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SOME RESULTS ON A RATIONAL RECURSIVE SEQUENCES

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ABSTRACT. In this paper, we study some results on the following rational recursive sequences:

$$x_{n+1} = \frac{x_{n-9}}{\pm 1 \pm x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}, \quad n = 0, 1, \cdots, \tag{0.1}$$

where the initial conditions are arbitrary real numbers.

1. INTRODUCTION

Our aim in this article is to propose some results concerning the behavior of the following rational recursive sequences:

$$x_{n+1} = \frac{x_{n-9}}{\pm 1 \pm x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}, \quad n = 0, 1, \cdots, \tag{1}$$

where the initial conditions are arbitrary real numbers.

The study of Difference Equations has been growing continuously for the last decade. This is largely due to the fact that difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing theory, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical network, quanta in radiation, genetics in biology, economics, psychology, sociology, etc. In fact, now it occupies a central position in applicable analysis and will no doubt continue to play an important role in mathematics as a whole

Recently there has been a great interest in studying the qualitative properties of rational difference equations. For the systematical studies of rational and non-rational difference equations, one can refer to the papers [1-44] and references therein. The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of non-linear difference equations of order greater than one come from the results for rational difference equations. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

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Aloqeili [5] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [80]-[10] investigated the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + a x_n x_{n-1}}, x_{n+1} = \frac{x_{n-1}}{-1 + a x_n x_{n-1}}, x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}$$

Elabbasy et al. [11] gave the solution of the following difference equations

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm x_{n-3} x_{n-7}}$$

Elabbasy et al. [12] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequences

$$x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}$$

Karatas et al. [24] got the form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}$$

Simsek et al. [34] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}$$

Elsayed [17] proposed the solutions of the following recursive sequences

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm x_{n-1} x_{n-3} x_{n-5} x_{n-7}}, \quad n = 0, 1, ...,$$

where the initial conditions are arbitrary real numbers.

Here, we recall some notations and results which will be useful in our investigation. Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions

$$x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_0 \in I,$$

the difference equation

(2) $$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ...,$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. (Equilibrium Point) A point $\bar{x} \in I$ is called an equilibrium point of Eq. (2) if

$$\bar{x} = f(\bar{x}, \bar{x}, ..., \bar{x}).$$

That is, $x_n = \bar{x}$ for $n \geq 0$, is a solution of Eq. (2), or equivalently, $\bar{x}$ is a fixed point of f.

Definition 2. (Stability)

- The equilibrium point $\bar{x}$ of Eq. (2) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_0 \in I$, with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + |x_{-k+2} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta,$$

we have

$$|x_n - \bar{x}| < \varepsilon, \quad for\,all\ n \geq -k$$

- The equilibrium point $\bar{x}$ of Eq. (2) is locally asymptotically stable if $\bar{x}$ is locally stable solution of Eq. (2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_0 \in I$, with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + |x_{-k+2} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta,$$

we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

- The equilibrium point $\bar{x}$ of Eq. (2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_0 \in I$ we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

- The equilibrium point $\bar{x}$ of Eq. (2) is globally asymptotically stable if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of Eq. (2).
- The equilibrium point $\bar{x}$ of Eq. (2) is unstable if $\bar{x}$ is not locally stable.

The linearised equation of Eq. (2) about the equilibrium $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, ..., \bar{x})}{\partial x_{n-i}} y_{n-i}$$

Theorem 1. Assume that $p, q \in \mathbb{R}$ and $k \in \{0, 1, 2, ...\}$. Then

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, n = 0, 1, ...$$

Remark 1. The theorem can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ..., \tag{3}$$

where $p_1, p_2, ..., p_k \in \mathbb{R}$ and $k \in \{0, 1, 2, ...\}$. Then Eq. (3) is asymptotically stable provided that

$$\sum_{i=0}^{k} |p_i| < 1.$$

Definition 3. **(Periodicity)** A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

2. First Case : $x_{n+1} = \dfrac{x_{n-9}}{1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}$

A specific form is proposed for the solutions of the following rational recursive sequences:

$$x_{n+1} = \frac{x_{n-9}}{1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}, \quad n = 0, 1, ..., \tag{4}$$

where the initial conditions are arbitrary real numbers.

Theorem 2. Let $\{x_n\}_{n=-9}^{\infty}$ be a solution of equations (4). Then for $n = 0, 1, ...,$

$$x_{10n-9} = \frac{t\prod_{i=0}^{n-1}(1+5ibdfht)}{\prod_{i=0}^{n-1}(1+(5i+1)bdfht)}, \qquad x_{10n-4} = \frac{e\prod_{i=0}^{n-1}(1+(5i+2)acegr)}{\prod_{i=0}^{n-1}(1+(5i+3)acegr)},$$

$$x_{10n-8} = \frac{r\prod_{i=0}^{n-1}(1+5iacegr)}{\prod_{i=0}^{n-1}(1+(5i+1)acegr)}, \qquad x_{10n-3} = \frac{d\prod_{i=0}^{n-1}(1+(5i+3)bdfht)}{\prod_{i=0}^{n-1}(1+(5i+4)bdfht)},$$

