ISSN: 2832-4293

Pan-American Journal of Mathematics 2 (2023), 10
https://doi.org/10.28919/cpr-pajm/2-10

ON NUMERICAL RANGES AND SPECTRA OF NORM ATTAINING OPERATORS IN C*-ALGEBRAS

SABASI OMAORO, BENARD OKELO*, AND OMOLO ONGATI

ABSTRACT. In this paper, we study norm attaining operators in C^*-algebras. We characterize their numerical ranges and spectra. In particular, we show that if a norm-attaining operator S is self-adjoint, then its spectrum lies in the interval $[-\|S\|, \|S\|]$.

1. INTRODUCTION

The theory of norm-attaining operators was influenced by the classical Bishop-Phelps Theorem [2] which states that the set of norm attaining functionals for a Banach space is dense in the dual space and the question on whether there was a possible extension of their result for operators was raised. This is the question Lindenstrauss [7] sought to answer when he initiated that systematic study which gave the first counter example and obtained several positive results. Acosta, Aguirre and Paya in [1] used a Banach space which had been considered by ([3]- [17] and the references therein) to improve some results on norm attaining operators. They managed to show that the norm attaining operators from this space to a strictly convex Banach space are of finite rank [8]. The same Banach space was also used to get a new example of a space which does not satisfy the denseness of the numerical radius attaining operators. This new counterexample improved and simplified the one previously obtained by [13] when he answered that open question asked by [5] followed by [14]. Acosta, Aguirre and Paya [1] proved that in $A_\infty(BX; X)$, the set of norm attaining elements contains numerical radius attaining elements and also when X is a finite-dimensional space they coincide. It was shown by [4] that the norm attaining paranormal operators have a non-trivial invariant space and the norm attaining quadratically hyponormal weighted shift is subnormal. It was proved [7] that the norm attaining operators mapping $L^1[0,1]$ to strictly convex Banach space are dense in the space of all linear operators from $L^1[0,1]$ to the convex Banach space. The authors in [5] managed to show examples of compact linear operators between Banach spaces which cannot be approximated by norm attaining operators. This was a negative answer to an open question posed in the 1970's. Any strictly convex Banach space failing the approximation property serves as the range space. Similarly there are examples in which the domain space has a Schauder basis. In [10] they constructed a compact metric space S and it was shown that there is a bounded linear operator $T : L^1[0,1] \to C(S)$ which could not be approximated by a norm attaining operator. Also it was established that there does not exist a retract of $L^\infty[0,1]$ onto its unit ball

DEPARTMENT OF PURE AND APPLIED MATHEMATICS, JARAMOGI OGINGA ODINGA UNIVERSITY OF SCIENCE AND TECHNOLOGY, BOX 210-40601, BONDO-KENYA
E-mail addresses: omaoro69@yahoo.com, bnyaare@yahoo.com, nomoloongati@yahoo.com.
Submitted on July 13, 2023.
2020 *Mathematics Subject Classification.* Primary 47B47; Secondary 47A30.
Key words and phrases. Numerical range, Spectrum, Norm-attainability, C^*-algebra.
*Corresponding author.
The first author was supported in part by NRF Grant 903601.

which is simultaneously weak* continuous and norm continuous. The work of [1] proved that if a strictly convex Banach space Y contains either a symmetric basic sequence which is not equivalent to the l_1 basis or a normalized sequence with an upper p-estimate, then there is a Banach space X such that the set of norm attaining operators is not dense in the Banach space of all bounded linear operators from X into Y. He deduced that no infinite-dimensional uniformly convex Banach space has Lindenstrauss property B. They also gave a new sufficient condition for a Banach space Y to satisfy Lindenstrauss's property B, namely the set of norm attining operators from any other Banach space X into Y is dense. Even in the finite-dimensional case, the result gave new examples of Banach spaces with property B. In [16] the research proved that every Banach space is isometric to a space with the property that the norm attaining operators are dense in the space of all operators into it, for any given domain space. Similarly, a super-reflexive space is arbitrarily nearly isometric to a space with this property. In this regard therefore, this study characterizes norm-attaining operators in-terms of their numerical ranges and spectra.

2. Preliminaries

We provide some definitions and some known preliminary results which are useful in the sequel.

Definition 2.1. ([14], Definition 5) Let H be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on H. An operator $T \in B(H)$ that satisfies the norm-attainability condition, that is, if there exists a unit vector $h \in H$ such that $\|Th\| = \|T\|$ is called a norm-attainable operator. We denote the class of all norm-attaining operators by $NA(H)$.

