ISSN: 2832-4293

Pan-American Journal of Mathematics 2 (2023), 9
https://doi.org/10.28919/cpr-pajm/2-9

THE SOLUTION OF A SYSTEM OF HIGHER-ORDER DIFFERENCE EQUATIONS IN TERMS OF BALANCING NUMBERS

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ABSTRACT. In this paper, we are interested in the closed-form solution of the following system of nonlinear difference equations of higher order,

$$u_{n+1} = \frac{1}{34 - v_{n-m}}, v_{n+1} = \frac{1}{34 - u_{n-m}}, n, m \in \mathbb{N}_0,$$

and the initial values u_{-j} and v_{-j}, $j \in \{0, 1, ..., m\}$ are real numbers do not equal 34. We show that the solutions of this system are associated with Balancing numbers. As consequence, these solutions are also associated with Pell numbers, Pell-Lucas numbers, and Lucas-Balancing numbers. It is shown that the global stability of positive solutions of this system holds. Our results are illustrated via numerical examples.

1. INTRODUCTION

Many researchers have interested in different types of difference equations, and we mention but are not limited to the homogeneous linear difference equation of the 2nd-order,

$$u_{n+1} = \alpha u_n + \beta u_{n-1}, n \geq 1,$$

where $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$ such that $\beta \neq 0$, in particular, we give information about Balancing (resp. Pell) sequence that establishes a significant part of our study, defined as follows

$$B_{n+1} = 6B_n - B_{n-1}, \text{ (resp. } P_{n+1} = 2P_n + P_{n-1}), n \geq 1,$$

with initial conditions $B_0 = P_0 = 0$ and $B_1 = P_1 = 1$. The following Binet formula of the Balancing (resp. Pell) numbers gives,

$$B_n = \frac{a^n - b^n}{a - b}, \; P_n = 2B_{2n}, \text{ (see., [11])}$$

where $a = 3 + 2\sqrt{2}$ and $b = 3 - 2\sqrt{2}$. The search for solutions in the closed form of difference equations or systems has attracted the attention of many mathematicians (see., [1]- [26]). So, in this paper, we seek to provide a class of system of nonlinear difference equations which can be solved in explicit form, but the solutions are expressed by Balancing numbers, is the following system of difference equations,

$$u_{n+1} = \frac{1}{34 - v_{n-m}}, v_{n+1} = \frac{1}{34 - u_{n-m}}, n, m \in \mathbb{N}_0, \tag{1.0}$$

and the initial values $u_{-m}, ..., u_0, v_{-m}, ..., v_0$ are real numbers do not equal 34.

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Submitted on April 28, 2023.
2020 *Mathematics Subject Classification.* Primary 39A10; Secondary 40A05.
Key words and phrases. Stability, Balancing numbers, Pell numbers, Lucas numbers, Binet formula, System of difference equations.
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2. Main results

To solve system (2.3) we require to utilize the following lemmas.

Lemma 2.1. *Consider the homogeneous linear difference equation with constant coefficients*

$$\omega_{n+1} - 34\omega_n + \omega_{n-1} = 0, n \geq 0, \tag{2.0}$$

with initial conditions $\omega_0, \omega_{-1} \in \mathbb{R}$. *Then,*

$$\forall n \geq 0,\ \omega_n = \frac{\omega_0}{4} B_{2n+2} - \frac{\omega_{-1}}{4} B_{2n} = \frac{\omega_0}{8} P_{4(n+1)} - \frac{\omega_{-1}}{8} P_{4n},$$

where $(B_n, n \geq 0)$ *is the Balancing sequence and* $(P_n, n \geq 0)$ *is the Pell sequence.*

