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CHARACTERIZATIONS OF BELLIGERENT INTERIOR GE-FILTERS IN GE-ALGEBRAS

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ABSTRACT. The notion of GE-algebra is introduced by Bandaru et al. as a generalization of Hilbert algebra. The consept of interior GE-algebra is introduced by Lee et al., and related properties are investigated. GE-filter study at GE-algebra using interior operator was conducted by Song et al. In this paper, various properties that belligerent interior GE-filter could perform were explored, and necessary conditions have been established for interior GE-filter to turn into belligerent interior GE-filter. As a result, the characterization of belligerent interior GE-filter was established.

1. INTRODUCTION

Hilbert algebras arises in the theory of von Neumann algebras in Commutation theorem and Tomita-–Takesaki theory. In mathematics, a commutation theorem explicitly identifies the commutant of a specific von Neumann algebra acting on a Hilbert space in the presence of a trace. In the theory of von Neumann algebras, a part of the mathematical field of functional analysis, Tomita-–Takesaki theory is a method for constructing modular automorphisms of von Neumann algebras from the polar decomposition of a certain involution. Diego [6] proved that Hilbert algebras form a variety which is locally finite. Bandaru et al. [1] attempted generalized research in Hilbert algebra, and as a result they introduced the concept of GE-algebra. They investigated several properties (see [1–4, 9]). The notion of interior operator is introduced by Vorster [12] in an arbitrary category, and it is used in [5] to study the notions of connectedness and disconnectedness in topology. Interior algebras are a certain type of algebraic structure that encodes the idea of the topological interior of a set, and are a generalization of topological spaces defined by means of topological interior operators. Rachůnek and Svoboda [8] studied interior operators on bounded residuated lattices, and Svrcek [11] studied multiplicative interior operators on GMV-algebras. Lee et al. [7] applied the interior operator theory to GE-algebras, and they introduced the concepts of (commutative, transitive, left exchangeable, belligerent, antisymmetric) interior GE-algebras and bordered interior GE-algebras, and investigated their relations and properties. Later, Bandaru et al. [10] introduced the notions of an interior GE-filter, a weak interior GE-filter and a belligerent interior GE-filter and investigated their relations and properties. We are well aware that every belligerent interior GE-filter is an interior GE-filter, and that the reverse is generally not valid by providing examples (see [10]).

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This paper is carried out with the following intentions:
- To explore properties which belligerent interior GE-filter can unfold.
- To find conditions under which interior GE-filter can be belligerent interior GE-filter.
- To form the characterization of belligerent interior GE-filter as a result.

2. Preliminaries

Definition 2.1 ([1]). By a *GE-algebra* we mean a non-empty set $\mathcal{L}$ with a constant 1 and a binary operation $*$ satisfying the following axioms:
(GE1) $x * x = 1$,
(GE2) $1 * x = x$,
(GE3) $x * (y * z) = x * (y * (x * z))$
for all $x, y, z \in \mathcal{L}$.

In a GE-algebra $\mathcal{L}$, a binary relation "$\leq$" is defined by

$$(\forall x, y \in \mathcal{L})\,(x \leq y \iff x * y = 1)\,. \tag{2.1}$$

Definition 2.2 ([1,2,4]). A GE-algebra $\mathcal{L}$ is said to be

- *transitive* if it satisfies:

$$(\forall x, y, z \in \mathcal{L})\,(x * y \leq (z * x) * (z * y))\,. \tag{2.2}$$

- *left exchangeable* if it satisfies:

$$(\forall x, y, z \in \mathcal{L})\,(x * (y * z) = y * (x * z))\,. \tag{2.3}$$

- *antisymmetric* if the binary relation "$\leq$" is antisymmetric.