Definition 2.2. ([6], Definition 2.7) Let B be a Banach algebra. Then B is called a C^*-algebra if it has an involution $*$ such that for all $b \in B, \|bb^*\| = \|b\|^2$.

Definition 2.3. ([1]) Let S be a linear operator. Then the set $W(S) = \{\langle S\xi, \xi\rangle : \xi \in H, \|\xi\| = 1\}$ in the complex plane is called the numerical range of S.

Definition 2.4. ([6]) Let S be an operator. The spectrum of S is denoted and given by $\sigma(S) = \{\lambda \in \mathbb{C} : (S - \lambda I)\text{does not have an inverse}\}$

Remark 2.5. Let $S \in B(H)$. Then the numerical range of S is convex. This is a known fact in [16], Theorem 4.1

Remark 2.6. [15, Corollary 4.23] Given a normed vector space A and a linear subspace B of A, $a^* \in A^*\backslash\{0\}$ is equivalent to B not being dense in A $\forall b \in B$.

3. Main results

In this section, results on the numerical ranges and spectra of norm-attaining operators are given. The set of all norm-attaining operators is denoted by $NA(H)$ unless otherwise stated. We begin with the following proposition on elements of the numerical range.

Proposition 3.1. *Let $S_o \in NA(H)$ then $W(S_o)$ is nonempty and $0 \in W(S_o)$ if S_o is compact.*

Proof. Suppose S_o is compact. Then there exists a sequence $\{\varsigma_n\} \in H$ of unit vectors and $\varsigma \in H$ with $\|\varsigma\| = 1$ for which $\{S_o\varsigma_n\} \to \varsigma$ and $\|S_o\varsigma\| = \|S_o\|$. Let $\lambda \in W(S_o)$, then $\lambda = \langle S_o\varsigma_n, \varsigma_n\rangle$. This gives

$$\begin{aligned} |\lambda| &= |\langle S_o\varsigma_n, \varsigma_n\rangle| \\ &\leq \|S_o\varsigma_n\|.\|\varsigma_n\| \\ &\leq \|S_o\|.\|\varsigma_n\|.\|\varsigma_n\| \\ &\leq \|S_o\| \end{aligned}$$

implying that $\lim_{n\to\infty}\langle S_o\varsigma_n,\varsigma_n\rangle = \|S_o\|$ and therefore $\{\langle S_o\varsigma_n,\varsigma_n\rangle\}_{n=1}^{\infty}$ is bounded. Now,

$$\begin{aligned}
\langle S_o\varsigma_n - \|S_o\|\varsigma_n, S_o\varsigma_n - \|S_o\|\varsigma_n\rangle &= \langle S_o\varsigma_n, S_o\varsigma_n\rangle - \langle S_o\varsigma_n, \|S_o\|\varsigma_n\rangle \\
&- \langle \|S_o\|\varsigma_n, S_o\varsigma_n\rangle + \langle \|S_o\|\varsigma_n, \|S_o\|\varsigma_n\rangle \\
&= \|S_o\varsigma_n\|^2 - \|S_o\|(\langle S_o\varsigma_n,\varsigma_n\rangle + \overline{\langle S_o\varsigma_n,\varsigma_n\rangle}) + \|S_o\|^2\|\varsigma_n\|^2 \\
&\leq \|S_o\|^2\|\varsigma_n\|^2 - 2\|S_o\||\langle S_o\varsigma_n,\varsigma_n\rangle| + \|S_o\|^2\|\varsigma_n\|^2 \\
&= 2\|S_o\|^2\|\varsigma_n\|^2 - 2\|S_o\|\langle S_o\varsigma_n,\varsigma_n\rangle \\
&= 2\|S_o\|^2\|\varsigma_n\|^2 - 2\|S_o\|^2\|\varsigma_n\|^2 \\
&= 0.
\end{aligned}$$