$$x_{10n-7} = \frac{h\prod_{i=0}^{n-1}(1+(5i+1)bdfht)}{\prod_{i=0}^{n-1}(1+(5i+2)bdfht)}, \qquad x_{10n-2} = \frac{c\prod_{i=0}^{n-1}(1+(5i+3)acegr)}{\prod_{i=0}^{n-1}(1+(5i+4)acegr)},$$

$$x_{10n-6} = \frac{g\prod_{i=0}^{n-1}(1+(5i+1)acegr)}{\prod_{i=0}^{n-1}(1+(5i+2)acegr)}, \qquad x_{10n-1} = \frac{b\prod_{i=0}^{n-1}(1+(5i+4)bdfht)}{\prod_{i=0}^{n-1}(1+(5i+5)bdfht)},$$

$$x_{10n-5} = \frac{f\prod_{i=0}^{n-1}(1+(5i+2)bdfht)}{\prod_{i=0}^{n-1}(1+(5i+3)bdfht)}, \qquad x_{10n} = \frac{a\prod_{i=0}^{n-1}(1+(5i+4)acegr)}{\prod_{i=0}^{n-1}(1+(5i+5)acegr)},$$

where $x_{-9} = t, x_{-8} = r, x_{-7} = h, x_{-6} = g, x_{-5} = f, x_{-4} = e, x_{-3} = d, x_{-2} = c, x_{-1} = b, x_0 = a$ and $\prod_{i=0}^{-1} A_i = 1$.

Proof: We use an inductive proof for this rational recursive sequences. It is easy to see that for $n = 0$, the result holds. Suppose that $n > 0$ and that the assumption is satisfied for $n - 1$. That is;

$$x_{10n-19} = \frac{t\prod_{i=0}^{n-2}(1+5ibdfht)}{\prod_{i=0}^{n-2}(1+(5i+1)bdfht)}, \qquad x_{10n-14} = \frac{e\prod_{i=0}^{n-2}(1+(5i+2)acegr)}{\prod_{i=0}^{n-2}(1+(5i+3)acegr)},$$

$$x_{10n-18} = \frac{r\prod_{i=0}^{n-2}(1+5iacegr)}{\prod_{i=0}^{n-2}(1+(5i+1)acegr)}, \qquad x_{10n-13} = \frac{d\prod_{i=0}^{n-2}(1+(5i+3)bdfht)}{\prod_{i=0}^{n-2}(1+(5i+4)bdfht)},$$

$$x_{10n-17}=\frac{h\prod_{i=0}^{n-2}(1+(5i+1)bdfht)}{\prod_{i=0}^{n-2}(1+(5i+2)bdfht)},\quad x_{10n-12}=\frac{c\prod_{i=0}^{n-2}(1+(5i+3)acegr)}{\prod_{i=0}^{n-2}(1+(5i+4)acegr)},$$

$$x_{10n-16}=\frac{g\prod_{i=0}^{n-2}(1+(5i+1)acegr)}{\prod_{i=0}^{n-2}(1+(5i+2)acegr)},\quad x_{10n-11}=\frac{b\prod_{i=0}^{n-2}(1+(5i+4)bdfht)}{\prod_{i=0}^{n-2}(1+(5i+5)bdfht)},$$

$$x_{10n-15}=\frac{f\prod_{i=0}^{n-2}(1+(5i+2)bdfht)}{\prod_{i=0}^{n-2}(1+(5i+3)bdfht)},\quad x_{10n-10}=\frac{a\prod_{i=0}^{n-2}(1+(5i+4)acegr)}{\prod_{i=0}^{n-2}(1+(5i+5)acegr)}.$$

Now, using the main Eq. (4), one has

$$x_{10n-9}=\frac{x_{10n-19}}{1+x_{10n-11}x_{10n-13}x_{10n-15}x_{10n-17}x_{10n-19}}$$

$$=\frac{\frac{t\prod_{i=0}^{n-2}(1+5ibdfht)}{\prod_{i=0}^{n-2}(1+(5i+1)bdfht)}}{1+\frac{b\prod_{i=0}^{n-2}(1+(5i+4)bdfht)}{\prod_{i=0}^{n-2}(1+(5i+5)bdfht)}\frac{d\prod_{i=0}^{n-2}(1+(5i+3)bdfht)}{\prod_{i=0}^{n-2}(1+(5i+4)bdfht)}\frac{f\prod_{i=0}^{n-2}(1+(5i+2)bdfht)}{\prod_{i=0}^{n-2}(1+(5i+3)bdfht)}\frac{h\prod_{i=0}^{n-2}(1+(5i+1)bdfht)}{\prod_{i=0}^{n-2}(1+(5i+2)bdfht)}\frac{t\prod_{i=0}^{n-2}(1+5ibdfht)}{\prod_{i=0}^{n-2}(1+(5i+1)bdfht)}}$$