Proposition 2.1 ([1]). *Every GE-algebra $\mathcal{L}$ satisfies the following items.*

$$(\forall x \in \mathcal{L})\,(x * 1 = 1)\,. \tag{2.4}$$

$$(\forall x, y \in \mathcal{L})\,(x * (x * y) = x * y)\,. \tag{2.5}$$

$$(\forall x, y \in \mathcal{L})\,(x \leq y * x)\,. \tag{2.6}$$

$$(\forall x, y, z \in \mathcal{L})\,(x * (y * z) \leq y * (x * z))\,. \tag{2.7}$$

$$(\forall x \in \mathcal{L})\,(1 \leq x \implies x = 1)\,. \tag{2.8}$$

$$(\forall x, y \in \mathcal{L})\,(x \leq (y * x) * x)\,. \tag{2.9}$$

$$(\forall x, y \in \mathcal{L})\,(x \leq (x * y) * y)\,. \tag{2.10}$$

$$(\forall x, y, z \in \mathcal{L})\,(x \leq y * z \implies y \leq x * z)\,. \tag{2.11}$$

If $\mathcal{L}$ is transitive, then

$$(\forall x, y, z \in \mathcal{L})\,(x \leq y \implies z * x \leq z * y,\; y * z \leq x * z)\,. \tag{2.12}$$

$$(\forall x, y, z \in \mathcal{L})\,(x * y \leq (y * z) * (x * z))\,. \tag{2.13}$$

$$(\forall x, y, z \in \mathcal{L})\,(x \leq y,\; y \leq z \implies x \leq z)\,. \tag{2.14}$$

Lemma 2.3 ([1]). *In a GE-algebra $\mathcal{L}$,* (2.13) *is equivalent the following argument.*

$$(\forall x, y, z \in \mathcal{L})\,(x * y \leq (z * x) * (z * y))\,. \tag{2.15}$$

Definition 2.4 ([1]). A subset $\mathcal{G}$ of a GE-algebra $\mathcal{L}$ is called a *GE-filter* of $\mathcal{L}$ if it satisfies:

(2.16) $$1 \in G,$$

(2.17) $$(\forall x, y \in \mathcal{L})(x * y \in \mathcal{G},\, x \in \mathcal{G} \implies y \in \mathcal{G}).$$

Lemma 2.5 ([1]). *In a GE-algebra $\mathcal{L}$, every GE-filter $\mathcal{G}$ of $\mathcal{L}$ satisfies:*

(2.18) $$(\forall x, y \in \mathcal{L})\,(x \leq jy,\, x \in \mathcal{G} \implies y \in \mathcal{G})\,.$$

Definition 2.6 ([2]). A subset $\mathcal{G}$ of a GE-algebra $\mathcal{L}$ is called a *belligerent GE-filter* of $\mathcal{L}$ if it satisfies (2.16) and

(2.19) $$(\forall x, y, z \in \mathcal{L})(x * (y * z) \in \mathcal{G},\, x * y \in \mathcal{G} \implies x * z \in \mathcal{G}).$$

Definition 2.7 ([7]). By an *interior GE-algebra* we mean a pair $(\mathcal{L}, \beta)$ in which $\mathcal{L}$ is a GE-algebra and $\beta : \mathcal{L} \longrightarrow \mathcal{L}$ is a mapping such that

(2.20) $$(\forall x \in \mathcal{L})(x \leq \beta(x)),$$

(2.21) $$(\forall x \in \mathcal{L})((\beta \circ \beta)(x) = \beta(x)),$$

(2.22) $$(\forall x, y \in \mathcal{L})(x \leq y \implies \beta(x) \leq \beta(y)).$$

Definition 2.8 ([10]). Let $(\mathcal{L}, \beta)$ be an interior GE-algebra. A GE-filter $\mathcal{G}$ of $\mathcal{L}$ is said to be *interior* if it satisfies:

(2.23) $$(\forall x \in \mathcal{L})(\beta(x) \in \mathcal{G} \implies x \in \mathcal{G}).$$

3. Belligerent interior GE-filters

In what follows, let $(\mathcal{L}, \beta)$ denote an interior GE-algebra unless otherwise specified.

Definition 3.1. An interior GE-algebra $(\mathcal{L}, \beta)$ is said to be *transitive* (resp., *left exchangeable* and *antisymmetric*) if $\mathcal{L}$ is a transitive (resp., left exchangeable and *antisymmetric*) GE-algebra.