This means $\{\varsigma_n\}_{n=1}^{\infty} \to 0$ weakly in H and therefore $\|S_o\|$ is an eigenvalue of S_o. Since each eigenvalue of S_o is contained in $W(S_o)$ then $W(S_o)$ is nonempty. Let $S_o = S_o^*$ (since by Hahn Banach Theorem, if S_o attains the norm S_o^* also does) and particularly let $\lambda = 0$ and ς be a unit vector with the aim that $S_o\varsigma = 0$ but $S_o^*\varsigma \neq 0$. Let $f = \frac{S_o^*\varsigma}{\|S_o^*\varsigma\|}$. As $\langle\varsigma, S_o^*\varsigma\rangle = \langle S_o\varsigma,\varsigma\rangle = \langle 0,\varsigma\rangle = 0$, it means ς, f is orthonormal and hence spans any subspace M which is two-dimensional. This therefore is an implication that the numerical range of S_o when restricted to M is contained in $W(S_o)$ that is, $W(S_o|_M) \subset W(S_o)$. Therefore showing that 0 is in int$(W(S_o|_M))$ is all what is needed in this case. Using orthonormal basis ς, f for M, $\begin{pmatrix} 0 & c \\ 0 & * \end{pmatrix}$ is the way the matrix of $S_o|_M$ appears where $c = \langle S_o|_M f,\varsigma\rangle$. It should be shown now that $c \neq 0$ so that $W(S_o|_M)$ is confirmed not to be a degenerate disk which is elliptical in form with a focus found at the origin. We have, $c = \langle S_o|_M f,\varsigma\rangle = \langle S_o f,\varsigma\rangle = \langle f, S_o^*\varsigma\rangle$. But $f = \frac{S_o^*\varsigma}{\|S_o^*\varsigma\|}$ which implies that $\frac{\langle S_o^*\varsigma, S_o^*\varsigma\rangle}{\|S_o^*\varsigma\|} = \|S_o^*\varsigma\| \neq 0$ as required. □

Proposition 3.2. *Suppose $S \in NA(H)$ and $[NA(H)]_1$ is the open unit disc of $NA(H)$ with center at the origin. Then $W(S) = [NA(H)]_1$ if S is a backward unilateral shift.*

Proof. Since $[NA(H)]_1$ is open, then for some $\lambda \in \mathbb{C}$, $\lambda \in [NA(H)]_1$ implies that $|\lambda| < 1$. As S is norm-attaining, there must be a vector $\xi \in H$ with $\|\xi\| = 1$ so that $\|S\xi\| = \|S\|$. If λ is in $W(S)$, then $|\lambda| = |\langle S\xi,\xi\rangle| \leq \|S\xi\|\|\xi\| \leq 1$ with equality satisfied whenever $S\xi$ and ξ are expressible as multiples of each other where $\|S\xi\| = 1$. Therefore this means that $S\xi = \lambda\xi$ where $|\lambda| = 1$. Let u_n be an orthonormal basis. Then $Su_1 = 0$ and $Su_n = u_{n-1}$ $\forall n \geq 2$. If $\xi = \sum_{n\geq 1} c_n u_n$, then $S\xi = \lambda\xi$ implies that $\lambda c_n = c_{n+1}$ $\forall n \in \mathbb{N}$ which is impossible since $1 = \|\xi\|^2 = \sum_{n\geq 1} |c_n|^2$. Hence $W(S) \subset [NA(H)]_1$. Now let $\lambda \in [NA(H)]_1$. This implies that $|\lambda| < 1$ and letting $\xi_0 = \sum_{n\geq 1} \lambda^n u_n \in H$ means $S\xi_0 = \lambda\xi_0$ and therefore λ is an eigenvalue for S. Since $W(S)$ contains all the eigenvalues of S, then $\lambda \in W(S)$ which means $[NA(H)]_1$ is a subset of $W(S)$ implying that $W(S) = [NA(H)]_1$. □

The relationship between the convex and infinity convex hulls of a set of complex numbers is shown in the proposition which follows.

Lemma 3.3. *Let $\alpha = \beta_n$ be any set of numbers which is countable. If $\beta_n \in \mathbb{C}$, then $co_\infty(\alpha) = co(\alpha)$.*

Proof. Since $co(\alpha) \subset co_\infty(\alpha)$ with $co_\infty(\alpha)$ being convex, then we have to show that any $q \in co_\infty(\alpha)$ is a convex combination of α's points. Since

$$co(k\alpha + d) = k\, co(\alpha) + d, \forall k, d \in \mathbb{C} \tag{3.1}$$

then it is the same for $co_\infty(\alpha)$ and therefore α can be replaced by $\alpha - q$ taking q to be equal to 0.
Assume $0 \notin co(\alpha)$. Then there is a line between 0 and $co(\alpha)$ and therefore α, $co(\alpha)$ and $co_\infty(\alpha)$ are assumed

to belong to the closed upper half of the complex-plane through using Equation 4.2.1 to have a rotation about the origin.
It is assumed that there exist infinitely many non-zero numbers k_n between 0 and 1 such that $0 = \sum\limits_{n=0}^{\infty} k_n\beta_n$, otherwise trivially $0 \in co(\alpha)$. Now, $0 = \sum\limits_{n=0}^{\infty} k_n Im(\beta_n)$ and since $Im\beta_n \geq 0$; $\forall n$, then there must be real β_n for every non-zero k_n. Hence there is some $\beta_n \in \mathbb{R} < 0$ and another $\beta_m \in \mathbb{R} > 0$. This therefore makes the origin to lie on this line between β_m and β_n and therefore is in α's convex hull. This is a contradiction since 0 was assumed not to belong to $co(\alpha)$. □

The result which follow relates a norm-attaining unitarily diagonalizable operator's numerical range to its eigenvalues' convex hull.