$$=\frac{t\prod_{i=0}^{n-2}(1+5ibdfht)}{\prod_{i=0}^{n-2}(1+(5i+1)bdfht)}\left(\frac{1}{1+\frac{bdfht}{\prod_{i=0}^{n-2}(1+(5i+5)bdfht)}\prod_{i=0}^{n-2}(1+5ibdfht)}\right)$$

$$=\frac{t\prod_{i=0}^{n-2}(1+5ibdfht)}{\prod_{i=0}^{n-2}(1+(5i+1)bdfht)}\left(\frac{1}{1+\frac{bdfht}{(1+(5n-5)bdfht)}}\right)$$

$$=\frac{t\prod_{i=0}^{n-2}(1+5ibdfht)}{\prod_{i=0}^{n-2}(1+(5i+1)bdfht)}\left(\frac{(1+(5n-5)bdfht)}{(1+(5n-5)bdfht+bdfht)}\right)$$

$$=\frac{t\prod_{i=0}^{n-2}(1+5ibdfht)}{\prod_{i=0}^{n-2}(1+(5i+1)bdfht)}\frac{(1+(5n-5)bdfht)}{(1+(5n-4)bdfht)}.$$

Hence, we have

$$x_{10n-9} = \frac{t\prod_{i=0}^{n-2}(1+5ibdfht)}{\prod_{i=0}^{n-2}(1+(5i+1)bdfht)} \frac{1+(5n-5)bdfht}{1+(5n-4)bdfht} = \frac{t\prod_{i=0}^{n-1}(1+5ibdfht)}{\prod_{i=0}^{n-1}(1+(5i+1)bdfht)}.$$

Similarly, using the main Eq. (4), one has

$$x_{10n-8} = \frac{x_{10n-18}}{1+x_{10n-10}x_{10n-12}x_{10n-14}x_{10n-16}x_{10n-18}}$$

$$= \frac{\frac{r\prod_{i=0}^{n-2}(1+5iacegr)}{\prod_{i=0}^{n-2}(1+(5i+1)acegr)}}{1+\frac{r\prod_{i=0}^{n-2}(1+(5i+4)acegr)}{\prod_{i=0}^{n-2}(1+(5i+5)acegr)}\frac{g\prod_{i=0}^{n-2}(1+(5i+3)acegr)}{\prod_{i=0}^{n-2}(1+(5i+4)acegr)}\frac{e\prod_{i=0}^{n-2}(1+(5i+2)acegr)}{\prod_{i=0}^{n-2}(1+(5i+3)acegr)}\frac{c\prod_{i=0}^{n-2}(1+(5i+1)acegr)}{\prod_{i=0}^{n-2}(1+(5i+2)acegr)}\frac{a\prod_{i=0}^{n-2}(1+5iacegr)}{\prod_{i=0}^{n-2}(1+(5i+1)acegr)}}$$

$$= \frac{r\prod_{i=0}^{n-2}(1+5iacegr)}{\prod_{i=0}^{n-2}(1+(5i+1)acegr)}\left(\frac{1}{1+\frac{acegr}{\prod_{i=0}^{n-2}(1+(5i+5)acegr)}\prod_{i=0}^{n-2}(1+5iacegr)}\right)$$

$$= \frac{r\prod_{i=0}^{n-2}(1+5iacegr)}{\prod_{i=0}^{n-2}(1+(5i+1)acegr)}\left(\frac{1}{1+\frac{acegr}{(1+(5n-5)acegr)}}\right)$$

$$= \frac{r\prod_{i=0}^{n-2}(1+5iacegr)}{\prod_{i=0}^{n-2}(1+(5i+1)acegr)}\left(\frac{(1+(5n-5)acegr)}{(1+(5n-5)acegr+acegr)}\right)$$

$$= \frac{r\prod_{i=0}^{n-2}(1+5iacegr)}{\prod_{i=0}^{n-2}(1+(5i+1)acegr)}\frac{(1+(5n-5)acegr)}{(1+(5n-4)acegr)}.$$

Hence, we have

$$x_{10n-8} = \frac{r\prod_{i=0}^{n-2}(1+5iacegr)}{\prod_{i=0}^{n-2}(1+(5i+1)acegr)} \frac{1+(5n-5)acegr}{1+(5n-4)acegr} = \frac{r\prod_{i=0}^{n-1}(1+5iacegr)}{\prod_{i=0}^{n-1}(1+(5i+1)acegr)}.$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 3. Eq. (4) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.

Proof: For the equilibrium points of Eq. (4), we can write

$$\bar{x} = \frac{\bar{x}}{1+\bar{x}^5}$$

Then $\bar{x} + \bar{x}^6 = \bar{x}$ or also $\bar{x}^6 = 0$.
Thus the equilibrium point of Eq. (4) is $\bar{x} = 0$.
Let $f : (0,\infty)^5 \to (0,\infty)$ be a function defined by

$$f(u,v,w,y,z) = \frac{u}{1+uvwyz}.$$

Therefore it follows that

$$\begin{aligned} f_u(u,v,w,y,z) &= \frac{1}{(1+uvwyz)^2}, \quad f_v(u,v,w,y,z) = \frac{-u^2wyz}{(1+uvwyz)^2}, \\ f_w(u,v,w,y,z) &= \frac{-u^2vyz}{(1+uvwyz)^2}, \quad f_y(u,v,w,y,z) = \frac{-u^2vwz}{(1+uvwyz)^2}, \\ f_z(u,v,w,y,z) &= \frac{-u^2vwy}{(1+uvwyz)^2}. \end{aligned}$$

We see that

$$f_u(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 1, f_v(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0, f_w(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0, f_y(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0, f_z(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0.$$

The proof follows by using Theorem 1.