Example 3.2. 1. Let $\mathcal{L} = \{1, a, b, c, d\}$ and define binary operation $*$ as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	1
b	1	d	1	1	d
c	1	a	1	1	a
d	1	1	c	c	1

If we define a mapping β on $\mathcal{L}$ as follows:

$$\beta : \mathcal{L}, \longrightarrow, \mathcal{L},\, x \longmapsto \begin{cases} 1 & \text{if } x = 1, \\ c & \text{if } x \in \{b, c\}, \\ d & \text{if } x \in \{a, d\}, \end{cases}$$

then $(\mathcal{L}, \beta)$ is a transitive interior GE-algebra.

2. Let $\mathcal{L} = \{1, a, b, c, d\}$ and define binary operation $*$ as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	d
b	1	a	1	d	d
c	1	a	b	1	1
d	1	a	1	1	1

If we define a mapping β on $\mathcal{L}$ as follows:

$$\beta : \mathcal{L} \longrightarrow \mathcal{L},\, x \longmapsto \begin{cases} 1 & \text{if } x \in \{1, b\}, \\ a & \text{if } x = a, \\ d & \text{if } x \in \{c, d\}, \end{cases}$$

then $(\mathcal{L}, \beta)$ is a left exchangeable interior GE-algebra.

3. Let $\mathcal{L} = \{1, a, b, c, d\}$ and define binary operation $*$ as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	c	1	c	1
c	1	b	b	1	d
d	1	a	b	c	1

If we define a mapping β on $\mathcal{L}$ as follows:

$$\beta : \mathcal{L} \longrightarrow \mathcal{L},\, x \longmapsto \begin{cases} 1 & \text{if } x \in \{1, c\} \\ b & \text{if } x \in \{a, b\}, \\ d & \text{if } x = d, \end{cases}$$

then $(\mathcal{L}, \beta)$ is an antisymmetric interior GE-algebra.

Definition 3.3 ([10]). A subset $\mathcal{G}$ of $\mathcal{L}$ in $(\mathcal{L}, \beta)$ is called a *belligerent interior GE-filter* in $(\mathcal{L}, \beta)$ if $\mathcal{G}$ is a belligerent GE-filter of $\mathcal{L}$ which satisfies the condition (2.23).

Lemma 3.4 ([10]). *Every belligerent interior GE-filter is an interior GE-filter in* $(\mathcal{L}, \beta)$.

Proposition 3.1. *Every interior GE-filter* $\mathcal{G}$ *in* $(\mathcal{L}, \beta)$ *satisfies:*

$$(\forall x, y \in \mathcal{L})(\beta(x * (x * y)) \in \mathcal{G} \implies x * y \in \mathcal{G}). \tag{3.1}$$

Proof. It is straightforward by using (2.5) and (2.23). □

Corollary 3.5. *Every belligerent interior GE-filter* $\mathcal{G}$ *in* $(\mathcal{L}, \beta)$ *satisfies* (3.1).

Corollary 3.6. *Every belligerent interior GE-filter* $\mathcal{G}$ *in* $(\mathcal{L}, \beta)$ *satisfies:*

$$(\forall x, y \in \mathcal{L})(x * (x * y) \in \mathcal{G} \implies x * y \in \mathcal{G}). \tag{3.2}$$

Proof. Let $x, y \in \mathcal{L}$ be such that $x * (x * y) \in \mathcal{G}$. Since $x * (x * y) \leq \beta(x * (x * y))$ by (2.20), it follows from Lemma 2.5 that $\beta(x * (x * y)) \in \mathcal{G}$. Hence $x * y \in \mathcal{G}$ by Corollary 3.5. □

Corollary 3.7. *Every belligerent interior GE-filter* $\mathcal{G}$ *in* $(\mathcal{L}, \beta)$ *satisfies:*

$$(\forall x, y \in \mathcal{L})(x * (x * y) \in G \implies \beta(x * y) \in \mathcal{G}).$$