Theorem 3.4. *Let $S \in NA(H)$ and $co(S)_{ev}$ be a convex hull of S's eigenvalues. Then $W(S) = co(S)_{ev}$ if S is unitarily diagonalizable.*

Proof. Let S be unitarily diagonalizable. Then we have a basis $\{e_j\}$ which is orthonormal for H together with a sequence $\{\lambda_j\}$ of complex numbers giving $Se_j = \lambda_j e_j$ for each integer $j \geq 0$. Hence

$$\begin{aligned}
W(S) &= \{\langle S\varsigma, \varsigma\rangle : \varsigma \in H, \|\varsigma\| = 1\} \\
&= \{\sum_{j=0}^{\infty} \langle S\langle \varsigma, e_j\rangle e_j, \varsigma\rangle\} \\
&= \{\sum_{j=0}^{\infty} \langle \varsigma, e_j\rangle \langle Se_j, \varsigma\rangle\} \\
&= \{\sum_{j=0}^{\infty} \langle \varsigma, e_j\rangle \langle \lambda_j e_j, \varsigma\rangle\} \\
&= \{\sum_{j=0}^{\infty} \lambda_j \langle \varsigma, e_j\rangle \overline{\langle \varsigma, e_j\rangle} \\
&= \{\sum_{j=0}^{\infty} \lambda_j |\langle \varsigma, e_j\rangle|^2 : \varsigma \in H, \|\varsigma\| = 1\} \\
&= \{\sum_{j=0}^{\infty} \lambda_j b_j : 0 \leq b_j \leq 1, \sum_{j=0}^{\infty} b_j = 1\} \\
&= co(S)_{ev}
\end{aligned}$$

where $(S)_{ev}$ is the collection of eigenvalues of S and by Lemma 3.3 the proof is complete. □

The result which follows shows that boundary points of a numerical range of a norm-attaining operator are its eigenvalues if the boundary's curvature is infinite.

Proposition 3.5. *Let $S \in NA(H)$ and λ be a point on $\partial W(S)$ at the part where its curvature is infinite. Then if S is reflexive, λ is its eigenvalue.*

Proof. Let S be reflexive. Then we have $\lambda = \langle S\xi, \xi\rangle$ with $\|\xi\| = 1$. Suppose y is orthogonal to ξ where $\|y\| = 1$. Consider a subspace $\mathcal{V}$ whose span is ξ and y. From Theorem **??** the compression $S_{\mathcal{V}}$ of S to $\mathcal{V}$ has its numerical range being a ellipse which is degenerate with $W(S)$ having λ in its boundary. Since $W(S_{\mathcal{V}})$ is contained in $W(S)$ which has no closed disc having λ, it means it does not contain a closed elliptical disc containing λ. This implies that $W(S_{\mathcal{V}})$ is a line segment whose endpoint is λ which is then an eigenvalue of $S_{\mathcal{V}}$ and therefore of S also. □

At this point, results on spectra for norm-attaining operators are given here starting with the following

Definition 3.6. The spectrum of an operator T denoted by $\sigma(T)$ is the set of complex numbers λ such that $(T - \lambda I)$ is not invertible.

Theorem 3.7. *Let $A \in NA(H)$, then*
$co(\sigma(A)) = \bigcap\{\overline{W}(TAT^{-1}) : T\ is\ invertible\ on\ NA(H)\}$.

Proof. Suppose λ is not in the convex hull of the spectrum of A. We need to show that there is an invertible operator $T \in NA(H)$ such that λ is not in $\overline{W}(T^{-1}AT)$. Since $conv\sigma(A)$ is compact, it implies that there is an open disc Θ that contains it but whose closure does not contain λ. Assume that Θ is the open unit disc, particularly that $r(A) < 1$. Then for $A \in NA(H)$, $r(A) = \inf\{\|TAT^{-1}\| : T\ is\ invertible\ on\ H\}$ implying that there is an invertible operator $T \in NA(H)$ such that $\|T^{-1}AT\| \leq \frac{(1+r(A))}{2} < 1$ and $\overline{W}(T^{-1}AT) \subset \Theta$. Hence $\lambda \in \overline{W}(T^{-1}AT)$. □

Proposition 3.8. *Let $S \in NA(H)$. Then the statements which follow are similar:*

(i). $\sigma(S) \subset H^+$.