Theorem 4. Every positive solution of Eq. (4) is bounded and

$$\lim_{n\to\infty} x_n = 0.$$

Proof: It follows from Eq. (4) that

$$x_{n+1} = \frac{x_{n-9}}{1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}} \leq x_{n-9}$$

Then the sub-sequences $\{x_{10n-9}\}_{n=0}^{\infty}, \{x_{10n-8}\}_{n=0}^{\infty}, \{x_{10n-7}\}_{n=0}^{\infty}, \{x_{10n-6}\}_{n=0}^{\infty}, \{x_{10n-5}\}_{n=0}^{\infty}$, $\{x_{10n-4}\}_{n=0}^{\infty}, \{x_{10n-3}\}_{n=0}^{\infty}, \{x_{10n-2}\}_{n=0}^{\infty}, \{x_{10n-1}\}_{n=0}^{\infty}, \{x_{10n}\}_{n=0}^{\infty}$, are decreasing and so are bounded from above by $M = \max\{x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0\}$.

In order to perform the results of this section, we consider numerical examples which represent different types of solutions to Eq. (4).

Example 1: Suppose for Eq. (4) that $x_{-9} = -17, x_{-8} = 20, x_{-7} = 40, x_{-6} = 20, x_{-5} = -4, x_{-4} = 0.6, x_{-3} = -2/84, x_{-2} = -1/6, x_{-1} = 7, x_0 = -19$. See Figure 1.

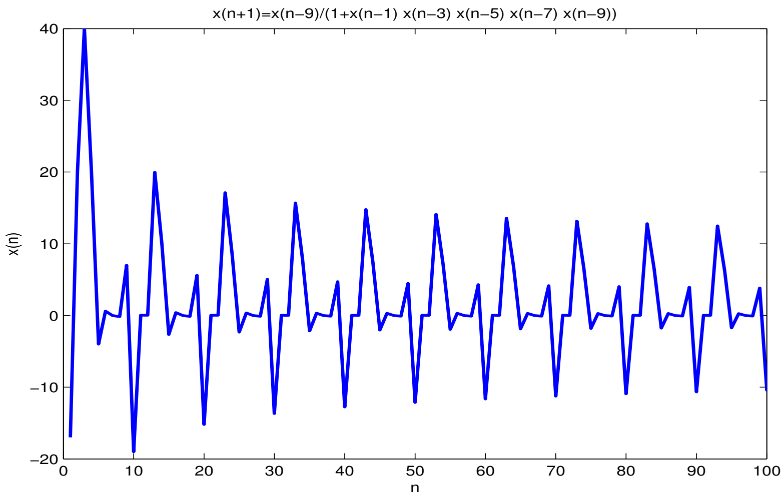

FIGURE 1

Example 2: See Figure 2 since we put for Eq. (4), $x_{-9} = 7, x_{-8} = 6, x_{-7} = 5, x_{-6} = 4, x_{-5} = 3, x_{-4} = 2, x_{-3} = 2, x_{-2} = 1, x_{-1} = 7, x_0 = 9$.

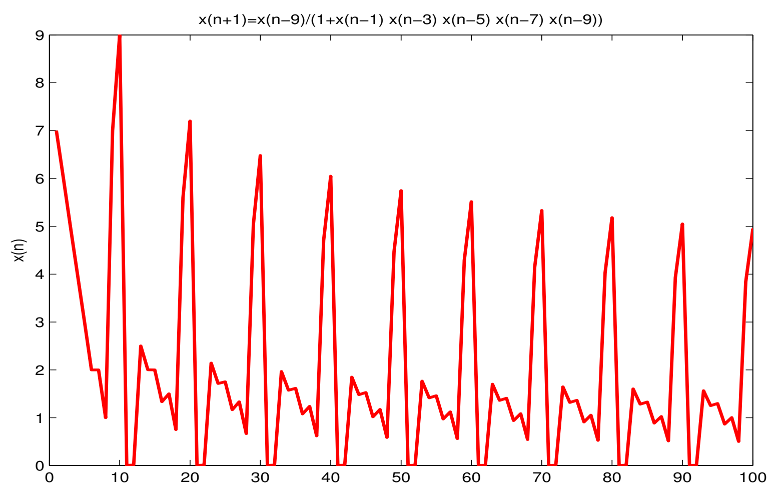

FIGURE 2

3. SECOND CASE : $x_{n+1} = \dfrac{x_{n-9}}{1 - x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}$

Consider the following rational recursive sequences :

$$x_{n+1} = \frac{x_{n-9}}{1 - x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}, \quad n = 0, 1, ..., . \tag{5}$$

where the initial conditions are arbitrary real numbers.