Proof. It is straightforward by the combination of Lemma 2.5, (2.20) and Corollary 3.6. □

Corollary 3.8. *Every belligerent interior GE-filter* $\mathcal{G}$ *in* $(\mathcal{L}, \beta)$ *satisfies:*

$$(\forall x, y \in \mathcal{L})(x * (x * y) \in \mathcal{G} \implies \beta(x * y) \in \mathcal{G}).$$

Proof. It is straightforward by the combination of Lemma 2.5, (2.20) and Corollary 3.6. □

In $(\mathcal{L}, \beta)$, we consider the following argument.

$$(\forall x, y, z \in \mathcal{L})(\beta(x * (y * z)) \in \mathcal{G} \implies (x * y) * (x * z) \in \mathcal{G}). \tag{3.3}$$

The following example shows that any belligerent interior GE-filter $\mathcal{G}$ in $(\mathcal{L}, \beta)$ does not satisfy the argument (3.3).

Example 3.9. 1. Let $\mathcal{L} = \{1, a, b, c, d\}$ and define binary operation $*$ as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	d	c	d
b	1	1	1	1	1
c	1	1	d	1	d
d	1	a	a	c	1

If we define a mapping β on $\mathcal{L}$ as follows:

$$\beta : \mathcal{L} \longrightarrow \mathcal{L},\ x \longmapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, c\}, \\ b & \text{if } x = b, \\ d & \text{if } x = d, \end{cases}$$

then $(\mathcal{L}, \beta)$ is a left exchangeable interior GE-algebra. But it is not transitive since

$$(b * c) * (d * b) * (d * c)) = 1 * (a * c) = 1 * c = c \neq 1.$$

Let $\mathcal{G} = \{1\}$. Then we can observe that $\mathcal{G}$ is a belligerent interior GE-filter in $(\mathcal{L}, \beta)$. But it does not satisfy (3.3) since $\beta(a * (b * c)) = \beta(a * 1) = \beta(1) = 1 \in \mathcal{G}$ but $(a * b) * (a * c) = d * c = c \notin \mathcal{G}$.

2. Let $\mathcal{L} = \{1, a, b, c, d, e\}$ and define binary operation $*$ as follows:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	d	c	d	d
b	1	1	1	1	1	1
c	1	1	e	1	e	e
d	1	a	a	c	1	1
e	1	a	a	c	1	1

If we define a mapping β on $\mathcal{L}$ as follows:

$$\beta : \mathcal{L} \longrightarrow \mathcal{L},\ x \longmapsto \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x = b, \\ a & \text{if } x \in \{a, c\}, \\ d & \text{if } x \in \{d, e\}, \end{cases}$$

then $(\mathcal{L}, \beta)$ is an interior GE-algebra. But it is neither transitive nor left exchangeable since

$$a * (c * b) = a * e = d \neq e = c * d = c * (a * b)$$

and

$$(b * c) * ((d * b) * (d * c)) = 1 * (a * c) = 1 * c = c \neq 1.$$

Let $\mathcal{G} = \{1\}$. Then we can observe that $\mathcal{G}$ is a belligerent interior GE-filter in $(\mathcal{L}, \beta)$. But it does not satisfy (3.3) since $\beta(a * (b * c)) = \beta(a * 1) = \beta(1) = 1 \in \mathcal{G}$ but $(a * b) * (a * c) = d * c = c \notin \mathcal{G}$.

Based on the example above, we present conditions under which belligerent interior GE-filter can satisfy the argument (3.3).