Proof. Difference equation (2.0) is ordinarily solved by using the following characteristic polynomial, $\lambda^2 - 34\lambda + 1 = 0$, roots of this equation are

$$\lambda_1 = 17 + 12\sqrt{2} = a^2, \lambda_2 = 17 - 12\sqrt{2} = b^2.$$

These roots are linked to the roots of the Balancing number sequence. Then the closed form of the general solution of the equation (2.0) is

$$\forall n \geq -1,\ \omega_n = c_1 a^{2n} + c_2 b^{2n},$$

where ω_0, ω_{-1} are initial values such that

$$\begin{cases} \omega_0 = c_1 + c_2 \\ \omega_{-1} = \dfrac{c_1}{a^2} + \dfrac{c_2}{b^2} \end{cases},$$

and we have

$$c_1 = \frac{a^2\omega_0 - \omega_{-1}}{24\sqrt{2}}, c_2 = \frac{-b^2\omega_0 + \omega_{-1}}{24\sqrt{2}},$$

after some calculations, we get

$$\begin{aligned} \omega_n &= \frac{a^2\omega_0 - \omega_{-1}}{24\sqrt{2}} a^{2n} + \frac{-b^2\omega_0 + \omega_{-1}}{24\sqrt{2}} b^{2n} \\ &= \frac{\omega_0}{4}\left(\frac{a^{2(n+1)} - b^{2(n+1)}}{a-b}\right) - \frac{\omega_{-1}}{4}\left(\frac{a^{2n} - b^{2n}}{a-b}\right). \end{aligned}$$

The lemma is proved. □

Lemma 2.2. *Consider the homogeneous linear difference equation with constant coefficients*

$$\theta_{n+1} + 34\theta_n + \theta_{n-1} = 0, n \geq 0, \tag{2.1}$$

with initial conditions $\theta_0, \theta_{-1} \in \mathbb{R}$. *Then,*

$$\forall n \geq 0,\ \theta_n = (-1)^n \left(\frac{\theta_0}{4} B_{2n+2} + \frac{\theta_{-1}}{4} B_{2n}\right) = (-1)^n \left(\frac{\theta_0}{8} P_{4(n+1)} + \frac{\theta_{-1}}{8} P_{4n}\right),$$

where $(B_n, n \geq 0)$ *is the Balancing sequence and* $(P_n, n \geq 0)$ *is the Pell sequence.*

Proof. The difference equation (2.1) is ordinarily solved by using the following characteristic polynomial, $\lambda^2 + 34\lambda + 1 = 0$, roots of this equation are

$$\lambda_1 = -\left(17 + 12\sqrt{2}\right) = -a^2, \lambda_2 = -\left(17 - 12\sqrt{2}\right) = -b^2.$$

These roots are linked to the roots of the Balancing number sequence. Then the closed form of the general solution of the equation (2.1) is

$$\forall n \geq -1,\ \theta_n = (-1)^n \left(\widetilde{c}_1 a^{2n} + \widetilde{c}_2 b^{2n}\right),$$

where θ_0, θ_{-1} are initial values such that

$$\begin{cases} \theta_0 = \widetilde{c}_1 + \widetilde{c}_2 \\ \theta_{-1} = -\dfrac{\widetilde{c}_1}{a^2} - \dfrac{\widetilde{c}_2}{b^2} \end{cases},$$

and we have

$$\widetilde{c}_1 = \frac{a^2\theta_0 + \theta_{-1}}{24\sqrt{2}}, \widetilde{c}_2 = \frac{-b^2\theta_0 - \theta_{-1}}{24\sqrt{2}},$$

after some calculations, we get

$$\begin{aligned} \theta_n &= (-1)^n \left(\frac{a^2\theta_0 + \theta_{-1}}{24\sqrt{2}} a^{2n} + \frac{-b^2\theta_0 - \theta_{-1}}{24\sqrt{2}} b^{2n} \right) \\ &= (-1)^n \left(\frac{\theta_0}{4} \left(\frac{a^{2(n+1)} - b^{2(n+1)}}{a-b} \right) + \frac{\theta_{-1}}{4} \left(\frac{a^{2n} - b^{2n}}{a-b} \right) \right). \end{aligned}$$

The lemma is proved. □

Lemma 2.3. *Consider the following system of rational difference equations*

$$\begin{cases} x_{n+1} = 34y_n - x_{n-1} \\ y_{n+1} = 34x_n - y_{n-1} \end{cases}, n \geq 0, \tag{2.3}$$