A specific form concerning the solutions of equations (5) will be proposed in the following theorem

Theorem 5. Let $\{x_n\}_{n=-9}^{\infty}$ be a solution of equations (5). Then for $n = 0, 1, ...,$

$$x_{10n-9} = \frac{t\prod_{i=0}^{n-1}(1-5ibdfht)}{\prod_{i=0}^{n-1}(1-(5i+1)bdfht)}, \quad x_{10n-4} = \frac{e\prod_{i=0}^{n-1}(1-(5i+2)acegr)}{\prod_{i=0}^{n-1}(1-(5i+3)acegr)},$$

$$x_{10n-8} = \frac{r\prod_{i=0}^{n-1}(1-5iacegr)}{\prod_{i=0}^{n-1}(1-(5i+1)acegr)}, \quad x_{10n-3} = \frac{d\prod_{i=0}^{n-1}(1-(5i+3)bdfht)}{\prod_{i=0}^{n-1}(1-(5i+4)bdfht)},$$

$$x_{10n-7} = \frac{h\prod_{i=0}^{n-1}(1-(5i+1)bdfht)}{\prod_{i=0}^{n-1}(1-(5i+2)bdfht)}, \quad x_{10n-2} = \frac{c\prod_{i=0}^{n-1}(1-(5i+3)acegr)}{\prod_{i=0}^{n-1}(1-(5i+4)acegr)},$$

$$x_{10n-6} = \frac{g\prod_{i=0}^{n-1}(1-(5i+1)acegr)}{\prod_{i=0}^{n-1}(1-(5i+2)acegr)}, \quad x_{10n-1} = \frac{b\prod_{i=0}^{n-1}(1-(5i+4)bdfht)}{\prod_{i=0}^{n-1}(1-(5i+5)bdfht)},$$

$$x_{10n-5} = \frac{f\prod_{i=0}^{n-1}(1-(5i+2)bdfht)}{\prod_{i=0}^{n-1}(1-(5i+3)bdfht)}, \quad x_{10n} = \frac{a\prod_{i=0}^{n-1}(1-(5i+4)acegr)}{\prod_{i=0}^{n-1}(1-(5i+5)acegr)},$$

where $j\,bdfht \neq 1, j\,acegr \neq 1$ for $j = 1, 2, 3, ...$.

Proof: It is similar to the proof of Theorem 2 and will be omitted.

Theorem 6. Eq. (5) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.

Example 3: For Eq. (5), consider $x_{-9} = -17, x_{-8} = 20, x_{-7} = 40, x_{-6} = 20, x_{-5} = -4, x_{-4} = 0.6, x_{-3} = -2/84, x_{-2} = -1/6, x_{-1} = 7, x_0 = -19$. See Figure 3.

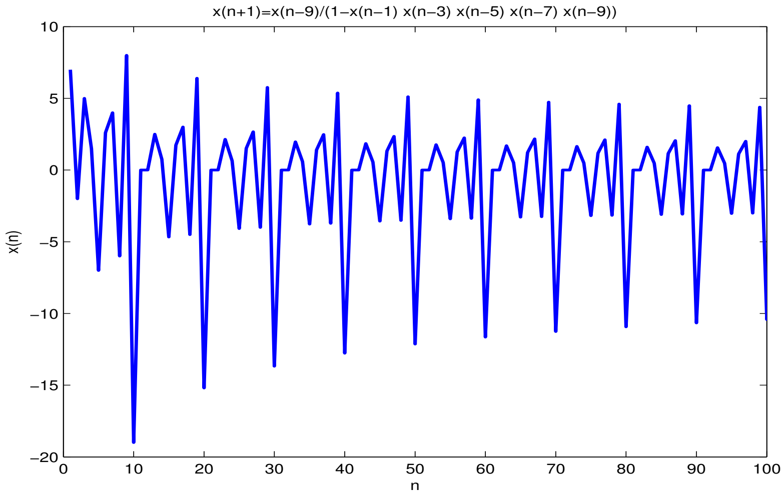

FIGURE 3

Example 4: By choosing $x_{-9} = 7, x_{-8} = 6, x_{-7} = 5, x_{-6} = 4, x_{-5} = 3, x_{-4} = 2, x_{-3} = 2, x_{-2} = 1, x_{-1} = 7, x_0 = 9$ for Eq. (5). See Figure 4.