Proposition 3.2. *If $(\mathcal{L}, \beta)$ is transitive, then every belligerent interior GE-filter $\mathcal{G}$ satisfies the argument* (3.3)*.*

Proof. Let $\mathcal{G}$ be a belligerent interior GE-filter in a transitive interior GE-algebra $(\mathcal{L}, \beta)$. Then $\mathcal{G}$ is an interior GE-filter in $(\mathcal{L}, \beta)$ by Lemma 3.4. Let $x, y, z \in \mathcal{L}$ be such that $\beta(x * (y * z)) \in \mathcal{G}$. Then $x * (y * z) \in \mathcal{G}$ by (2.23). Using (2.2), (2.5) and (2.7), we have

$$x * (y * z) \leq y * (x * z) \leq (x * y) * (x * (x * z)) = (x * y) * (x * z).$$

Hence $x * (y * z) \leq (x * y) * (x * z)$ by (2.14). It follows from Lemma 2.5 that $(x * y) * (x * z) \in \mathcal{G}$. □

Corollary 3.10. *Every belligerent interior GE-filter $\mathcal{G}$ in a transitive interior GE-algebra $(\mathcal{L}, \beta)$ satisfies:*

(3.4) $$(\forall x, y, z \in \mathcal{L})(x * (y * z) \in \mathcal{G} \implies (x * y) * (x * z) \in \mathcal{G}).$$

Corollary 3.11. *Every belligerent interior GE-filter $\mathcal{G}$ in a transitive interior GE-algebra $(\mathcal{L}, \beta)$ satisfies:*

(3.5) $$(\forall x, y, z \in \mathcal{L})(x * (y * z) \in \mathcal{G} \implies \beta((x * y) * (x * z)) \in \mathcal{G}).$$

The following example shows that any interior GE-filter $\mathcal{G}$ in $(\mathcal{L}, \beta)$ does not satisfy the condition (3.3).

Example 3.12. Let $\mathcal{L} = \{1, a, b, c, d\}$ and define binary operation $*$ as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	d	c	d
b	1	a	1	1	1
c	1	a	1	1	1
d	1	a	1	c	1

Then $(\mathcal{L}, \beta)$ is an interior GE-algebra where β is given as follows:

$$\beta : \mathcal{L} \longrightarrow \mathcal{L}\, x \longmapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x = a, \\ b & \text{if } x \in \{b, c, d\}, \end{cases}$$

We can observe that $G := \{1, a\}$ is an interior GE-filter in $(\mathcal{L}, \beta)$. But $\mathcal{G}$ does not satisfy (3.3) since $\beta(a * (b * c)) = \beta(a * 1) = \beta(1) = 1 \in \mathcal{G}$ but $(a * b) * (a * c) = d * c = c \notin \mathcal{G}$.

We explore the conditions under which GE-filter can be belligerent GE-filter.

Theorem 3.13. *Let $\mathcal{G}$ be an interior GE-filter in $(\mathcal{L}, \beta)$. If it satisfies the condition* (3.3)*, then $\mathcal{G}$ is a belligerent interior GE-filter in $(\mathcal{L}, \beta)$.*

Proof. Let $\mathcal{G}$ be an interior GE-filter in $(\mathcal{L}, \beta)$ which satisfies the condition (3.3). It is clear that $\mathcal{G}$ satisfies (2.23). Let $x * (y * z) \in \mathcal{G}$ and $x * y \in \mathcal{G}$. Since $x * (y * z) \leq \beta(x * (y * z))$ by (2.20), it follows from Lemma 2.5 that $\beta(x * (y * z)) \in \mathcal{G}$. Hence $(x * y) * (x * z) \in \mathcal{G}$ by (3.3), and so $x * z \in \mathcal{G}$ since $\mathcal{G}$ is a GE-filter of $\mathcal{L}$. This shows that $\mathcal{G}$ is a belligerent GE-filter of $\mathcal{L}$. Therefore $\mathcal{G}$ is a belligerent interior GE-filter in $(\mathcal{L}, \beta)$. □

Proposition 3.3. *If $(\mathcal{L}, \beta)$ is transitive and left exchangeable, then every interior GE-filter $\mathcal{G}$ in $(\mathcal{L}, \beta)$ satisfies the condition* (3.3)*.*