(ii). *An invertible norm-attaining operator B exists such that $\overline{W(B^{-1}SB)} \subset H^+$.*

(iii). *An invertible norm-attaining operator C which is positive exists in order for $\overline{W(C^{-1}SC)} \subset H^+$.*

(iv). *An invertible norm-attaining operator C which is positive exists in order for $\overline{W(SC)} \subset H^+$.*

Proof. $(i) \Rightarrow (ii)$. Follows since $\sigma(S)$ is in the interior of a subset of an open set which is convex and $\overline{W(B^{-1}SB)}$ is in the set. $(iii) \Rightarrow (ii)$ is trivial. By polar decomposition, let $B = P_oU_o$ with P_o being invertible and positive and U_o being unitary. This gives $W(P_o^{-1}SP_o) = W(U_o^T P_o^T SP_oU_o) = W(B^{-1}SB)$ which is, under unitary transformations, invariant. $(iii) \Rightarrow (iv)$ follows due to the identity $\overline{\langle SP_o^2y, y\rangle} = \langle(P_o^{-1}SP_o), (P_oy)\rangle$ showing that, for some $\delta > 0$, $W(SP_o^2) \subset \{z : Re\ z \geq \frac{\delta}{\|P_o^{-1}\|^2}\}$ whenever $\overline{W(P_o^{-1}SP_o)} \subset \{z : Re\ z \geq \delta\}$. $(iv) \Rightarrow (i)$. To show this, we apply [**??**, Theorem 1] to have $\sigma(S) = \sigma(SP_oP_o^1) \subset \frac{\overline{W(SP_o)}}{\overline{W(P_o)}}$ and $\frac{\overline{W(SP_o)}}{\overline{W(P_o)}} \subset H^+$ which together with the fact that the positive real axis contains $\overline{W(P_o)}$ implies $\sigma(S) \subset H^+$. □

If an operator is equivalent to a norm-attaining operator then the convex hull of the operator's spectrum is the same as the closure of its numerical range as established in the result which follows:

Proposition 3.9. *Let $S \in NA(H)$. Then $co(\sigma(S)) = \bigcap\{\overline{W(TST^{-1})}$:*
$T\ is\ invertible\ on\ H\}$.

Proof. Suppose $\lambda \notin co(\sigma(S))$. We need to show that $T \in NA(H)$ which is invertible exists such that $\lambda \notin \overline{W(T^{-1}ST)}$. Since $co(\sigma(S))$ is compact, then an open disc Θ containing this exists with λ not in its closure. With the existence of Θ, particularly that $r(S) < 1$, Corollary **??** implies the existence of $T \in NA(H)$ which is invertible giving $\|T^{-1}ST\| \leq \frac{(1+r(S))}{2} < 1$ which means that $\overline{W(T^{-1}ST)} \subset \Theta$ and hence $\lambda \notin \overline{W(T^{-1}ST)}$. □

For a closed norm-attaining operator, the residual spectrum and the point spectrum being equal implies that the operators's spectrum is equal to its adjoint's spectrum under certain conditions established in the result which follows.

Proposition 3.10. *Let $S \in NA(H)$ be closed on H. Then $\sigma_r(S) = \sigma_p(S^*)$ iff $\sigma(S) = \sigma(S^*)$ and $R(\lambda_o, S)^* = R(\lambda_o, S^*)$ for all λ_o in $\rho(S)$.*

Proof. By Corollary 2.6 the set $(\lambda_oI - S)D(S)$ cannot be dense in H if some vector $y^* \in H^* \setminus \{0\}$ exists so that we have, for each $x \in D(S)$, $\langle(\lambda_oI - S)x, y^*\rangle = 0$, which is similar to $\langle Sx, y^*\rangle = \langle x, \lambda_oy^*\rangle$. This means that y^* is in $D(S^*) \setminus \{0\}$ and $S^*y^* = \lambda_oy^*$, hence $\lambda_o \in \sigma_p(S^*)$.
Conversely, let $\lambda_o \in \rho(S)$ and take $y \in D(S),\ x^* \in H^*$ and set $y^* = R(\lambda_o, S)^*x^*$. Then $\langle(\lambda_oI - S)x, y^*\rangle = \langle R(\lambda_o, S)(\lambda_oI - S)x, x^*\rangle = \langle x, x^*\rangle$. Thus, $y^* \in D(S^*)$ with $x^* = (\lambda_oI - S)^*y^* = (\lambda_oI - S^*)y^*$ which implies