with initial conditions $x_0, x_1, y_0, y_1 \in \mathbb{R}$. *Then,*

$$\begin{aligned} x_{2n} &= \tfrac{x_0}{4} B_{4n+2} - \tfrac{y_{-1}}{4} B_{4n} = \tfrac{x_0}{8} P_{8n+4} - \tfrac{y_{-1}}{8} P_{8n}, \\ y_{2n} &= \tfrac{y_0}{4} B_{4n+2} - \tfrac{x_{-1}}{4} B_{4n} = \tfrac{y_0}{4} P_{8n+4} - \tfrac{x_{-1}}{4} P_{8n}, \\ x_{2n+1} &= \tfrac{y_0}{4} B_{4n+4} - \tfrac{x_{-1}}{4} B_{4n+2} = \tfrac{y_0}{4} P_{8n+8} - \tfrac{x_{-1}}{4} P_{8n+4}, \\ y_{2n+1} &= \tfrac{x_0}{4} B_{4n+4} - \tfrac{y_{-1}}{4} B_{4n+2} = \tfrac{x_0}{4} P_{8n+8} - \tfrac{y_{-1}}{4} P_{8n+4}. \end{aligned}$$

Proof. From system (2.3), we get the following system

$$\begin{cases} x_{n+1} + y_{n+1} = 34\,(x_n + y_n) - (x_{n-1} + y_{n-1}) \\ x_{n+1} - y_{n+1} = -34\,(x_n - y_n) - (x_{n-1} - y_{n-1}) \end{cases}, n \geq 0, \tag{2.4}$$

Using the change of variables $\omega_n = x_n + y_n$ and $\theta_n = x_n - y_n$, we can write (2.4) as

$$\begin{cases} \omega_{n+1} = 34\omega_n - \omega_{n-1} \\ \theta_{n+1} = -34\theta_n - \theta_{n-1} \end{cases}, n \geq 0,$$

by Lemmas $2.1 - 2.2$, we have

$$\begin{aligned} &\forall n \geq 0,\ \omega_n = \frac{\omega_0}{4} B_{2n+2} - \frac{\omega_{-1}}{4} B_{2n} = \frac{\omega_0}{8} P_{4(n+1)} - \frac{\omega_{-1}}{8} P_{4n}, \\ &\forall n \geq 0,\ \theta_n = (-1)^n \left(\frac{\theta_0}{4} B_{2n+2} + \frac{\theta_{-1}}{4} B_{2n} \right) = (-1)^n \left(\frac{\theta_0}{8} P_{4(n+1)} + \frac{\theta_{-1}}{8} P_{4n} \right), \end{aligned}$$

hence, the closed form of general solution of the system (2.3) is $(x_n, y_n) = \left(\frac{\omega_n + \theta_n}{2}, \frac{\omega_n - \theta_n}{2} \right)$, $n \geq 0$. The lemma is proved. □

2.1. **On the system** (2.3)**.** In this subsection, we consider the following system of difference equations of 1st-order,

$$u_{n+1} = \frac{1}{34 - v_n}, v_{n+1} = \frac{1}{34 - u_n}, n \in \mathbb{N}_0. \tag{2.3}$$

To find the closed form of the solutions of the system (2.3) we consider the following change variables

$$u_n = \frac{y_{n-1}}{x_n}, v_n = \frac{x_{n-1}}{y_n},$$

then the system (2.3) becomes

$$\begin{cases} x_{n+1} = 34y_n - x_{n-1} \\ y_{n+1} = -34x_n - y_{n-1} \end{cases}, n \geq 0.$$

By Lemma 2.3, the closed form of general solution of the equation (2.3) is easily obtained, in the following Theorem

Theorem 2.1. *Let $\{u_n, v_n, n \geq 0\}$ be a solution of equation (2.3). Then,*

$$u_{2n} = \frac{B_{4n+2} - v_0 B_{4n}}{B_{4n+4} - v_0 B_{4n+2}} = \frac{P_{8n+4} - v_0 P_{8n}}{P_{8n+8} - v_0 P_{8n+4}},$$
$$u_{2n+1} = \frac{B_{4n+4} - u_0 B_{4n+2}}{B_{4n+6} - u_0 B_{4n+4}} = \frac{P_{8n+8} - u_0 P_{8n+4}}{P_{8n+12} - u_0 P_{8n+8}},$$
$$v_{2n} = \frac{B_{4n+2} - u_0 B_{4n}}{B_{4n+4} - u_0 B_{4n+2}} = \frac{P_{8n+4} - u_0 P_{8n}}{P_{8n+8} - u_0 P_{8n+4}},$$
$$v_{2n+1} = \frac{B_{4n+4} - v_0 B_{4n+2}}{B_{4n+6} - v_0 B_{4n+4}} = \frac{P_{8n+8} - v_0 P_{8n+4}}{P_{8n+12} - v_0 P_{8n+8}},$$

where $(B_n, n \geq 0)$ is the Balancing sequence and $(P_n, n \geq 0)$ is the Pell sequence.