Proof. Let $x, y, z \in \mathcal{L}$ be such that $\beta(x * (y * z)) \in \mathcal{G}$. Then $x * (y * z) \in \mathcal{G}$ by (2.23). Using (2.2) and (2.12), we have $x * (y * z) \leq x * ((x * y) * (x * z))$. Since $\mathcal{G}$ is a GE-filter of $\mathcal{L}$, it follows from (2.3) and Lemma 2.5 that

$$x * (x * ((x * y) * z)) = x * ((x * y) * (x * z)) \in \mathcal{G}.$$

This induces $\beta(x * (x * ((x * y) * z))) \in \mathcal{G}$ by combining (2.20) and Lemma 2.5. Using (2.3) and (3.1), we get $(x * y) * (x * z) = x * ((x * y) * z) \in \mathcal{G}$. □

Corollary 3.14. *If* $(\mathcal{L}, \beta)$ *is transitive and antisymmetric, then every interior GE-filter* $\mathcal{G}$ *in* $(\mathcal{L}, \beta)$ *satisfies the condition* (3.3).

Combining Theorem 3.13 and Proposition 3.3, we obtain the following theorem.

Theorem 3.15. *In a transitive and left exchangeable interior GE-algebra* $(\mathcal{L}, \beta)$*, every interior GE-filter* $\mathcal{G}$ *is a belligerent interior GE-filter in* $(\mathcal{L}, \beta)$*.*

Corollary 3.16. *In a transitive and antisymmetric interior GE-algebra* $(\mathcal{L}, \beta)$*, every interior GE-filter* $\mathcal{G}$ *is a belligerent interior GE-filter in* $(\mathcal{L}, \beta)$*.*

Proposition 3.4. *If* $(\mathcal{L}, \beta)$ *is transitive and left exchangeable, then every belligerent interior GE-filter* $\mathcal{G}$ *satisfies:*

$$(\forall x, y, z \in \mathcal{L})(\beta(x) \in \mathcal{G},\ \beta(x * (y * (y * z))) \in \mathcal{G} \implies y * z \in \mathcal{G}). \tag{3.6}$$

Proof. Let $\mathcal{G}$ be a belligerent interior GE-filter in $(\mathcal{L}, \beta)$. Then it is an interior GE-filter in $(\mathcal{L}, \beta)$ by Lemma 3.4. Let $x, y, z \in \mathcal{G}$ be such that $\beta(x) \in \mathcal{G}$ and $\beta(x * (y * (y * z))) \in \mathcal{G}$. Then $x \in \mathcal{G}$ by (2.23) and

$$\beta(y * (y * (x * z))) = \beta(x * (y * (y * z))) \in \mathcal{G}$$

by (2.3), which implies from (GE1), (GE2) and Proposition 3.2 that

$$x * (y * z) = y * (x * z) = 1 * (y * (x * z)) = (y * y) * (y * (x * z)) \in \mathcal{G}.$$

Since $\mathcal{G}$ is a GE-filter of $\mathcal{L}$ in $(\mathcal{L}, \beta)$, we have $y * z \in \mathcal{G}$. □

Corollary 3.17. *If* $(\mathcal{L}, \beta)$ *is transitive and left exchangeable, then every belligerent interior GE-filter* $\mathcal{G}$ *satisfies:*

$$(\forall x, y, z \in \mathcal{L})(\beta(x) \in \mathcal{G},\ x * (y * (y * z)) \in \mathcal{G} \implies y * z \in \mathcal{G}). \tag{3.7}$$

Corollary 3.18. *If* $(\mathcal{L}, \beta)$ *is transitive and left exchangeable, then every belligerent interior GE-filter* $\mathcal{G}$ *satisfies:*

$$(\forall x, y, z \in \mathcal{L})(x \in \mathcal{G},\ \beta(x * (y * (y * z))) \in \mathcal{G} \implies y * z \in \mathcal{G}). \tag{3.8}$$

Corollary 3.19. *If* $(\mathcal{L}, \beta)$ *is transitive and left exchangeable, then every belligerent interior GE-filter* $\mathcal{G}$ *satisfies:*

$$(\forall x, y, z \in \mathcal{L})(\beta(x) \in \mathcal{G},\ \beta(x * (y * (y * z))) \in \mathcal{G} \implies \beta(y * z) \in \mathcal{G}). \tag{3.9}$$

Corollary 3.20. *If* $(\mathcal{L}, \beta)$ *is transitive and antisymmetric, then every belligerent interior GE-filter* $\mathcal{G}$ *satisfies the arguments* (3.6), (3.7), (3.8) *and* (3.9).