that $\lambda_o I - S^*$ is surjective. Now taking $x^* \in D(S^*)$ for $x \in H$ and using the fact that $R(\lambda_o, S)y \in D(S)$ we have

$$\begin{aligned}\langle y, R(\lambda_o, S)^*(\lambda_o I - S^*)x^*\rangle &= \langle R(\lambda_o, S)x, (\lambda_o I - S^*)x^*\rangle \\ &= \langle(\lambda_o I - S)R(\lambda_o, S)x, x^*\rangle \\ &= \langle y^*, y\rangle.\end{aligned}$$

This means that $R(\lambda_o, S)^*(\lambda_o I - S^*)x^* = x^*$ and hence $\lambda_o I - S^*$ is injective. Therefore there exists $R(\lambda_o, S^*) = R(\lambda_o, S)^*$.

Similarly, let $\lambda_o \in \rho^*(S)$ and take $x \in D(S)$ so that given each $x^* \in H^*$, it follows that

$$\begin{aligned}\langle(\lambda_o I - S)x, R(\lambda_o, S^*)x^*\rangle &= \langle x, (\lambda_o I - S^*)R(\lambda_o, S^*)x^*\rangle \\ &= \langle x, x^*\rangle.\end{aligned}$$

By Corollary 2.6, $\langle x, y^*\rangle = \|y\|$ whenever y^* is a unit vector giving $\|y\| = \langle(\lambda_o I - S)x, R(\lambda_o, S^*)y^*\rangle \leq \|R(\lambda_o, S^*)\|\|\lambda_o x - Sx\|$ which implies that $\lambda_o \notin \sigma_{ap}(S)$ and $\lambda_o \notin \sigma_p(S^*) = \sigma_r(S)$, hence $\lambda_o \notin \sigma(S)$. □

The spectrum of a norm-attaining operator is bounded under certain conditions as established in the following.

Theorem 3.11. *Let $S \in NA(H)$ and $\|S\| < |\lambda|$. Then $\sigma(S)$ is bounded.*

Proof. Define $R_{\lambda,j} \in NA(H)$ by $R_{\lambda,j} = -\frac{1}{\lambda}\sum_{n=0}^{j}\frac{S^n}{\lambda^n}$. Since $\frac{\|S\|}{|\lambda|} < 1$, then $\sum_{n=0}^{\infty}\frac{\|S\|^n}{|\lambda|^n}$ is a convergent geometric series. Therefore $R_{\lambda,j}$ is Cauchy and converges to some $A_\lambda \in NA(H)$. So

$$\begin{aligned}\|A_\lambda(S - \lambda I) - I\| &\leq \|A_\lambda(S - \lambda I) - R_{\lambda,j}(S - \lambda I)\| + \|R_{\lambda,j}(S - \lambda I) - I\| \\ &\leq \|A_\lambda - R_{\lambda,j}\|\|S - \lambda I\| + \| - \frac{S}{\lambda}\sum_{n=0}^{j}\frac{S^n}{\lambda^n} + \sum_{n=0}^{j}\frac{S^n}{\lambda^n} - I\| \\ &= \|A_\lambda - R_{\lambda,j}\|\|S - \lambda I\| + \|\frac{S^{j+1}}{\lambda^{j+1}}\| \\ &\leq \|A_\lambda - R_{\lambda,j}\|\|S - \lambda I\| + (\frac{\|S\|}{|\lambda|})^{j+1},\end{aligned}$$

which tends to 0 as $j \to \infty$. Hence, $A_\lambda(S - \lambda I) = I$ and

$$\begin{aligned}\|(S - \lambda I)A_\lambda - I\| &\leq \|(S - \lambda I)A_\lambda - (S - \lambda I)R_{\lambda,j}\| + \|(S - \lambda I) - I\| \\ &\leq \|S - \lambda I\|\|A_\lambda - R_{\lambda,j}\| + \|\frac{\|S\|}{|\lambda|}\|^{j+1}\end{aligned}$$

where $(S - \lambda I)A_\lambda = I$ implying that $A_\lambda = (S - \lambda I)^{-1}$. Thus if $\|S\| < |\lambda|$, then $\lambda \in \rho(S)$ and therefore $\sigma(S)$ is in the disc $|\lambda| \leq \|S\|$ which means it is bounded as required. □