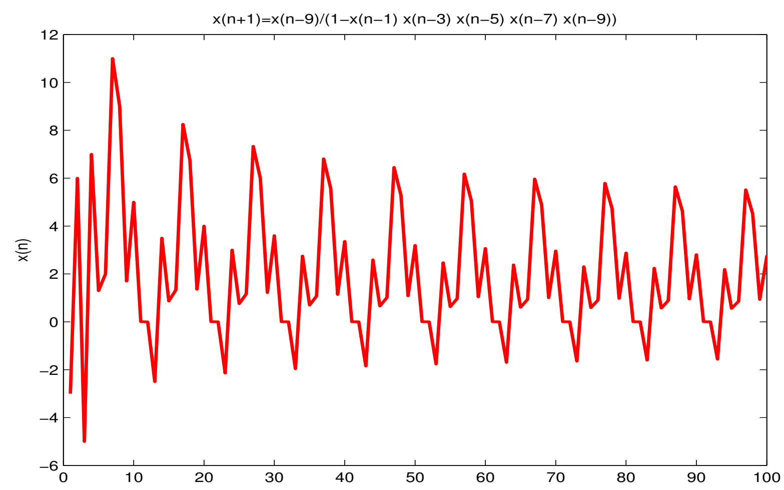

FIGURE 4

4. THIRD CASE : $x_{n+1} = \dfrac{x_{n-9}}{-1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}$

Now by consider the following rational recursive sequences, a specific form for the solutions will be given in Theorem 7.

(6)
$$x_{n+1} = \frac{x_{n-9}}{-1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}, \quad n = 0, 1, ...,$$

where the initial conditions are arbitrary real numbers such that

$$x_{-9}x_{-7}x_{-5}x_{-3}x_{-1} \neq 1, \quad x_{-8}x_{-6}x_{-4}x_{-2}x_0 \neq 1.$$

Theorem 7. Let $\{x_n\}_{n=-9}^{\infty}$ be a solution of equations (6). Then Eq. (6) has unbounded solutions and for $n = 0, 1, \dots, .$

$$x_{10n-9} = \frac{t}{(-1+bdfht)^n}, \qquad x_{10n-4} = \frac{e}{(-1+acegr)^n},$$
$$x_{10n-8} = \frac{r}{(-1+acegr)^n}, \qquad x_{10n-3} = d(-1+bdfht)^n,$$
$$x_{10n-7} = h(-1+bdfht)^n, \qquad x_{10n-2} = c(-1+acegr)^n,$$
$$x_{10n-6} = g(-1+acegr)^n, \qquad x_{10n-1} = \frac{b}{(-1+bdfht)^n},$$
$$x_{10n-5} = \frac{f}{(-1+bdfht)^n}, \qquad x_{10n} = \frac{a}{(-1+acegr)^n}.$$

Proof: For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$x_{10n-19} = \frac{t}{(-1+bdfht)^{n-1}}, \qquad x_{10n-14} = \frac{e}{(-1+acegr)^{n-1}},$$
$$x_{10n-18} = \frac{r}{(-1+acegr)^{n-1}}, \qquad x_{10n-13} = d(-1+bdfht)^{n-1},$$
$$x_{10n-17} = h(-1+bdfht)^{n-1}, \qquad x_{10n-12} = c(-1+acegr)^{n-1},$$
$$x_{10n-16} = g(-1+acegr)^{n-1}, \qquad x_{10n-11} = \frac{b}{(-1+bdfht)^{n-1}}$$
$$x_{10n-15} = \frac{f}{(-1+bdfht)^{n-1}}, \qquad x_{10n-10} = \frac{a}{(-1+acegr)^{n-1}}.$$

Now, it follows from Eq. (6) that

$$x_{10n-9} = \frac{x_{10n-19}}{-1 + x_{10n-11}x_{10n-13}x_{10n-15}x_{10n-17}x_{10n-19}}$$
$$= \frac{\frac{t}{(-1+bdfht)^{n-1}}}{-1 + \frac{b}{(-1+bdfht)^{n-1}} d(-1+bdfht)^{n-1} \frac{f}{(-1+bdfht)^{n-1}} h(-1+bdfht)^{n-1} \frac{t}{(-1+bdfht)^{n-1}}}$$
$$= \frac{\frac{t}{(-1+bdfht)^{n-1}}}{-1+bdfht}.$$

Hence, we have

$$x_{10n-9} = \frac{t}{(-1+bdfht)^n}.$$

Similary, it follows from Eq. (6) that

$$x_{10n-8} = \frac{x_{10n-18}}{-1 + x_{10n-10}x_{10n-12}x_{10n-14}x_{10n-16}x_{10n-18}}$$
$$= \frac{\frac{r}{(-1+acegr)^{n-1}}}{-1 + \frac{a}{(-1+acegr)^{n-1}} c(-1+acegr)^{n-1} \frac{g}{(-1+acegr)^{n-1}} e(-1+acegr)^{n-1} \frac{r}{(-1+acegr)^{n-1}}}$$
$$= \frac{\frac{r}{(-1+acegr)^{n-1}}}{-1+acegr}$$

Hence, we have

$$x_{10n-8} = \frac{r}{(-1+acegr)^n}$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 8. Eq. (6) has three equilibrium points which are $0, \pm\sqrt[5]{2}$ and these equilibrium points are not locally asymptotically stable.

Proof: It is similar to the proof of Theorem 2 and will be omitted.

Theorem 9. Eq. (6) has a periodic solutions of period ten iff $acegr = bdfht = 2$ and will take the form $\{t, r, h, g, f, e, d, c, b, a, t, r, h, g, f, e, d, c, b, a, ...\}$.