We provide conditions for a subset to be a belligerent interior GE-filter.

Theorem 3.21. *Let* $(\mathcal{L}, \beta)$ *be a transitive and left exchangeable interior GE-algebra and let* $\mathcal{G}$ *be a subset of* $\mathcal{L}$ *containing the constant* 1 *in* $(\mathcal{L}, \beta)$ *which satisfies* (2.18). *If it satisfies* (2.23) *and* (3.6), *then* $\mathcal{G}$ *is a belligerent interior GE-filter in* $(\mathcal{L}, \beta)$*.*

Proof. Assume that a subset $\mathcal{G}$ of $\mathcal{L}$ in $(\mathcal{L}, \beta)$ satisfies the conditions (2.18), (2.23) and (3.6). Let $x, y \in \mathcal{L}$ be such that $x * y \in \mathcal{G}$ and $x \in \mathcal{G}$. Then $x * y \leq \beta(x * y)$ and $x \leq \beta(x)$ by (2.20). It follows from (2.18) that $\beta(x * y) \in \mathcal{G}$ and $\beta(x) \in \mathcal{G}$. Using (GE2), we have $\beta(x * (1 * (1 * y))) = \beta(x * y) \in \mathcal{G}$. This induces $y = 1 * y \in \mathcal{G}$ by combining (GE2) and (3.6). Hence $\mathcal{G}$ is an interior GE-filter in $(\mathcal{L}, \beta)$. Let $x, y, z \in \mathcal{L}$ be such that $x * (y * z) \in \mathcal{G}$ and $x * y \in \mathcal{G}$. Since $x * (y * z) \leq \beta(x * (y * z))$ and $x * y \leq \beta(x * y)$ by (2.23), we have $\beta(x * (y * z)) \in \mathcal{G}$ and $\beta(x * y) \in \mathcal{G}$ by (2.18). Since $(\mathcal{L}, \beta)$ is transitive and left exchangeable, we get

$$x * (y * z) = y * (x * z) \leq (x * y) * (x * (x * z)) \leq \beta((x * y) * (x * (x * z)))).$$

It follows from (2.18) that $\beta((x * y) * (x * (x * z)))) \in \mathcal{G}$. We use (3.6) to induce $x * z \in \mathcal{G}$. This shows that $\mathcal{G}$ is a belligerent GE-filter of $\mathcal{L}$, and consequently $\mathcal{G}$ is a belligerent interior GE-filter in $(\mathcal{L}, \beta)$. □

Corollary 3.22. *Let $(\mathcal{L}, \beta)$ be a transitive and antisymmetric interior GE-algebra and let $\mathcal{G}$ be a subset of $\mathcal{L}$ containing the constant 1 in $(\mathcal{L}, \beta)$ which satisfies* (2.18). *If it satisfies* (2.23) *and* (3.6), *then $\mathcal{G}$ is a belligerent interior GE-filter in $(\mathcal{L}, \beta)$.*

We provide examples illustrating Theorem 3.21 as follows.

Example 3.23. Let $\mathcal{L} = \{1, a, b, c, d\}$ and define binary operation $*$ as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	1	1
b	1	d	1	d	d
c	1	1	b	1	1
d	1	1	b	1	1

Then $(\mathcal{L}, \beta)$ is a transitive and left exchangeable interior GE-algebra where β is given as follows:

$$\beta : \mathcal{L} \longrightarrow \mathcal{L}, x \longmapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, c\}, \\ b & \text{if } x = b, \\ d & \text{if } x = d, \end{cases}$$

Let $\mathcal{G} := \{1, b\}$. Then $\mathcal{G}$ satisfies (2.18), (2.23) and (3.6). Also, we can observe that $\mathcal{G}$ is a belligerent interior GE-filter in $(\mathcal{L}, \beta)$.

