \text{ and } \sum_{k=2}^{\infty} k|a_k - c_k| \leqq \delta,\ \delta \geqq 0 \right\} \tag{2.24}$$

and for the function $h(z) = z \in \mathcal{A}$, the δ-neighbourhood is defined by the set

$$\mathcal{N}_\delta(h) = \left\{ F : F(z) = z + \sum_{k=2}^{\infty} c_k z^k \in \mathcal{A} \text{ and } \sum_{k=2}^{\infty} k|c_k| \leqq \delta,\ \delta \geqq 0 \right\}. \tag{2.25}$$

The concept of neighbourhood of analytic functions was initiated by Goodman [5] where it was proved that $\mathcal{N}_1(h) \subset \mathcal{S}^\star$. In 1981, Ruscheweyh [27] presented the sets in (2.24) and (2.25) which was an extension of Goodman's idea.

Definition 2.24. Let $f \in \mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$. Then there is a function

$$F \in \mathcal{S}_{\tau,\sigma,r,\gamma}^{m,\alpha,\beta,\mu}(U,V) \quad (-1 \leqq V < U \leqq 1)$$

such that

$$\left| \frac{f(z)}{F(z)} - 1 \right| \leqq 1 - \gamma \quad (|z| < 1,\ 0 \leqq \gamma < 1).$$

Theorem 2.25. *Let*

$$F(z) = z + \sum_{k=2}^{\infty} c_k z^k \in \mathcal{S}_{\tau,\sigma,r,\gamma}^{m,\alpha,\beta,\mu}(U,V) \quad (-1 \leqq V < U \leqq 1,\ |z| < 1)$$

and

$$\gamma := 1 - \frac{\delta \mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{2[\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu) - (U-V)]},$$

then $\mathcal{N}_\delta(f) \subset \mathcal{S}_{\tau,\sigma,r,\gamma}^{m,\alpha,\beta,\mu}(U,V)$.

Proof. Suppose $f \in \mathcal{N}_\delta(f)$, then from (2.24),

$$\sum_{k=2}^{\infty} k|a_k - c_k| \leqq \delta \implies \sum_{k=2}^{\infty} |a_k - c_k| \leqq \frac{\delta}{2}. \tag{2.26}$$

Also, since $F(z) = z + \sum_{k=2}^{\infty} c_k z^k \in \mathcal{S}_{\tau,\sigma,r,\gamma}^{m,\alpha,\beta,\mu}(U,V)$, then from (2.2)

$$\sum_{k=2}^{\infty} |c_k| \leqq \frac{(U-V)}{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)} \quad (-1 \leqslant V < U \leqq 1). \tag{2.27}$$

Definition 2.24 implies that

$$\left| \frac{f(z)}{F(z)} - 1 \right| = \left| \frac{\sum_{k=2}^{\infty} (a_k - c_k) z^{k-1}}{1 + \sum_{k=2}^{\infty} c_k z^{k-1}} \right| \leqq \frac{\sum_{k=2}^{\infty} |a_k - c_k|}{1 - \sum_{k=2}^{\infty} |c_k|}. \tag{2.28}$$

Putting (2.26) and (2.27) into (2.28) shows that

$$\left| \frac{f(z)}{F(z)} - 1 \right| \leqq \frac{\delta \mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{2[\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu) - (U-V)]} = 1 - \gamma$$

which completes the proof. □

2.8. **Subordinating Factor Sequence.**

Definition 2.26 ([31]). The sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is called a *subordinating factor sequence* if whenever

$$g(z)=\sum_{k=1}^{\infty} b_k z^k \quad (b_1=1,\ |z|<1)$$

is analytic and univalently convex in $|z|<1$, $\sum_{k=1}^{\infty} c_k b_k \prec g(z)$.

Lemma 2.27 ([31]). *The sequence $\{c_k\}_{k=1}^{\infty}$ is called a subordinating factor sequence if and only if*

$$\Re\left(1+2\sum_{k=1}^{\infty} c_k z^k\right)>0 \quad (|z|<1).$$

Theorem 2.28. *Let $f\in\mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$ and $g(z)$ be a convex function, then*

$$\frac{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{2[(A-B)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)]}(f\star g)(z)\prec g(z) \tag{2.29}$$

for

$$\Re f>-\frac{(A-B)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}. \tag{2.30}$$

Note that the constant factor

$$\frac{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{2[(A-B)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)]} \tag{2.31}$$

cannot be replaced by a bigger value.

Proof. Let $f\in\mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$ and suppose $g(z)=z+\sum_{k=2}^{\infty} b_k z^k$ is a convex function, then from (2.29),

$$\begin{aligned}&\frac{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{2[(A-B)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)]}(f\star g)(z)\\&=\frac{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{2[(A-B)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)]}\left(z+\sum_{k=2}^{\infty} a_k b_k z^k\right)\\&=\sum_{k=1}^{\infty}\frac{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{2[(A-B)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)]}a_k b_k z^k.\end{aligned}$$

Clearly by Definition 2.26, the subordination result (2.29) holds if

$$\left\{\frac{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{2[(A-B)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)]}a_k\right\}_{k=1}^{\infty}$$

is a *subordinating factor sequence* for $a_1=1$. Application of Lemma 2.27 shows that an equivalence inequality

$$\Re\left(1+\sum_{k=1}^{\infty}\frac{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{(A-B)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}a_k z^k\right)>0. \tag{2.32}$$

Observe that $\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)$ is an increasing function for $k\geqq 2$, so

$$\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)\leqq\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)\quad\forall k\geqq 2$$

hence, it follows by $|z| = r < 1$, triangle inequality and inequality (2.2) that

$$\begin{aligned}
&\Re\left(1+\sum_{k=1}^{\infty}\frac{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{(A-B)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}a_k z^k\right)\\
&\quad=\Re\left(1+\frac{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{(A-B)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}\sum_{k=1}^{\infty}a_k z^k\right)\\
&\quad\geqq 1-\frac{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{(A-B)(1-\lambda)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}r-\frac{\sum_{k=2}^{\infty}\mho_r^m(k,\alpha,\beta,\sigma,\tau,\mu)|a_k|}{(A-B)(1-\lambda)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}r^k\\
&\quad> 1-\frac{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{(A-B)(1-\lambda)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}r-\frac{(A-B)(1-\lambda)}{(A-B)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}r\\
&\quad=1-r>0.
\end{aligned}$$

This evidently proves the inequality (2.32) and as well as the subordination result (2.29). Also, the inequality (2.30) follows from (2.29) by taking the convex function

$$g_0(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k.$$

To prove the sharpness of the constant (2.31), consider the function

$$f_2(z)=z+\frac{(A-B)}{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}z^2\in\mathcal{S}_{\tau,\sigma,r}^{m,\alpha,\beta,\mu}(A,B)$$

so that by using (2.29) we get

$$\frac{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{2[(A-B)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)]}f_2(z)\prec g_0(z)=\frac{z}{1-z}. \tag{2.33}$$

It can *easily* be verified for $f_2(z)$ that

$$\min_{|z|\leqq r}\left\{\Re\left(\frac{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{2[(A-B)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)]}f_2(z)\right)\right\}=-\frac{1}{2}\quad(|z|<1)$$

which shows that the constant $\frac{\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)}{2[(A-B)+\mho_r^m(2,\alpha,\beta,\sigma,\tau,\mu)]}$ cannot be replaced by any bigger value. □

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