Proof. Straightforward and hence omitted. □

2.2. **On the system** (1.0)**.** In this paper, we study the System (1.0), which is an extension of System (2.3). Therefore, the System (1.0) can be written as follows

$$u_{(m+1)(n+1)-t} = \frac{1}{34 - v_{(m+1)n-t}}, v_{(m+1)(n+1)-t} = \frac{1}{34 - u_{(m+1)n-t}},$$

for $t \in \{0, 1, ..., m\}$ and $n \in \mathbb{N}$. Now, using the following notation,

$$u_{n,t} = u_{(m+1)n-t}, v_{n,t} = v_{(m+1)n-t}, t \in \{0, 1, ..., m\},$$

we can get $(m+1)$ −systems similar to System (2.3),

$$u_{n+1,t} = \frac{1}{34 - v_{n,t}}, v_{n+1,t} = \frac{1}{34 - v_{n,t}}, n \in \mathbb{N}_0,$$

for $t \in \{0, 1, ..., m\}$. Through the above discussion, we can introduce the following Theorem

Theorem 2.2. *Let $\{u_n, v_n, n \geq -m\}$ be a solution of equation (1.0). Then, for $t \in \{0, 1, ..., m\}$,*

$$u_{2(m+1)n-t} = \frac{B_{4n+2} - v_{-t} B_{4n}}{B_{4n+4} - v_{-t} B_{4n+2}} = \frac{P_{8n+4} - v_{-t} P_{8n}}{P_{8n+8} - v_{-t} P_{8n+4}},$$
$$u_{(m+1)(2n+1)-t} = \frac{B_{4n+4} - u_{-t} B_{4n+2}}{B_{4n+6} - u_{-t} B_{4n+4}} = \frac{P_{8n+8} - u_{-t} P_{8n+4}}{P_{8n+12} - u_{-t} P_{8n+8}},$$
$$v_{2(m+1)n-t} = \frac{B_{4n+2} - u_{-t} B_{4n}}{B_{4n+4} - u_{-t} B_{4n+2}} = \frac{P_{8n+4} - u_{-t} P_{8n}}{P_{8n+8} - u_{-t} P_{8n+4}},$$
$$v_{(m+1)(2n+1)-t} = \frac{B_{4n+4} - v_{-t} B_{4n+2}}{B_{4n+6} - v_{-t} B_{4n+4}} = \frac{P_{8n+8} - v_{-t} P_{8n+4}}{P_{8n+12} - v_{-t} P_{8n+8}},$$

where $(B_n, n \geq 0)$ is the Balancing sequence and $(P_n, n \geq 0)$ is the Pell sequence.

Proof. The proof of Theorem 2.2 is based on Theorem 2.1 for $(m+1)$ −systems (1.0). □

Corollary 2.1. *Let $\{u_n, v_n, n \geq -m\}$ be a solution of equation (1.0). Then, for $t \in \{0, 1, ..., m\}$,*

$$u_{2(m+1)n-t} = \frac{P_{4n+2} Q_{4n+2} - v_{-t} P_{4n} Q_{4n}}{P_{4n+4} Q_{4n+4} - v_{-t} P_{4n+2} Q_{4n+2}},$$
$$u_{(m+1)(2n+1)-t} = \frac{P_{4n+4} Q_{4n+4} - u_{-t} P_{4n+2} Q_{4n+2}}{P_{4n+6} Q_{4n+6} - u_{-t} P_{4n+4} Q_{4n+4}},$$

$$v_{2(m+1)n-t} = \frac{P_{4n+2}Q_{4n+2} - u_{-t}P_{4n}Q_{4n}}{P_{4n+4}Q_{4n+4} - u_{-t}P_{4n+2}Q_{4n+2}},$$
$$v_{(m+1)(2n+1)-t} = \frac{P_{4n+4}Q_{4n+4} - v_{-t}P_{4n+2}Q_{4n+2}}{P_{4n+6}Q_{4n+6} - v_{-t}P_{4n+4}Q_{4n+4}},$$

where $(P_n, n \geq 0)$ *is the Pell sequence and* $(Q_n, n \geq 0)$ *is the Pell-Lucas sequence.*