Given a norm-attaining operator, its spectrum is a set which is closed in the complex plane as proved in this result:

Proposition 3.12. *Let $S \in NA(H)$. Then $\sigma(S) \in \mathbb{C}$ is closed.*

Proof. Suppose $\lambda \in \rho(S)$ and $(\kappa - \lambda) < \|R_\lambda\|^{-1}$. Let $R_{\kappa,j} \in NA(H)$ be defined by $R_{\kappa,j} = R_\lambda \sum_{n=0}^{j}(\kappa - \lambda)^n R_\lambda^n$. Since $|(\kappa - \lambda)| < \|R_\lambda\|^{-1}$, then $R_{\lambda,\kappa}$ is Cauchy converging to some $A_\kappa \in NA(H)$. Therefore, since $R_\lambda =$

$(S-\lambda I)^{-1}$ we get

$$\begin{aligned}\|A_\kappa(S-\kappa I)-I\| &\leq \|A_\kappa(S-\kappa I)-R_{\kappa,j}(S-\kappa I)\|+\|R_{\kappa,j}(S-\kappa I+\lambda I-\lambda I)-I\| \\ &\leq \|A_\kappa-R_{\kappa,j}\|\|S-\kappa I\|+\|\sum_{n=0}^{j}(\kappa-\lambda)^n R_\lambda^n \\ &\quad -(\kappa-\lambda)R_\lambda\sum_{n=0}^{j}(\kappa-\lambda)^n R_\lambda^n-I\| \\ &= \|A_\kappa-R_{\kappa,j}\|\|S-\kappa I\|+\|-(\kappa-\lambda)^{j+1}R_\lambda^{j+1}\| \\ &= \|A_\kappa-R_{\kappa,j}\|\|S-\kappa I\|+|\kappa-\lambda|^{j+1}R_\lambda^{j+1}\end{aligned}$$

and this tends to zero as $j\to\infty$. Thus $A_\kappa(S-\kappa I)=I$.
Similarly $(S-\kappa I)A_\kappa=I$ which implies that $(S-\kappa I)^{-1}=A_\kappa$. Therefore, $\kappa\in\rho(S)$ meaning $\rho(S)$ is open. Hence, $\sigma(S)$ is closed. □

If a norm-attaining operator S is self-adjoint, then its spectrum lies in the interval $[-\|S\|,\|S\|]$ as shown in the proposition below.

Proposition 3.13. *Let $S\in NA(H)$ and $S=S^*$. Then $\sigma(S)\in\mathbb{R}$ and $\sigma(S)\in[-\|S\|,\|S\|]$.*

Proof. Since $r(S)\leq\|S\|$, then we need only to show that $\sigma(S)\in\mathbb{R}$. Suppose $\lambda_o=\beta+i\theta\in\mathbb{C}$ $\beta,\theta\in\mathbb{R}$ with $\theta\neq 0$. Then given $\pi\in H$ we get

$$\begin{aligned}\|(S-\lambda_o I)\pi\|^2 &= \langle(S-\lambda_o I)\pi,(S-\lambda_o I)\pi\rangle \\ &= \langle(S-\beta I)\pi,(S-\beta I)\pi\rangle+\langle(-i\theta)\pi,(-i\theta)y\rangle \\ &\quad +\langle S\pi,(-i\theta)\pi\rangle+\langle(-i\theta)y,S\pi\rangle \\ &= \|(S-\beta I)\pi\|^2+\theta^2\|\pi\|^2 \\ &\geq \theta^2\|\pi\|^2.\end{aligned}$$

This estimate implies that $S-\lambda_o I$ is a one-to-one operator whose range is closed. Now suppose range$(S-\lambda I)\neq H$, then λ_o is in the residual spectrum of S such that $\overline{\lambda_o}=\beta-i\theta$ is an eigenvalue of S implying that S has an eigenvalue which is not a real number hence contradicting the property that eigenvalues of bounded self-adjoint operators are real. Therefore λ_o is in $\rho(S)$ since it is not real. □

Proposition 3.14. *Let $S\in NA(H)$ and $p_1(\lambda)=\beta_0+\beta_1\lambda+...+\beta_n\lambda^n$ be a polynomial. If $p_2(S)=\beta_0+\beta_1\varsigma+...+\beta_n\varsigma^n$, then $\sigma(p_2(S))=p_1(\sigma(S))=\{p_1(\lambda):\lambda\in\sigma(S)\}$.*