Proof: First suppose that there exists a prime period ten solution

$$t, r, h, g, f, e, d, c, b, a, t, r, h, g, f, e, d, c, b, a, ...$$

of Eq. (6), we see from Eq. (6) that

$$\begin{aligned} t &= \frac{t}{(-1+bdfht)^n}, & e &= \frac{e}{(-1+acegr)^n} \\ r &= \frac{r}{(-1+acegr)^n}, & d &= d(-1+bdfht)^n \\ h &= h(-1+bdfht)^n, & c &= c(-1+acegr)^n \\ g &= g(-1+acegr)^n, & b &= \frac{b}{(-1+bdfht)^n} \\ f &= \frac{f}{(-1+bdfht)^n}, & a &= \frac{a}{(-1+acegr)^n} \end{aligned}$$

or

$$(-1+bdfht)^n = 1, \quad (-1+acegr)^n = 1.$$

Then

$$bdfht = 2, \quad acegr = 2.$$

Second suppose $bdfht = acegr = 2$. Then we see from Eq. (6) that $x_{10n} = a, x_{10n-1} = b, x_{10n-2} = c, x_{10n-3} = d, x_{10n-4} = e, x_{10n-5} = f, x_{10n-6} = g, x_{10n-7} = h, x_{10n-8} = r, x_{10n-9} = t$. Thus we have a period ten solution and the proof is complete.

Example 5: For Eq. (6), consider $x_{-9} = -17, x_{-8} = 20, x_{-7} = 40, x_{-6} = 20, x_{-5} = -4, x_{-4} = 0.6, x_{-3} = -2/84, x_{-2} = -1/6, x_{-1} = 7, x_0 = -19$. See Figure 5.

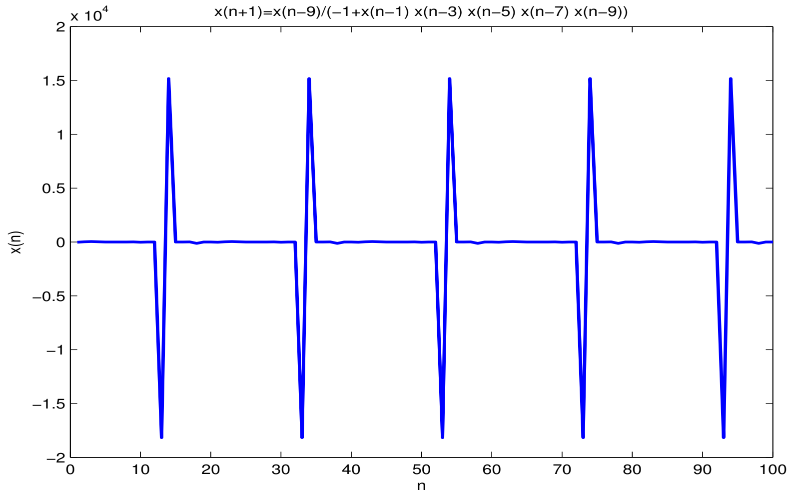

FIGURE 5

Example 6: By choosing $x_{-9} = \dfrac{2}{x_{-7}x_{-5}x_{-3}x_{-1}}$ and $x_{-8} = \dfrac{2}{x_{-6}x_{-4}x_{-2}x_0}$ where $x_{-7} = 5, x_{-6} = 4, x_{-5} = 3, x_{-4} = 2, x_{-3} = 2, x_{-2} = 1, x_{-1} = 7, x_0 = 9$ for Eq. (6). See Figure 6.

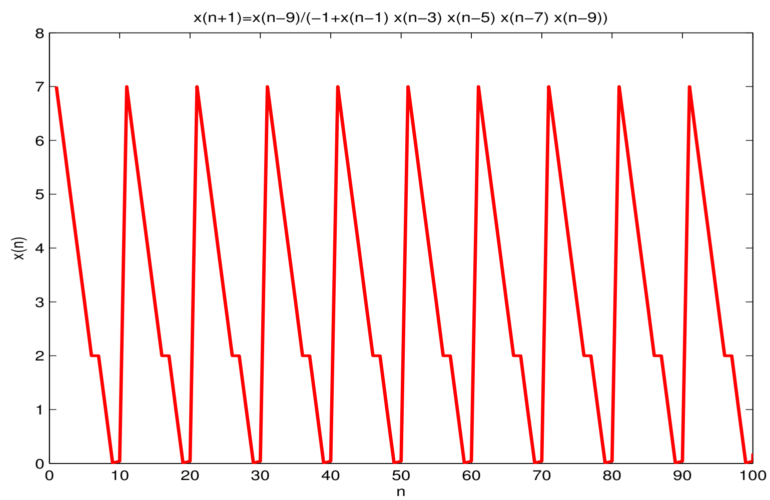

FIGURE 6

5. FORTH CASE : $x_{n+1} = \dfrac{x_{n-9}}{-1 - x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}$

The last case that we consider here is given by:

$$x_{n+1} = \frac{x_{n-9}}{-1 - x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}, \quad n = 0, 1, ..., \tag{7}$$

where the initial conditions are arbitrary real numbers satisfying $x_{-9}x_{-7}x_{-5}x_{-3}x_{-1} \neq -1$, and $x_{-8}x_{-6}x_{-4}x_{-2}x_0 \neq -1$.