The following example shows that Theorem 3.21 does not have to be valid for an interior GE-algebra $(\mathcal{L}, \beta)$ which is neither transitive nor left exchangeable.

Example 3.24. Let $\mathcal{L} = \{1, a, b, c, d, e, f, g, h\}$ and define binary operation $*$ as follows:

$*$	1	a	b	c	d	e	f	g	h
1	1	a	b	c	d	e	f	g	h
a	1	1	1	c	d	e	1	g	g
b	1	1	1	c	c	1	1	1	1
c	1	a	a	1	e	e	f	e	e
d	1	1	1	1	1	1	1	1	1
e	1	1	1	1	1	1	1	1	1
f	1	a	a	1	e	e	1	h	h
g	1	1	1	1	1	1	f	1	1
h	1	1	1	1	1	1	1	1	1

Then $(\mathcal{L}, \beta)$ is an interior GE-algebra where β is given as follows:

$$\beta : \mathcal{L} \longrightarrow \mathcal{L},\, x \longmapsto \begin{cases} 1 & \text{if } x \in \{1, c\} \\ a & \text{if } x \in \{a, b, d, e, g, h\}, \\ f & \text{if } x = f. \end{cases}$$

Since $a * (f * g) = a * h = g \neq h = f * g = f * (a * g)$ and

$$(a * f) * ((g * a) * (g * f)) = 1 * (1 * f) = 1 * f = f \neq 1,$$

we know that $(\mathcal{L}, \beta)$ is neither transitive nor left exchangeable. Let $\mathcal{G} := \{1, c\}$. Then $\mathcal{G}$ satisfies (2.18), (2.23) and (3.6). But $\mathcal{G}$ is not a belligerent interior GE-filter in $(\mathcal{L}, \beta)$, since $a*(b*d) = a*c = c \in \mathcal{G}$ and $a*b = 1 \in \mathcal{G}$ but $a * d = d \notin \mathcal{G}$.

By aggregating the above, we obtain the following characterizations:

Theorem 3.25. *Let $(\mathcal{L}, \beta)$ be a transitive and left exchangeable interior GE-algebra. Given a subset $\mathcal{G}$ of $\mathcal{L}$, the following arguments are equivalent.*

(i) *$\mathcal{G}$ is a belligerent interior GE-filter in $(\mathcal{L}, \beta)$.*
(iii) *$\mathcal{G}$ is an interior GE-filter in $(\mathcal{L}, \beta)$ which satisfies* (3.3).
(iv) *$\mathcal{G}$ contains the constant 1 and satisfies* (2.18), (2.23) *and* (3.6).

Corollary 3.26. *Let $(\mathcal{L}, \beta)$ be a transitive and antisymmetric interior GE-algebra. Given a subset $\mathcal{G}$ of $\mathcal{L}$, the following arguments are equivalent.*

(i) *$\mathcal{G}$ is a belligerent interior GE-filter in $(\mathcal{L}, \beta)$.*
(iii) *$\mathcal{G}$ is an interior GE-filter in $(\mathcal{L}, \beta)$ which satisfies* (3.3).
(iv) *$\mathcal{G}$ contains the constant 1 and satisfies* (2.18), (2.23) *and* (3.6).

4. Conclusion

In 2021, Bandaru et. al. have studied GE-algebras which are a generalization of Hilbert algebras, and Lee et al. have used interior operator to introduce interior GE-algebra. Song et al. have introduced the "(belligerent) interior GE-filter" by studying the filter theory in the interior GE-algebras. Based on these study, in this manuscript, we have explored the various properties that belligerent interior GE-filter could perform. Using these properties, we have found conditions under which interior GE-filter could convert to belligerent interior GE-filter, and as a result we have established characterization of belligerent interior GE-filter.

Using the results or ideas in this paper, we will first study different forms of filter theory in GE-algebra, and further study interior substructures in another algebraic structure in the same context as GE-algebra.

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