Proof. We see that it suffices to remark

$$B_n = \frac{a_1^n - b_1^n}{a_1 - b_1}\frac{a_1^n + b_1^n}{a_1 + b_1} = \frac{1}{2}P_nQ_n \text{ (see., [11]).}$$

□

Corollary 2.2. *Let* $\{u_n, v_n, n \geq -m\}$ *be a solution of equation* (1.0)*. Then, for* $t \in \{0, 1, ..., m\}$,

$$u_{2(m+1)n-t} = \frac{C_{4n+3} - C_{4n+1} - v_{-t}\left(C_{4n+1} - C_{4n-1}\right)}{C_{4n+5} - C_{4n+3} - v_{-t}\left(C_{4n+3} - C_{4n+1}\right)B_{4n+2}},$$
$$u_{(m+1)(2n+1)-t} = \frac{C_{4n+5} - C_{4n+3} - u_{-t}\left(C_{4n+3} - C_{4n+1}\right)}{C_{4n+7} - C_{4n+5} - u_{-t}\left(C_{4n+5} - C_{4n+3}\right)},$$

$$v_{2(m+1)n-t} = \frac{C_{4n+3} - C_{4n+1} - u_{-t}\left(C_{4n+1} - C_{4n-1}\right)}{C_{4n+5} - C_{4n+3} - u_{-t}\left(C_{4n+3} - C_{4n+1}\right)B_{4n+2}},$$
$$v_{(m+1)(2n+1)-t} = \frac{C_{4n+5} - C_{4n+3} - v_{-t}\left(C_{4n+3} - C_{4n+1}\right)}{C_{4n+7} - C_{4n+5} - v_{-t}\left(C_{4n+5} - C_{4n+3}\right)},$$

where $(C_n, n \geq 0)$ *is the Lucas-Balancing sequence.*

Proof. We see that it suffices to remark $16B_n = C_{n+1} - C_{n-1}$ (see., [11]). □

Remark 2.1. *There are many systems whose solutions can be expressed by Pell, Balancing and Lucas-Balancing numbers, which are*

$$u_{n+1} = \frac{1}{\delta_k - v_{n-m}}, v_{n+1} = \frac{1}{\delta_k - u_{n-m}}, n, m \in \mathbb{N}_0, k \geq 1,$$

where $\delta_k = a^k + b^k \in \{6; 34; 198; 1154; 6726; ...\}, k \geq 1$*. Using the results of Theorem* 2.2*, we get*

$$u_{2(m+1)n-t} = \frac{B_{(2n+1)k} - v_{-t}B_{2kn}}{B_{2k(n+1)} - v_{-t}B_{k(2n+1)}},$$
$$u_{(m+1)(2n+1)-t} = \frac{B_{2k(n+1)} - u_{-t}B_{(2n+1)k}}{B_{(2n+3)k} - u_{-t}B_{2k(n+1)}},$$

$$v_{2(m+1)n-t} = \frac{B_{(2n+1)k} - u_{-t}B_{2kn}}{B_{2k(n+1)} - u_{-t}B_{(2n+1)k}},$$
$$v_{(m+1)(2n+1)-t} = \frac{B_{2k(n+1)} - v_{-t}B_{(2n+1)k}}{B_{(2n+3)k} - v_{-t}B_{2k(n+1)}}, k \geq 1.$$

3. Global stability of positive solutions of (1.0)

In the following, we will study the global stability character of the solutions of system (1.0). Obviously, the positive equilibriums of system (1.0) are

$$U_1 = (\overline{u}_1, \overline{v}_1) = a^2(1, 1) \text{ and } U_2 = (\overline{u}_2, \overline{v}_2) = b^2(1, 1).$$

Let the functions $f_1, f_2 : (0, +\infty)^{2(m+1)} \to (0, +\infty)$ defined by

$$f_1\left(\underline{x}'_{0:m}, \underline{y}'_{0:m}\right) = \frac{1}{34 - y_{n-m}}, f_2\left(\underline{x}'_{0:m}, \underline{y}'_{0:m}\right) = \frac{1}{34 - x_{n-m}},$$

where $\underline{z}_{0:m} = (z_0, z_1, ..., z_m)'$. Now, it is usually useful to linearized system (1.0) around the equilibrium point U_2 in order to facilitate its study. For this purpose, introducing the vectors $\underline{X}'_n := \left(\underline{X}'_n, Y'_n\right)$ where