Proof. Since $n=0$ is obvious, we let $n\geq 1$ and take $\lambda_0\in\sigma(S)$ such that λ_0-S is not invertible. Then $p_1(\lambda_0)-p_2(S)$ is not invertible since

$$\begin{aligned}p_1(\lambda_0)-p_2(S) &= \sum_{k=0}^{n}\beta_k(\lambda_0^k-S^k) \\ &= \sum_{k=1}^{n}\beta_k(\lambda_0^k-S^k) \\ &= (\lambda_0-S)\sum_{k=1}^{n}\beta_k\sum_{i=1}^{k-1}\lambda_0^{k-i}S^{i-1}\end{aligned}$$

with λ_0-S and $\sum_{k=1}^{n}\beta_k\sum_{i=1}^{k-1}\lambda_0^{k-i}S^{i-1}$ commuting. This implies that $p_1(\sigma(S))\subset\sigma(p_2(S))$. Also, if $\mu\notin\{p_1(\lambda):\lambda\in\sigma(S)\}$ and $\lambda_1,\lambda_2,...,\lambda_n$ are solutions to the polynomial $\mu-p_1(\lambda)$ then $\lambda_1,\lambda_2,...,\lambda_n\notin\sigma(S)$. Additionally, $\mu-p_1(\lambda)=\xi(\lambda_1-\lambda)^{m_1}(\lambda_2-\lambda)^{m_2}...(\lambda_n-\lambda)^{m_n}$ where $m_1,m_2,...m_n\in\mathbb{N}$ and $\xi\neq 0$ which means that
$\mu-p_2(S)=\xi(\lambda_1-S)^{m_1}(\lambda_2-S)^{m_2}...(\lambda_n-S)^{m_n}$.
Hence $\mu-P_2(S)$ is invertible(being a product of invertible operators) and therefore $\mu\notin\sigma(P(S))$. This shows that $p_1(\sigma(S))\supset\sigma(p_2(S))$ as required. □

4. Conclusion

In conclusion, we have studied norm attaining operators in C^*-algebras. We have characterized their numerical ranges and spectra.

References

[1] M. D. Acosta, F. J. Aguirre and R. Paya, A space by W. Gowers and new results on norm and numerical radius attaining operators, Acta Univ. Carol. Math. Phys. 33 (1992), 5–14.

[2] E. Bishop and R. R. Phelps, A proof that every Banach space is subreflexive, Bull. Amer. Math. Soc. 67 (1961), 97–98.

[3] H. K. Du, Y. Q. Wang, and G. B. Gao, Norms of elementary operators, Proc. Amer. Soc. 36 (2008), 1337–1348.

[4] N. Dunford and J. T. Schwartz, Linear operators, Interscience, New York, 1958.

[5] B. A. Frasin, Coefficient inequality for certain classes of Sakaguchi type functions, Int. J. Nonlinear Sci. 10 (2010), 206–211.

[6] E. Kreyzig, Introductory functional analysis with applications, John Wiley and Sons, New York, 1978.

[7] J. Lindenstrauss, On operators which attain their norm, J. Israel J. Math. 1 (1963), 139-148.

[8] N. B. Okelo, J. O. Agure and D. O. Ambogo, Norms of elementary operators and characterization of norm-attainable operators, Int. J. Math. Anal. 24 (2010), 1197–1204.

[9] N. B. Okelo, The norm-attainability of some elementary operators, Appl. Math. E-Notes 13 (2013), 1–7.

[10] N. B. Okelo, α-Supraposinormality of operators in dense norm-attainable classes, Univ. J. Math. Appl. 2 (2019), 42–43.

[11] N. B. Okelo, On orthogonality of elementary operators in norm-attainable classes, Taiwan. J. Math. 24 (2020), 119–130.

[12] N. B. Okelo, M. O. Okongo and S. A. Nyakiti, On projective tensor norm and norm-attainable α-derivations, Int. J. Contemp. Math. Sci. 5 (2010), 1969–1975.

[13] N. B. Okelo, J. O. Agure and P. O. Oleche, Various notions of orthogonality in normed spaces, Acta Math. Sci. 33 (2013), 1387–1397.

[14] N. B. Okelo, On norm-attainable operators in Banach spaces, J. Funct. Spaces, 2020 (2020), 8926214.

[15] R. Schnaubelt, Functional analysis, Class Notes, Karlsruhe, 2010.

[16] J. H. Shapiro, Notes on the numerical range, Michigan state University, 2010.

[17] S. Polidoro and M. A. Ragusa, Harnack inequality for hypoelliptic ultraparabolic equations with a singular lower order term, Rev. Mat. Iberoamer. 24 (2008), 1011–1046.