A specific form concerning the solutions is given in Theorem 10.

Theorem 10. Let $\{x_n\}_{n=-9}^{\infty}$ be a solution of equations (7). Then Eq. (7) has unbounded solutions and for $n = 0, 1, ...,$.

$$\begin{aligned}
x_{10n-9} &= \frac{t(-1)^n}{(1+bdfht)^n}, & x_{10n-4} &= \frac{e(-1)^n}{(1+acegr)^n},\\
x_{10n-8} &= \frac{r(-1)^n}{(1+acegr)^n}, & x_{10n-3} &= d(-1)^n(1+bdfht)^n,\\
x_{10n-7} &= h(-1)^n(1+bdfht)^n, & x_{10n-2} &= c(-1)^n(1+acegr)^n,\\
x_{10n-6} &= g(-1)^n(1+acegr)^n, & x_{10n-1} &= \frac{b(-1)^n}{(1+bdfht)^n},\\
x_{10n-5} &= \frac{f(-1)^n}{(1+bdfht)^n}, & x_{10n} &= \frac{a(-1)^n}{(1+acegr)^n}.
\end{aligned}$$

Theorem 11. Eq. (7) has two equilibrium points which are zero and $-\sqrt[5]{2}$. These equilibrium points are not locally asymptotically stable.

Theorem 12. Eq. (7) has a periodic solutions of period ten iff $acegr = bdfht = -2$ and will take the form $\{t, r, h, g, f, e, d, c, b, a, t, r, h, g, f, e, d, c, b, a, ...\}$.

Example 7: For Eq. (7), consider $x_{-9} = -17, x_{-8} = 20, x_{-7} = 40, x_{-6} = 20, x_{-5} = -4, x_{-4} = 0.6, x_{-3} = -2/84, x_{-2} = -1/6, x_{-1} = 7, x_0 = -19$. See Figure 7.

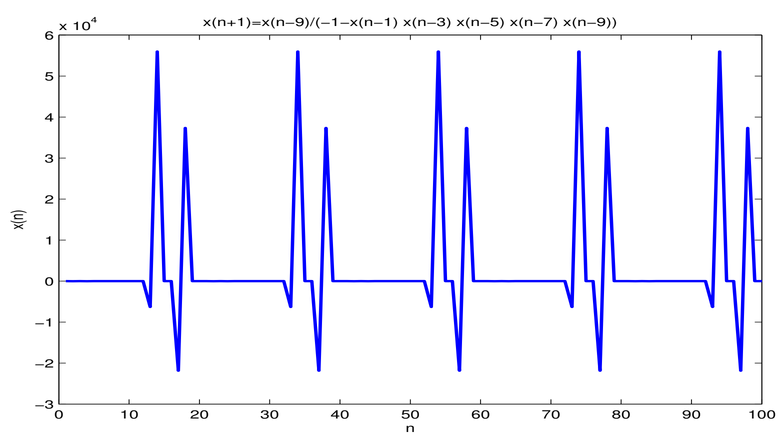

FIGURE 7

Example 8: By choosing $x_{-9} = \dfrac{-2}{x_{-7}x_{-5}x_{-3}x_{-1}}$ and $x_{-8} = \dfrac{-2}{x_{-6}x_{-4}x_{-2}x_0}$ where $x_{-7} = 5, x_{-6} = 4, x_{-5} = 3, x_{-4} = 2, x_{-3} = 2, x_{-2} = 1, x_{-1} = 7, x_0 = 9$ for Eq. (7). See Figure 8.

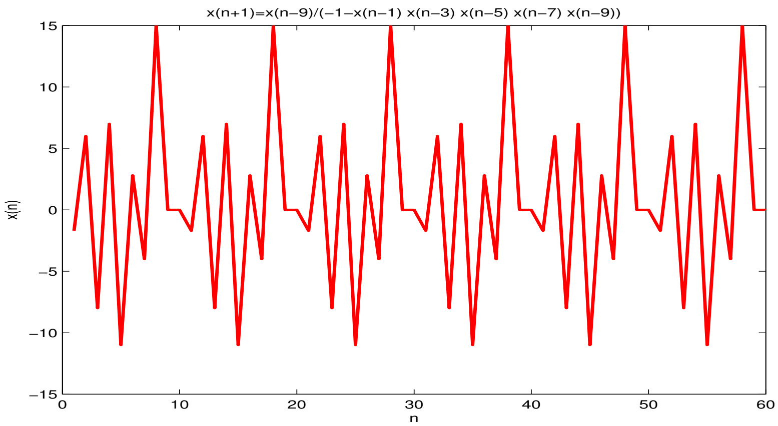

FIGURE 8

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