$\underline{X}'_n = (x_n, x_{n-1}, ..., x_{n-m})$ and $\underline{Y}'_n = (y_n, y_{n-1}, ..., y_{n-m})$. With these notations, we obtain the following representation

$$\underline{X}_{n+1} = F_m \underline{X}_n, \tag{3.1}$$

where

$$F_m = \begin{pmatrix} \underline{O}'_{(m-1)} & 0 & \underline{O}'_{(m-1)} & b^4 \\ I_{(m-1)} & \underline{O}_{(m-1)} & O_{(m-1)} & \underline{O}_{(m-1)} \\ \underline{O}'_{(m-1)} & b^4 & O_{(m-1)} & 0 \\ O_{(m-1)} & \underline{O}_{(m-1)} & I_{(m-1)} & \underline{O}_{(m-1)} \end{pmatrix},$$

with $O_{(k,l)}$ denotes the matrix of order $k \times l$ whose entries are zeros, for simplicity, we set $O_{(k)} := O_{(k,k)}$ and $\underline{O}_{(k)} := O_{(k,1)}$ and $I_{(m)}$ is the $m \times m$ identity matrix. We summarize the above discussion in the following theorem

Theorem 3.1. *The positive equilibrium point U_2 is locally asymptotically stable.*

Proof. After some preliminary calculations, the characteristic polynomial of F_m is

$$P_{F_m}(\lambda) = \det\left(F_m - \lambda I_{(2(m+1))}\right) = \Lambda_1(\lambda) - \Lambda_2(\lambda),$$

where $\Lambda_1(\lambda) = \lambda^{2(m+1)}$ and $\Lambda_2(\lambda) = b^8$, then $|\Lambda_2(\lambda)| < |\Lambda_1(\lambda)|, \forall\lambda : |\lambda| = 1$. So, according to Rouche's Theorem, all zeros of $\Lambda_1(\lambda) - \Lambda_2(\lambda) = 0$ lie in the unit disc $|\lambda| < 1$. Thus, the positive equilibrium point U_2 is locally asymptotically stable. □

Corollary 3.1. *For every well defined solution of system* (1.0), *we have* $\lim u_n = \lim v_n = b^2$.

Proof. From Theorem 2.2, we have

$$\begin{aligned} \lim u_{2(m+1)n-t} &= \lim \frac{B_{4n+2} - v_{-t}B_{4n}}{B_{4n+4} - v_{-t}B_{4n+2}} \\ &= \lim \frac{1 - v_{-t}\frac{B_{4n}}{B_{4n+2}}}{\frac{B_{4n+4}}{B_{4n+2}} - v_{-t}} \\ &= \frac{1 - v_{-t}b^2}{a^2 - v_{-t}} = b^2, \end{aligned}$$

$$\begin{aligned} \lim u_{(m+1)(2n+1)-t} &= \lim \frac{B_{4n+4} - u_{-t}B_{4n+2}}{B_{4n+6} - u_{-t}B_{4n+4}} \\ &= \lim \frac{1 - u_{-t}\frac{B_{4n+2}}{B_{4n+4}}}{\frac{B_{4n+6}}{B_{4n+4}} - u_{-t}} \\ &= \frac{1 - u_{-t}b^2}{a^2 - u_{-t}} = b^2. \end{aligned}$$

Rest of the proof of $\lim v_n$ is similar to the proof of $\lim u_n$, which completes the proof of Corollary 3.1. □

The following result is an immediate consequence of Theorem 3.1 and Corollary 3.1.

Corollary 3.2. *The unique positive equilibrium point U_2 is globally asymptotically stable.*

4. Numerical Examples

In order to clarify and shore theoretical results of the previous section, we consider some interesting numerical examples in this section.

Example 4.1. *We consider interesting numerical example for the difference equations system* (1.0) *when* $m = 1$ *with the initial conditions* $u_{-1} = 2/3$, $u_0 = 4$, $v_{-1} = 0.4$ *and* $v_0 = -2/3$. *The plot of the solutions is shown in Figure 1.*

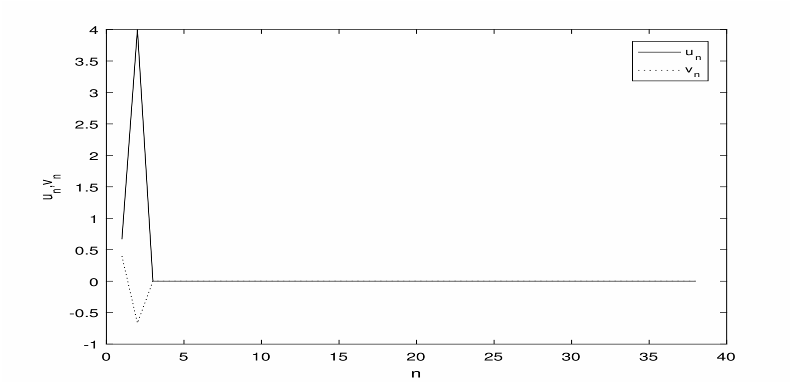

Figure1.The plot of the solutions of system (1.0), *when* $m = 1$ *and we put the initial*

conditions $u_{-1} = 2/3, u_0 = 4, v_{-1} = 0.4$ *and* $v_0 = -2/3$.

Example 4.2. *We consider interesting numerical example for the difference equations system* (1.0) *when* $m = 2$ *with the initial conditions*

i	0	1	2
u_{-i}	0	4	-1.32
v_{-i}	2	5	0.16

Table 1. The initial conditions.

The plot of the solutions is shc

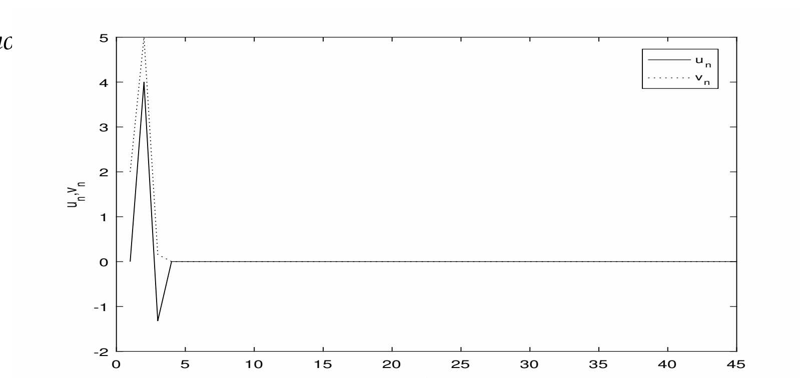

Figure 2. The plot of the solutions of system (1.0); when we put the initial conditions in Table 1.

Example 4.3. *We consider interesting numerical example for the difference equations system* (1.0) *when* $m = 3$ *with the initial conditions*

i	0	1	2	3
u_{-i}	3	4	2	0.2
v_{-i}	0.4	2/3	−1	3.9

Table 2. The initial conditions.

The plot of the solutions is shown in Figure 3.

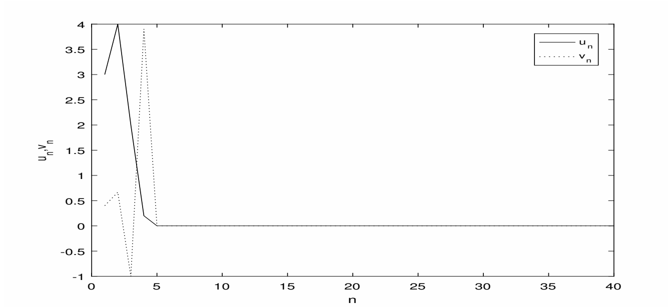

Figure 3. The plot of the solutions of system (1.0); when we put the initial conditions in Table 2.

In these examples, we show that the solutions of the system (1.0) for some cases are globally asymptotically stable.

Conflicts of Interest

The corresponding author declares no conflict of interest.

Acknowledgments

We should like to thank the Editor in Chief of the journal, an Associate Editor and the anonymous referees for their constructive comments and very useful suggestions and remarks which were most valuable for improvement in the final version of the paper. We would also like to thank our colleague Dr. Walid BenKhelifa for his important help and encouragement.

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